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Similarity Reductions of Nonisospectral KP Equation by a Direct Method*

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Abstract *On bases of the direct method developed by Clarkson and Kruskal [J. Math. Phys.* **27** *(1989) 2201], the (2+1)-dimensional nonisospectral Kadomtsev–Petviashvili (KP) equation has been reduced to three types of (1+1)-dimensional partial differential equations. We focus on solving the third type of reduction and dividing them into three subcases, from which we obtain rich solutions including some arbitrary functions.*

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1 Introduction

We know that the Kadomtsev–Petviashvili (KP) equation[1]

$$-u_{tx} + 6\,{u_x}^2 + 6\,uu_{xx} + u_{xxxx} + u_{yy} = 0\,,$$

plays an important role in many fields of physics, particularly in fluid mechanics, plasma physics, gas dynamics, etc. It is also of considerable importance in mathematics because it is one of a few equations in more than (1+1) dimensions that are completely integrable. In 1991, Lou[2] made use of the direct method presented by Clarkson and Kruskal[3] to reduce KP equation to three types of (1+1)-dimensional partial differential equations, which are equivalent to the three types of the similarity reduction equations obtained by the classical Lie approach but with different independent variables. Because the solutions obtained contain more arbitrary functions missed by the classical Lie approach, Lou's results show that the results obtained by the CK direct method contain those obtained by the classical Lie group approach and the results of the nonclassical Lie approach include those of the direct method.

Recently, more and more physicists and mathematicians are interested in studying the nonisospectral and variable coefficients generalizations of completely integrable nonlinear evolution equations.[4−12] The need for studying them is due to the fact that the physical situations in which equations with constant coefficients arise, tend to be highly idealized so that equations with variable coefficients and nonisospectral parameters may provide more realistic models, for example, in the propagation of (small-amplitude) surface waves in straits or large channels of (slowly) varying depth and width and nonvanishing vorticity.[4,5]

Based on the direct method extended by Lou, we would like to investigate the nonisospectral KP equation[13]

$$(4u_t + yu_{xxx} + 6yuu_x + 2xu_y)_x + 3yu_{yy} + 4u_y = 0\,, \quad (1)$$

which was introduced by Chen *et al.*[13] By constructing the symmetries and their algebraic structures for isospectral and nonisospectral evolution equations of (2+1)-dimensional systems, Chen *et al.* introduced the implicit representations of the isospectral flows K_m and nonisspectral flows σ_n in the high-dimensional cases. Here, as a result, we reduce the nonisospectral KP equation to three types of (1+1)-dimensional partial differential equations. We focus on solving the third type of reduction found in this paper and dividing them into three subcases, from which we obtain rich solutions including some arbitrary functions.

This paper is organized as follows. In Sec. 2 the nonisospectral KP equation is reduced to three types of (1+1)-dimensional partial differential equations. In Sec. 3 the third type of reduction is investigated and rich similarity reduction solutions are obtained. The summary and conclusion will be given finally.

2 Symmetry Reductions of Nonisospectral KP Equation

Firstly, for the nonisospectral KP equation (1), we show here that it is sufficient to seek a similarity reduction in the special form

$$u(x,y,t) = \alpha(x,y,t) + \beta(x,y,t) \times w(\xi(x,y,t), \eta(x,y,t))\,, \quad (2)$$

rather than the more general form

$$u(x,y,t) = U(x,y,t,w(\xi(x,y,t),\eta(x,y,t)))\,. \quad (3)$$

Substituting Eq. (3) into Eq. (1), we can get (because the

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formula is too long, just part of it is shown here)

$$y{\xi_x}^4U_w w_{\xi\xi\xi\xi} + 4y{\xi_x}^3\eta_x U_w w_{\xi\xi\xi\eta} + 4y{\xi_x}^4 U_{ww} w_{\xi\xi\xi} w_\xi + 6y{\xi_x}^2\xi_{xx}U_\xi w_{\xi\xi\xi} + 4y{\xi_x}^3\eta_x U_{ww} w_{\xi\xi\xi} w_\eta + \cdots \quad (4)$$

To make Eq. (4) a (1+1)-dimensional partial differential equation in $w(\xi,\eta)$, the ratios of different derivatives of $w(\xi,\eta)$ must be functions of w, ξ, and η. Using the coefficient of $w_{\xi\xi\xi\xi}$ as the normalizing coefficient, the coefficient of $w_{\xi\xi\xi}w_\xi$ requires that

$$4y{\xi_x}^4 U_{ww} = y{\xi_x}^4 U_w \Gamma(w,\xi,\eta)\,, \quad (5)$$

where $\Gamma(w,\xi,\eta)$ is a function to be determined. Hence

$$\Gamma(w,\xi,\eta) = \frac{4U_{ww}}{U_w}\,,$$

which after two integrations yields

$$U(x,y,t,w) = \Theta(x,y,t)\Gamma(w,\xi,\eta) + \Phi(x,y,t) \quad (6)$$

with $\Theta(x,y,t)$ and $\Phi(x,y,t)$ being arbitrary functions. Therefore it is sufficient to seek similarity reductions of the nonisospectral KP equation (1) in the form (2).

