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Research paper

Dynamics of rogue waves in the partially $\mathcal{PT}$-symmetric nonlocal Davey–Stewartson systems

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ABSTRACT

In this work, we study the dynamics of rogue waves in the partially $\mathcal{PT}$-symmetric nonlocal Davey–Stewartson(DS) systems. Using the Darboux transformation method, general rogue waves in the partially $\mathcal{PT}$-symmetric nonlocal DS equations are derived. For the partially $\mathcal{PT}$-symmetric nonlocal DS-I equation, the solutions are obtained and expressed in term of determinants. For the partially $\mathcal{PT}$-symmetric DS-II equation, the solutions are represented as quasi-Gram determinants. It is shown that the fundamental rogue waves in these two systems are rational solutions which arises from a constant background at $t \to -\infty$, and develops finite-time singularity on an entire hyperbola in the spatial plane at the critical time. It is also shown that the interaction of several fundamental rogue waves is described by the multi rogue waves. And the interaction of fundamental rogue waves with dark and anti-dark rational travelling waves generates the novel hybrid-pattern waves. However, no high-order rogue waves are found in this partially $\mathcal{PT}$-symmetric nonlocal DS systems. Instead, it can produce some high-order travelling waves from the high-order rational solutions.

1. Introduction

It is known to us all that the integrable nonlinear evolution equations are exactly solvable models which play an important role in a lot of branches of nonlinear science, especially in the study of nonlinear physical systems, including water waves, nonlinear optics, Bose–Einstein condensates and plasma physics. There are numerous celebrated continuous and discrete integrable systems that are physically revelent. In particular, the nonlinear Schrödinger (NLS) [1] and the Davey–Stewartson (DS) [2] equations are classical examples of generic integrable PDEs. The NLS-type equations are the essential models describing optical wave propagation in nonlinear optics. The DS equations, which can be seen as the multi-dimensional extension of NLS equation, are also the universal models governing the evolution of two-dimensional wave packet on water of finite depth.

In the last several years, $\mathcal{PT}$-symmetric systems, which allow for lossless-like propagation due to their balance of gain and loss, have attracted considerable attention and triggered renewed interest in integrable systems. Quite a lot of work were done on the new nonlocal integrable systems [3–22]. These nonlocal integrable equations are different from local integrable equations and could produce novel patterns of solution dynamics and intrigue new physical applications. Among

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these models, the $\mathcal{PT}$-symmetric NLS equation was the first nonlocal integrable equation proposed in [3]:

$$iq_t(x,t) = q_{xx}(x,t) + V(x,t)q(x,t), \tag{1}$$

with $V(x,t) = -2\sigma q(x,t)q^*(-x,t),\ \sigma = \pm 1$. It is shown to be an integrable infinite dimensional Hamiltonian equation with a self-induced potential satisfying the $\mathcal{PT}$-symmetry condition: $V(x,t) = V^*(-x,t)$. The nonlocality occurs in the form that one of the nonlinear terms is dependent on variable evaluated at $-x$. One-soliton solution with singularity for the focusing nonloc-NLS Eq. (1) has been obtained via the inverse scattering transform (IST). More detailed study of the inverse scattering theory for Eq. (1) was developed and the Cauchy problem was formulated in [7] via the Riemann–Hilbert problem (RHP).

As an integrable multidimensional versions of the nonlocal nonlinear Schrödinger equation, a new integrable nonlocal Davey–Stewartson (DS) equation is recently introduced in Refs [9,12].

$$\begin{aligned} &iu_t + \frac{1}{2}\alpha^2 u_{xx} + \frac{1}{2}u_{yy} + (uv - w)u = 0, \\ &w_{xx} - \alpha^2 w_{yy} - 2[(uv)]_{xx} = 0, \end{aligned} \tag{2}$$

where u, v and w are functions of x, y, t, $\alpha^2 = \pm 1$ is the equation-type parameter(with $\alpha^2 = 1$ being the DS-I and $\alpha^2 = -1$ being DS-II). With different symmetry reductions of potential function u and v, this equation contains two nonlocal versions: (i). $\mathcal{PT}-$ symmetric nonlocal reduction : $v(x,y,t) = \epsilon\bar{u}(-x,-y,t)$; (ii). Partially $\mathcal{PT}-$ symmetric nonlocal reduction : $v(x,y,t) = \epsilon\bar{u}(-x,y,t)$, here the sign $\bar{u}$ represents the complex conjugation of this function, and $\epsilon = \pm 1$ is the sign of nonlinearity. For these two nolocal versions, several results have been obtained in [15–17] by Darboux transformation or the Hirota bilinear method. Furthermore, other versions of nonlocal DS equations are also proposed and studied in [12] according to different types of time-space coupling. Especially, when $v(x,y,t) = \epsilon\bar{u}(x,-y,t)$, it produces another version of partially $\mathcal{PT}$-symmetric nonlocal DS equations. Especially, Denoting $V(x,y) = u(x,y,t)\bar{u}(-x,y,t)$. Then $V(x, y)$ is the partially $\mathcal{PT}$-symmetric potential satisfying the condition $\bar{V}(x,y) = V(-x,y)$. It is shown in [25] that the partially $\mathcal{PT}$ symmetric potentials can also possess all-real spectra and continuous soliton families, and they may find interesting applications in optics [25,30]. Moreover, these partially $\mathcal{PT}$-symmetric DS equations are the two-dimensional extensions for the $\mathcal{PT}$ symmetric NLS Eq. (1), which expands the concept of $\mathcal{PT}$-symmetry into multi-dimensions.