Then substituting Eq. (2) into Eq. (1) yields

$$\gamma_0 w_{\xi\xi\xi\xi} + \gamma_1 w_{\eta\eta\eta\eta} + 6\gamma_2({w_\xi}^2 + ww_{\xi\xi}) + 6\gamma_3({w_\eta}^2 + ww_{\eta\eta}) + 12\gamma_4(w_\xi w_\eta + ww_{\xi\eta}) + 4\gamma_5 w_{\xi\xi\xi\eta} + 4\gamma_6 w_{\eta\eta\eta\xi} + 6\gamma_7 w_{\xi\xi\eta\eta} + 6\gamma_8 w^2 + 6\gamma_9 ww_\xi + 6\gamma_{10} ww_\eta + \gamma_{11} w_{\xi\xi\xi} + \gamma_{12} w_{\eta\eta\eta} + \gamma_{13} w_{\xi\xi\eta} + \gamma_{14} w_{\eta\eta\xi} + \gamma_{15} + \gamma_{16} w + \gamma_{17} w_\xi + \gamma_{18} w_\eta + \gamma_{19} w_{\xi\xi} + \gamma_{20} w_{\eta\eta} + \gamma_{21} w_{\xi\eta} = 0\,, \quad (7)$$

where

$$\gamma_0 = y\beta\xi_x^4\,, \quad (8)$$
$$\gamma_1 = y\beta\eta_x^4\,, \quad (9)$$
$$\gamma_2 = y\beta^2\xi_x^2\,, \quad (10)$$
$$\gamma_3 = y\beta^2\eta_x^2\,, \quad (11)$$
$$\gamma_4 = y\beta^2\xi_x\eta_x\,, \quad (12)$$
$$\gamma_5 = y\beta\xi_x^3\eta_x\,, \quad (13)$$
$$\gamma_6 = y\beta\xi_x\eta_x^3\,, \quad (14)$$
$$\gamma_7 = y\beta\xi_x^2\eta_x^2\,, \quad (15)$$
$$\gamma_8 = y(\beta_x^2 + \beta\beta_{xx})\,, \quad (16)$$
$$\gamma_9 = y(4\beta\beta_x\xi_x + \beta^2\xi_{xx})\,, \quad (17)$$
$$\gamma_{10} = y(4\beta\beta_x\eta_x + \beta^2\eta_{xx})\,, \quad (18)$$
$$\gamma_{11} = y(4\beta_x\xi_x^3 + 6\beta\xi_x^2\xi_{xx})\,, \quad (19)$$
$$\gamma_{12} = y(4\beta_x\eta_x^3 + 6\beta\eta_x^2\eta_{xx})\,, \quad (20)$$
$$\gamma_{13} = y(12\beta\eta_x\xi_x\xi_{xx} + 12\beta_x\eta_x\xi_x^2 + 6\beta\eta_{xx}\xi_x^2)\,, \quad (21)$$
$$\gamma_{14} = y(12\beta\eta_x\xi_x\eta_{xx} + 12\beta_x\eta_x\eta_x^2 + 6\beta\xi_{xx}\eta_x^2)\,, \quad (22)$$
$$\gamma_{15} = 6y\alpha\alpha_{xx} + 3y\alpha_{yy} + 6\alpha_y + +4\alpha_{xt} + y\alpha_{xxxx} + 6y\alpha_x^2 + 2x\alpha_{xy}\,, \quad (23)$$
$$\gamma_{16} = 6y\alpha\beta_{xx} + 3y\beta_{yy} + 6\beta_y + +4\beta_{xt} + y\beta_{xxxx} + 6y\beta\alpha_{xx} + 2x\beta_{xy} + 12y\alpha_x\beta_x\,, \quad (24)$$
$$\gamma_{17} = 4\beta_t\xi_x + 4\beta_x\xi_t + 6\beta\xi_y + 4\beta\xi_{xt} + 4y\beta_{xxx}\xi_x + y\beta\xi_{xxxx} + 6y\beta_{xx}\xi_{xx} + 4y\beta_x\xi_{xxx} + 2x\beta_x\xi_y + 2x\beta_y\xi_x + 2x\beta\xi_{xy} + 6y\beta_y\xi_y + 3y\beta\xi_{yy} + 12y\alpha_x\beta\xi_x + 12y\alpha\xi_x\beta_x + 6y\alpha\beta\xi_{xx}\,, \quad (25)$$
$$\gamma_{18} = 4\beta_t\eta_x + 4\beta_x\eta_t + 6\beta\eta_y + 4\beta\eta_{xt} + 4y\eta_{xxx}\beta_x + y\beta\eta_{xxxx} + 6y\eta_{xx}\beta_{xx} + 4y\eta_x\beta_{xxx} + 2x\beta_x\eta_y + 2x\beta_y\eta_x + 2x\beta\eta_{xy} + 6y\beta_y\eta_y + 3y\beta\eta_{yy} + 12y\alpha_x\beta\eta_x + 12y\alpha\beta_x\eta_x + 6y\alpha\beta\eta_{xx}\,, \quad (26)$$
$$\gamma_{18} = 4\beta\xi_t\xi_x + 3y\beta\xi_{xx}^2 + 6y\beta_{xx}\xi_x^2 + 3y\beta\xi_y^2 + 12y\beta_x\xi_x\xi_{xx} + 4y\beta\xi_x\xi_{xxx} + 6y\alpha\beta\xi_x^2 + 2x\beta\xi_x\xi_y\,, \quad (27)$$
$$\gamma_{20} = 4\beta\eta_t\eta_x + 3y\beta\eta_{xx}^2 + 6y\beta_{xx}\eta_x^2 + 3y\beta\eta_y^2 + 12y\beta_x\eta_x\eta_{xx} + 4y\beta\eta_x\eta_{xxx} + 6y\alpha\beta\eta_x^2 + 2x\beta\eta_x\eta_y\,, \quad (28)$$
$$\gamma_{21} = 4\beta\xi_t\eta_x + 4\beta\eta_t\xi_x + 6y\beta\xi_{xx}\eta_{xx} + 12y\beta_{xx}\xi_x\eta_x + 12y\beta_x\xi_x\eta_{xx} + 12y\beta_x\eta_x\xi_{xx} + 4y\beta\eta_x\xi_{xxx} + 6y\beta\eta_y\xi_y + 4y\beta\eta_{xxx}\xi_x + 12y\alpha\beta\eta_x\xi_x + 2x\beta\xi_x\eta_y + 2x\beta\xi_y\eta_x\,. \quad (29)$$

(i) If $\xi_x \neq 0$, these conditions read

$$\gamma_i = \gamma_0\Gamma_i(\xi,\eta), \quad (i = 1,2,\ldots,21)\,, \quad (30)$$

where $\Gamma_i(\xi,\eta)$ $(i = 1,2,\ldots,21)$ are some functions of ξ and η to be determined later. Firstly we give some freedoms of α, β, ξ, η, and w without loss of generality:

Remark 1 If $\alpha(x,y,t)$ has the form $\alpha = \alpha_0(x,y,t) + \beta(x,y,t)\Omega(\xi,\eta)$, we can take $\Omega(\xi,\eta) = 0$, then $\alpha = \alpha_0(x,y,t)$.

Remark 2 if $\beta = \beta_0(x,y,t)\Omega(\xi,\eta)$, we can take $\Omega = \Omega_0 =$ constant, then $\beta = c\beta_0(x,y,t)$.

Remark 3 if $\xi = \xi(\xi_0(x,y,t),\eta)$ (or $\eta = \eta(\xi,\ \eta_0(x,y,t))$), then we can take $\xi = \xi_0$ (or $\eta = \eta_0$).

Remark 4 if $\xi(x,y,t)$ (or $\eta(x,y,t)$) is determined by an equation of the form $\Omega(\xi) = \xi_0(x,y,t)$ (or $\Omega(\eta) = \eta_0(x,y,t)$), then we can take $\xi = \xi_0(x,y,t)$ (or $\eta = \eta_0(x,y,t)$).

Then we will use the four remarks to determine Γ_i $(i = 1,2,\ldots,21)$.

Combining (30) with $i = 2$, we see

$$\beta = \xi_x^2\Gamma_2(\xi,\eta)\,.$$

Visa Remark 2, we can get

$$\beta = \xi_x^2, \qquad \Gamma_2 = 1\,. \quad (31)$$

Substituting Eq. (31) into Eq. (30) with $i = 5$, we have

$$\eta_x = \xi_x\Gamma_5(\xi,\eta)\,, \quad (32)$$

Integrating Eq. (32) with respect to x leads to

$$\eta = \int^x \xi_x(x,y,t)\Gamma_5(\xi(x,y,t),\eta(x,y,t))dx + \eta_0(y,t) = \Gamma_5(\xi,\eta)\xi - \int^x \xi[\Gamma_{5\xi}(\xi,\eta)\xi_x + \Gamma_{5\eta}(\xi,\eta)\eta_x]dx + \eta_0(y,t) \equiv \Gamma_5(\xi,\eta) + \eta_0(y,t)\,. \quad (33)$$

Remark 3 for η tells us that if $\eta = \Omega(\xi,\eta(\xi,\eta_0)$ one can take $\eta = \eta_0$, so from Eq. (33), we can take

$$\Gamma_5(\xi,\eta) = 0, \qquad \eta = \eta_0(y,t)\,. \quad (34)$$

Because of Eq. (31), equation (30) with $i = 9$ becomes

$$(\ln\xi_x)_x = \frac{1}{9}\Gamma_9(\xi,\eta)\xi_x\,. \quad (35)$$

Due to Eq. (34), integrating Eq. (35) with respect to x we can get

$$\Omega(\xi,\eta)\equiv\int^{\xi}\exp\Big(-\frac{1}{9}\int^{\xi_1}\Gamma_9(\xi',\eta)\mathrm{d}\xi'\Big)\mathrm{d}\xi_1 = \theta(y,t)x+\sigma(y,t)\equiv\xi_0(x,y,t)\,. \tag{36}$$

Remark 3 for ξ tells us we can take

$$\Gamma_9(\xi,\eta)\equiv 0\,,\quad \xi=\xi_0=\theta(y,t)x+\sigma(y,t)\,. \tag{37}$$

Substituting Eqs. (31), (34), and (37) into Eq. (30) with $i=19$ and using Remark 1, we can get

$$\alpha=-\frac{1}{6}\frac{4\,\theta\theta_t x+4\,\theta\sigma_t+3\,y{\theta_y}^2x^2+6\,y\theta_y x\sigma_y+3\,y{\sigma_y}^2+2\,x^2\theta\theta_y+2\,x\theta\sigma_y}{\theta^2 y}\,. \tag{38}$$

Due to Eqs. (31), (34), (37), and (38), we can get

$$\Gamma_1=\Gamma_3=\Gamma_4=\Gamma_5=\Gamma_6=\Gamma_7=\Gamma_8=\Gamma_9 =\Gamma_{10}=\Gamma_{11}=\Gamma_{12}=\Gamma_{13}=\Gamma_{14}=0\,. \tag{39}$$

Due to Eqs. (31), (34), and (37) , the remaining equations in Eq. (30) read

$$6\,\alpha_y+4\,\alpha_{tx}+6\,y\alpha\alpha_{xx}+6\,y{\alpha_x}^2+2\,x\alpha_{xy}+3\,y\alpha_{yy} =\Gamma_{15}\,y\theta^6\,, \tag{40}$$

$$12\,\theta\theta_y+6\,y{\theta_y}^2+6\,y\theta\theta_{yy}+6\,y\theta^2\alpha_{xx}=\Gamma_{16}\,y\theta^6\,, \tag{41}$$

$$12\,x\theta\theta_y+6\,\theta\sigma_y+12\,\theta\theta_t+12\,y{\theta_y}^2x+12\,y\theta_y\sigma_y +3\,y\theta\theta_{yy}x+3\,y\theta\sigma_{yy}+12\,y\alpha_x\theta^2=\Gamma_{17}\,y\theta^5\,, \tag{42}$$

$$6\theta\eta_y+12y\theta_y\eta_y+3y\theta\eta_{yy}=\Gamma_{18}y\theta^5\,, \tag{43}$$

$$3\,y{\eta_y}^2=\Gamma_{20}\,y\theta^4\,, \tag{44}$$

$$4\eta_t\theta+2x\theta\eta_y+6y\xi_y\eta_y=\Gamma_{21}y\theta^4\,. \tag{45}$$

Then we only need to solve Eqs. (40) $\sim$ (45). There are two possibilities to discuss further.

(ia) $\Gamma_{20}\neq 0$.

In this case, equation (44) can be rewritten as

$$\pm\sqrt{3}\eta_y\Gamma_{20}(\eta)^{-1/2}=\theta^2(y,t)\,. \tag{46}$$

Integrating Eq. (46) with respect to y and due to Remark 4 for η, we have

$$\Gamma_{20}=3\,, \tag{47}$$

$$\eta_y=\theta^2\quad\text{or}\quad \eta=\int^y\theta^2(y,t)\mathrm{d}y+\eta_0(t)\,. \tag{48}$$

Substituting Eq. (48) into Eq. (43) yields

$$\theta_y=\frac{1}{18}\theta^3\Gamma_{18}(\eta)-\frac{\theta}{3y}\equiv Z_1(\eta)\theta^3-\frac{\theta}{3y}\,. \tag{49}$$

Here $\Gamma_{18}=18Z_1$ is an arbitrary function of η and independent of ξ. Substituting Eqs. (38) and (49) into Eq. (41), we can easily get

$$\Gamma_{16}(\xi,\eta)=\Gamma_{16}(\eta)=18Z_1^2(\eta)+6Z_{1\eta}\,. \tag{50}$$

The left-hand sides of Eqs. (42) and (45) are x-dependent only in the linear forms, η is independent of x, and ξ is also x-dependent only in the linear forms, so we can take

$$\Gamma_{21}(\xi,\eta)=Z_3(\eta)\xi+Z_2(\eta)\,, \tag{51}$$

$$\Gamma_{17}(\xi,\eta)=Z_4(\eta)\xi+Z_5(\eta)\,. \tag{52}$$

Substituting Eqs. (51) and (52) into Eqs. (45) and (42), we can straight get

$$Z_3=6Z_1,\quad Z_4=9Z_1^2+3Z_{1\eta}\,,\quad Z_5=\frac{3}{2}Z_1Z_2+\frac{1}{2}Z_{2\eta}\,. \tag{53}$$

Here $Z_2(\eta)$ remains free. We can also have

$$\sigma_y=-\frac{2}{3}\frac{\eta_t}{y\theta}+\theta^2Z_1\sigma+\frac{1}{6}\theta^2Z_2\,, \tag{54}$$

$$\sigma=\Big[\int^y\frac{1}{6}\frac{(-4\,\eta_t+y\theta^3Z_2(\eta))\exp(-\int^y\theta^2Z_1(\eta)\mathrm{d}y)}{y\theta}\mathrm{d}y +\sigma_0(t)\Big]\exp^{\int^y\theta^2Z_1(\eta)\mathrm{d}y)} \tag{55}$$

with $\sigma_0(t)$ being an arbitrary function of t.

Similarly, we can take the form of Γ_{15} in Eq. (40) as

$$\Gamma_{15}(\xi,\eta)=Z_6(\eta)\xi^2+Z_7(\eta)\xi+Z_8(\eta)\,. \tag{56}$$

Substituting Eqs. (38), (48), (49), (55), and (56) into Eq. (40), we can get

$$Z_6=-27Z_1^4-3\,Z_{1\eta}^2-3\,Z_1Z_{1\eta\eta}-36Z_1^2Z_{1\eta}\,, \tag{57}$$

$$Z_7=-\frac{1}{2}Z_{2\eta\eta}Z_1-Z_{2\eta}Z_{1\eta}-\frac{1}{2}Z_2Z_{1\eta\eta}-9Z_1^3Z_2-\frac{9}{2}Z_{2\eta}Z_1^2-\frac{15}{2}Z_{1\eta}Z_1Z_2\,, \tag{58}$$

$$Z_8=-\frac{1}{4}Z_2^2Z_{1\eta}-\frac{3}{4}Z_1Z_2Z_{2\eta}-\frac{1}{12}Z_2Z_{2\eta\eta}-\frac{3}{4}Z_2^2Z_1^2-\frac{1}{12}Z_{2\eta}^2\,. \tag{59}$$

Now collecting all the results we have obtained, we will get the first type of similarity reduction as follows:

$$u=\Big(-\frac{1}{2}\theta^4Z_1^2+\frac{1}{18}y^{-2}\Big)x^2+\Big(-\frac{2}{3}\frac{\theta_t}{y\theta}+\frac{2}{3}\frac{Z_1\eta_t}{y}-\theta^3Z_1^2\sigma-\frac{1}{6}\theta^3Z_1Z_2\Big)x+\frac{2}{3}\frac{Z_1\eta_t\sigma}{y\theta}+\frac{1}{9}\frac{\eta_tZ_2}{y\theta} -\frac{2}{3}\frac{\sigma_t}{y\theta}-\frac{2}{9}\frac{\eta_t^2}{y^2\theta^4}-\frac{1}{72}\theta^2{Z_2}^2-\frac{1}{2}\theta^2{Z_1}^2\sigma^2-\frac{1}{6}\theta^2Z_1\sigma Z_2+\theta^2w(\xi,\eta)\,,$$

where

$$\alpha=\Big(-\frac{1}{2}\theta^4Z_1^2+\frac{1}{18}y^{-2}\Big)x^2+\Big(-\frac{2}{3}\frac{\theta_t}{y\theta}+\frac{2}{3}\frac{Z_1\eta_t}{y}-\theta^3Z_1^2\sigma-\frac{1}{6}\theta^3Z_1Z_2\Big)x$$

$$+\frac{2}{3}\frac{Z_1\eta_t\sigma}{y\theta}+\frac{1}{9}\frac{\eta_t Z_2}{y\theta}-\frac{2}{3}\frac{\sigma_t}{y\theta}-\frac{2}{9}\frac{\eta_t^2}{y^2\theta^4}-\frac{1}{72}\theta^2{Z_2}^2-\frac{1}{2}\theta^2{Z_1}^2\sigma^2-\frac{1}{6}\theta^2 Z_1\sigma Z_2\,,$$

$$\beta=\theta^2,\qquad \xi=\theta x+\sigma,\qquad \theta_y=Z_1(\eta)\theta^3-\frac{\theta}{3y}\,,$$

$$\sigma=\Big[\int^y\frac{1}{6}\frac{(-4\,\eta_t+y\theta^3 Z_2(\eta))\exp(-\int^y\theta^2 Z_1(\eta)\,\mathrm{d}y)}{y\theta}\,\mathrm{d}y+\sigma_0(t)\Big]\exp\Big(\int^y\theta^2 Z_1(\eta)\,\mathrm{d}y\Big)\,,$$

$$\eta=\int^y\theta^2(y,t)\,\mathrm{d}y+\eta_0(t)\,.$$

and $w(\xi,\eta)$ is determined by

$$w_{\xi\xi\xi\xi}+6(ww_\xi)_\xi+3w_{\eta\eta}+(6Z_1\xi+Z_2)w_{\xi\eta}+18Z_1w_\eta+\Big[(9Z_1^2+3Z_{1\eta})\xi+\Big(\frac{3}{2}Z_1Z_2+\frac{1}{2}Z_{2\eta}\Big)\Big]w_\xi$$
$$+(18Z_1^2+6Z_{1\eta})w+(-27Z_1^4-3\,Z_{1\eta}^2-3\,Z_1Z_{1\eta\eta}-36Z_1^2Z_{1\eta})\xi^2$$
$$+\Big(-\frac{1}{2}Z_{2\eta\eta}Z_1-Z_{2\eta}Z_{1\eta}-\frac{1}{2}Z_2Z_{1\eta\eta}-9Z_1^3Z_2-\frac{9}{2}Z_{2\eta}Z_1^2-\frac{15}{2}Z_{1\eta}Z_1Z_2\Big)\xi$$
$$-\frac{1}{4}Z_2^2Z_{1\eta}-\frac{3}{4}Z_1Z_2Z_{2\eta}-\frac{1}{12}Z_2Z_{2\eta\eta}-\frac{3}{4}Z_2^2Z_1^2-\frac{1}{12}Z_{2\eta}^2=0\,.$$

Here $Z_1(\eta)$ and $Z_2(\eta)$ are arbitrary functions of η and $\eta_0(t)$ and $\sigma_0(t)$ are any functions of t.

(ib) $\Gamma_{20}=0$.

In this case, equations (44) and (43) lead to

$$\eta_y=0,\quad \Gamma_{18}=0\,, \tag{60}$$

i.e.

$$\eta=\eta(t),\quad \text{or}\quad t=h(\eta) \tag{61}$$

with h being the inverse function of $\eta(t)$.

Substituting (61) into (45), we can get

$$4\eta_t=\Gamma_{21}(\eta)\theta^3 y\,.$$

Then for convenience taking

$$\theta^3 y=f^3(t)\,. \tag{62}$$

We have

$$\theta=\frac{f(t)}{y^{1/3}}\,, \tag{63}$$

$$4\Gamma_{21}^{-1}(\eta)\,\mathrm{d}\eta=f(t)\,\mathrm{d}t\,. \tag{64}$$

Integrating Eq. (63) and due to Remark 4 for η, we have

$$\eta=\int^t f(t')\,\mathrm{d}t'\,, \tag{65}$$

$$\Gamma_{21}=4\,. \tag{66}$$

Because of Eqs. (63) and (38), we have

$$\alpha=\frac{1}{18}\frac{-12\,f(t)f_txy-12\,f(t)\sigma_ty^{4/3}+(f(t))^2x^2-9\,y^{8/3}{\sigma_y}^2}{y^2(f(t))^2}\,. \tag{67}$$

Substituting Eqs. (67) and (63) into Eqs. (41) and (42) yields

$$\Gamma_{16}=0\,, \tag{68}$$

$$\sigma=\frac{3}{2}y^{2/3}g(t)+3\sigma_1(t)\sqrt[3]{y}+\sigma_2(t) \tag{69}$$

with

$$\Gamma_{17}(\eta)=\frac{g(t)+4f_t}{f(t)^4}\,, \tag{70}$$

and $\sigma_1(t)$ and $\sigma_2(t)$ are some arbitrary functions of t, while equation (40) yields the result

$$\Gamma_{15}(\eta)=-\frac{1}{3}\frac{2g_tf(t)+8f_{tt}f(t)+g(t)^2-16f_t^2}{f(t)^8}\,, \tag{71}$$

and equation (67) at last becomes

$$\alpha=\frac{1}{18}\frac{x^2}{y^2}-\frac{2}{3}\frac{f_tx}{yf(t)}-\frac{g_t}{f(t)}-2\frac{\sigma_{1t}}{\sqrt[3]{y}f(t)}-\frac{2}{3}\frac{\sigma_{2t}}{y^{2/3}f(t)}$$
$$-\frac{1}{2}\frac{(g(t))^2}{(f(t))^2}-\frac{g(t)\sigma_1(t)}{\sqrt[3]{y}t(f(t))^2}-\frac{1}{2}\frac{(\sigma_1(t))^2}{y^{2/3}(f(t))^2}\,. \tag{72}$$

Now recollecting all the results we have obtained, we will get the second type of similarity reduction as follows:

$$u=\frac{1}{18}\frac{x^2}{y^2}-\frac{2}{3}\frac{f_tx}{yf(t)}-\frac{g_t}{f(t)}-2\frac{\sigma_{1t}}{\sqrt[3]{y}f(t)}-\frac{2}{3}\frac{t\sigma_{2t}}{y^{2/3}f(t)}$$
$$-\frac{1}{2}\frac{(g(t)t)^2}{(f(t))^2}-\frac{g(t)\sigma_1(t)}{\sqrt[3]{y}(f(t))^2}-\frac{1}{2}\frac{(\sigma_1(t))^2}{y^{2/3}(f(t))^2}+\theta^2w\,,$$

where

$$\xi=\theta x+\sigma,\quad \theta=\frac{f(t)}{y^{1/3}}\,,$$

$$\sigma=\frac{3}{2}y^{2/3}g(t)+3\sigma_1(t)\sqrt[3]{y}+\sigma_2(t)\,,$$

$$\eta=\int^t f(t')\,\mathrm{d}t'\,,$$

and

$$[w_{\xi\xi\xi}+6ww_\xi-w_\eta]_\xi+\frac{g(t)+4f_t}{f(t)^4}w_\xi$$
$$-\frac{1}{3}\frac{2g_tf(t)+8f_{tt}f(t)+g(t)^2-16f_t^2}{f(t)^8}=0\,.$$

Here $f(t)$, $g(t)$, $\sigma_1(t)$, and $\sigma_2(t)$ are some arbitrary functions of t.

(ii) $\xi_x=0$

In this case, we also suppose that $\eta_x = 0$, otherwise exchange of ξ and η will lead to the first and the second types of reductions again. Furthermore, we suppose that $\xi_y \neq 0$(or $\eta_y \neq 0$), otherwise w would only be a function of t.

From $\xi_x = 0$, we know $\xi = \xi(y,t)$, so we can solve y reading $y = y(\xi,t)$. Hence

$$\eta = \eta(y,t) = \eta(y(\xi,t),t) \equiv \eta_1(\xi,t) \equiv \eta_1(\xi,\eta_0)\,. \tag{73}$$

Due to Remark 3, we have

$$\eta = \eta_0 = t\,. \tag{74}$$

And then

$$\xi = \xi(y,t) \equiv \xi(\xi_0,\eta)\,. \tag{75}$$

Using Remark 3 for ξ we get

$$\xi = y\,. \tag{76}$$

Substituting Eqs. (74) and (76) into Eqs. (8) $\sim$ (29), and (7), we will get

$$\begin{aligned}&3y\beta w_{\xi\xi} + 4\beta_x w_t + (6\beta + 2x\beta_x + 6y\beta_y)w_\xi \\ &\quad + 6(y\beta_x^2 + y\beta\beta_{xx})w^2 + (y\beta_{xxxx} + 6\beta_y + 4\beta_{tx} \\ &\quad + 2x\beta_{xy} + 3y\beta_{yy} + 12y\alpha_x\beta_x \\ &\quad + 6y\alpha\beta_{xx} + 6y\beta\alpha_{xx})w + 6y\alpha\alpha_{xx} + 3y\alpha_{yy} + 6\alpha_y \\ &\quad + 4\alpha_{tx} + y\alpha_{xxxx} + 6y\alpha_x^2 + 2x\alpha_{xy} = 0\,. \end{aligned}\tag{77}$$

Since $\beta \neq 0$, equation (77) can be written as

$$4\beta_x = 3y\beta\Gamma_A(\xi,\eta)\,, \tag{78}$$

$$6\beta + 2x\beta_x + 6y\beta_y = 3y\beta\Gamma_B(\xi,\eta)\,, \tag{79}$$

$$6(y\beta_x^2 + y\beta\beta_{xx}) = 3y\beta\Gamma_C(\xi,\eta)\,, \tag{80}$$

$$\begin{aligned}&y\beta_{xxxx} + 6\beta_y + 4\beta_{tx} + 2x\beta_{xy} + 3y\beta_{yy} + 12y\alpha_x\beta_x \\ &\quad + 6y\alpha\beta_{xx} + 6y\beta\alpha_{xx} = 3y\beta\Gamma_D(\xi,\eta)\,,\end{aligned} \tag{81}$$

$$\begin{aligned}&6y\alpha\alpha_{xx} + 3y\alpha_{yy} + 6\alpha_y + 4\alpha_{tx} + y\alpha_{xxxx} \\ &\quad + 6y\alpha_x^2 + 2x\alpha_{xy} = 3y\beta\Gamma_E(\xi,\eta)\,.\end{aligned} \tag{82}$$

From Eq. (78), we can solve

$$\beta = \beta_0(y,t)\exp\Bigl(\frac{3}{4}\Gamma_A(y,t)xy\Bigr)\,. \tag{83}$$

Substituting Eq. (83) into Eq. (80), we will have

$$\begin{aligned}&\frac{9}{8}y^3\beta(y,t)^2\Gamma_A(y,t)^2\exp\Bigl(\frac{3}{2}\Gamma_A(y,t)yx\Bigr) \\ &\quad = 3\Gamma_C(y,t)\beta_0(y,t)\exp\Bigl(\frac{3}{4}\Gamma_A(y,t)yx\Bigr)\,.\end{aligned} \tag{84}$$

Equation (84) is true for any x and y only for

$$\Gamma_A(y,t) = \Gamma_C(y,t) = 0\,. \tag{85}$$

Due to $\Gamma_A(y,t) = 0$ and Remark 2 for β, equation (83) becomes

$$\beta = \beta_0(y,t) = \beta_0(\xi,\eta) = 1\,. \tag{86}$$

Since $\beta = 1$, equation (79) becomes

$$\Gamma_B = \frac{2}{y}\,, \tag{87}$$

while equation (81) becomes

$$\Gamma_D(y,t) = 2\alpha_{xx}\,. \tag{88}$$

From Eq. (88) and Remark 1 for α, we can suppose $\alpha = \alpha_2(t)x^2 + \alpha_1(t)x$, equation (88) then becomes

$$\Gamma_D(y,t) = 4\alpha_2(y,t)\,, \tag{89}$$

and yielding equation (82)

$$\Gamma_E = \frac{4}{3}\frac{\alpha_{1t}(y,t)}{y} + 2\alpha_1(y,t)^2\,, \tag{90}$$

where

$$\frac{10}{3}\frac{\alpha_{2y}}{y} + \alpha_{2yy} + 12\alpha_2^2 = 0\,, \tag{91}$$

$$\frac{8}{3}\frac{\alpha_{2t}}{y} + \frac{8}{3}\frac{\alpha_{1y}}{y} + \alpha_{1yy} + 12\alpha_2\alpha_1 = 0\,. \tag{92}$$

At last, collecting all the results obtained in this case, we get the third type of similarity reduction. It reads

$$u = \alpha_2(y,t)x^2 + \alpha_1(y,t)x + w(y,t)\,, \tag{93}$$

and $\alpha_2(y,t)$, $\alpha_1(y,t)$, and $w(y,t)$ satisfy

$$\frac{10}{3}\frac{\alpha_{2y}}{y} + \alpha_{2yy} + 12\alpha_2^2 = 0\,, \tag{94}$$

$$\frac{8}{3}\frac{\alpha_{2t}}{y} + \frac{8}{3}\frac{\alpha_{1y}}{y} + \alpha_{1yy} + 12\alpha_2\alpha_1 = 0\,, \tag{95}$$

$$w_{yy} + \frac{2}{y}w_y + 4\alpha_2 w + \frac{4}{3}\frac{\alpha_{1t}}{y} + 2\alpha_1^2 = 0\,. \tag{96}$$

3 Solutions of Third Type

We will give out all the solutions of the third type obtained.

(i) The first type of special solution of Eqs. (93) $\sim$ (95) reads

$$\alpha_2 = 0\,, \tag{97}$$

$$\alpha_1 = \frac{A_2(t)}{y^{5/3}} + A_1(t)\,. \tag{98}$$

Substituting the results into Eq. (96), we can solve

$$\begin{aligned}w = &-\frac{1}{3}A_1(t)^2y^2 - 9A_1(t)A2(t)\sqrt[3]{y} - \frac{9}{2}\frac{A_2(t)^2}{y^{4/3}} + 6\frac{A_{2t}}{y^{2/3}} \\ &- \frac{A_3(t)}{y} - \frac{2}{3}A_{1t}y + A_4(t)\,.\end{aligned} \tag{99}$$

So we have

$$\begin{aligned}u(x,y,t) = &\Bigl[\frac{A_2(t)}{y^{5/3}} + A_1(t)\Bigr]x - \frac{1}{3}A_1(t)^2y^2 \\ &- 9A_1(t)A2(t)\sqrt[3]{y} - \frac{9}{2}\frac{A_2(t)^2}{y^{\frac{4}{3}}} + 6\frac{A_{2t}}{y^{2/3}} - \frac{A_3(t)}{y} \\ &- \frac{2}{3}A_{1t}y + A_4(t)\,.\end{aligned} \tag{100}$$

where $A_1(t)$, $A_2(t)$, $A_3(t)$, and $A_4(t)$ are all arbitrary functions of t.

(ii) The second type of special solution of Eqs. (93) $\sim$ (96) is

$$\alpha_2 = \frac{1}{18}y^{-2}\,, \tag{101}$$

$$\alpha_1 = B_1(t)y^{-\frac{2}{3}} + \frac{B_2(t)}{y}\,. \tag{102}$$

Substituting the results into Eq. (96), we can solve

$$\begin{aligned}w = &-\frac{3}{2}B_2(t)^2y^{2/3} - 6B_1(t)B_2(t)y^{1/3} - 2B_{2t}(t)y^{1/3} \\ &+ 3B_3(t)y^{-1/3} + B_4(t)y^{-2/3} - 9B_1(t)^2 - 6B_{1t}\,.\end{aligned} \tag{103}$$

So we can get

$$u = \frac{1}{18}y^{-2}x^2 + \Big(B_1(t)y^{-2/3} + \frac{B_2(t)}{y}\Big)x - \frac{3}{2}B_2(t)^2y^{2/3} - 6B_1(t)B_2(t)y^{1/3} - 2B_{2t}(t)y^{1/3} + 3B_3(t)y^{-1/3} + B_4(t)y^{-2/3} - 9B_1(t)^2 - 6B_{1t}\,, \tag{104}$$

where $B_1(t)$, $B_2(t)$, $B_3(t)$, and $B_4(t)$ are all arbitrary functions of t.

The general solutions of equations (93) $\sim$ (96) can be expressed by

$$\alpha_2 = \frac{-\wp(\sqrt{2}(3y^{1/3} + y_0(t)); 0; g_3(t))}{y^{4/3}} + \frac{1}{18y^2}\,. \tag{105}$$

α_1 satisfies

$$\alpha_{1yy} = \frac{1}{3y}\left[\frac{8\sqrt{8\wp^3 - g_3(t)}}{y^{4/3}}\Big(y_{0t} + \frac{1}{2}\int^{-\wp}\frac{g_{3t}(t)}{\sqrt{(-8f^3 - g_3(t))^3}}\mathrm{d}f\Big) - 8\alpha_{1y} + \frac{2\alpha_1(18\wp y^{2/3} - 1)}{y}\right]. \tag{106}$$

Here $y_0(t)$ and $g_3(t)$ are arbitrary functions of t. $\wp(\tau; g_2; g_3)$ is the Weierstrass elliptic function which is defined by

$$\Big(\frac{\mathrm{d}\wp}{d\tau}\Big)^2 = 4\wp^3 - g_2\wp - g_3\,.$$

While $w(y,t)$ satisfies a Lamé equation (96) with α_2 and α_1 given by Eqs. (105) and (106). The details of the solutions of the Lamé equation can be referred to Ref. [15].

4 Conclusions

Since the direct method was presented by Clarkson and Kruskal, there are many researchers to use and develop it to some important mathematical and physical equations. In particular, Lou's excellent work further extended this direct method and studied many nonlinear evolution equation successfully.[16,17] These applications show this fact that the solutions obtained by this method contain the results obtained by the classic Lie group approach, i.e., additional arbitrary functions contained in similarity reductions are missing by the classical approach.

Here, we have successfully reduced the (2+1)-dimensional nonisospectral KP equation to three types of (1+1)-dimensional partial differential equations. Compared with the three types of the similarity reductions of the (2+1)-dimensional isospectral KP equation obtained by Lou,[2] the results reduced by us are more complex for that the equation investigated is nonisospectral. Here we have solved the third type of reduction found in this paper and divided them into three subcases, from which we have obtained rich solutions including some arbitrary functions. With regards to the first and the second types of reduction equations, some transformations of dependent and independent variables are needed to seek for simplifications. This will be our further work.

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