Therefore, motivated by the potential physical applications of partially $\mathcal{PT}$-symmetric systems in multi dimensions [25–30]. In this article, we focus on the nonlocal DS systems with partially $\mathcal{PT}$-symmetric potential (i.e., the nonlocal version (ii)). Using the Darboux transformation method, general rogue waves in the partially $\mathcal{PT}$-symmetric nonlocal DS equations are derived. On the one hand, solutions of the partially $\mathcal{PT}$-symmetric nonlocal DS-I equation are obtained and expressed in terms of determinants. On the other hand, through the binary DT, solutions of the partially $\mathcal{PT}$-symmetric DS-II equation are constructed and represented as quasi-Gram determinants.

With different parameters chosen in the fundamental rational solutions, it is shown that the fundamental rogue waves in these two systems are rational solutions which arises from a constant background at $t \to -\infty$, and develops finite-time singularity on an entire hyperbola in the spatial plane at the critical time. It is also shown that the interaction of several fundamental rogue waves is described by the multi rogue waves, which are generated from multi-rational solutions, and the singular time points in these multi rogue waves appear in pairs or in a time interval. It is further shown that the interaction of fundamental rogue waves with dark and anti-dark rational travelling waves generates the hybrid-pattern waves. This novel pattern, which contains three different wave patterns in one solution, to the best of our knowledge, has never been reported in the local and nonlocal DS systems.

As we know, the local DS systems possess several patterns of high-order rogue waves [23,24]. However, in this partially $\mathcal{PT}$-symmetric nonlocal DS systems, we can not find any high-order rogue waves, even though there are some high-order rational travelling waves produced from the high-order rational solutions. This might because the singularity in the fundamental rogue waves will quickly increase if one proceeds the iteration through the high-order DT. While the iteration of N-fold DT only increase the numbers or the range of singularities, as what we have shown in the multi-rogue waves. It is found in refs. [25–30] that some possible applications in optics have been shown in the partially $\mathcal{PT}$-symmetric physical systems. We expect these rogue-wave solutions could have interesting implications for partially $\mathcal{PT}$-symmetric in multi-dimensions.

2. Darboux transformation for nonlocal DS system

In this section, we first work on the form of Darboux transformation in the general Davey-Stewartson system with the partially $\mathcal{PT}-$symmetric nonlocal reduction. For Eq. (2), the corresponding auxiliary linear system is reduced from the (2+1) dimensional AKNS system:

$$\Phi_y = J\Phi_x + P\Phi, \tag{3}$$

$$\Phi_t = \sum_{j=0}^{n} V_{n-j}\partial^j\Phi, \tag{4}$$

where $\partial = \partial/\partial x$, V_j are $N \times N$ matrices, J is $N \times N$ constant diagonal matrix, and P is a $N \times N$ off-diagonal matrix.

Taking $N=2,\ n=2$ in (3)-(4), it generate the following Lax-pair for systems (2) :

$$L\Phi=0,\quad L=\partial_y-J\partial_x-P, \tag{5}$$

$$M\Phi=0,\quad M=\partial_t-\sum_{j=0}^{2}V_{2-j}\partial^j=\partial_t-i\alpha^{-1}J\partial_x^2-i\alpha^{-1}P\partial_x-\alpha^{-1}V, \tag{6}$$

where,

$$J=\alpha^{-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ P=\begin{pmatrix}0 & u\\ -v & 0\end{pmatrix},$$

$$V=\frac{i}{2}\begin{pmatrix}\omega_1 & u_x+\alpha u_y\\ -v_x+\alpha v_y & \omega_2\end{pmatrix},$$

with

$$w=uv-\frac{1}{2\alpha}(\omega_1-\omega_2). \tag{7}$$

With the partially $\mathcal{PT}-$ symmetric reduction $v(x,y,t)=\epsilon\bar{u}(-x,y,t)$, the integrability condition:
$\Phi_{y,t}=\Phi_{t,y}$ leads to the partially $\mathcal{PT}-$ symmetric nonlocal DS equations.

2.1. Darboux transformation for partially $\mathcal{PT}$-symmetric nonlocal DS-I

It is already known in [31,32] that for any invertible matrix θ such that $L(\theta)=M(\theta)=0$, the operator

$$G_\theta=\theta\partial\theta^{-1},\quad \partial=\partial_x, \tag{8}$$

makes L and M form invariant under the elementary Darboux transformation:

$$L\to\tilde{L}=G_\theta LG_\theta^{-1},\ M\to\tilde{M}=G_\theta MG_\theta^{-1}.$$

Next, we introduce some notations. For operator L and its adjoint operator $L\dagger$, defining the space S and $S\dagger$ which stand for the sets of nontrivial solutions in the kernel of the operator, i.e.,:

$$S=\{\theta,\ \theta\text{ is nonsingular}:L(\theta)=0\},$$
$$S^\dagger=\{\rho,\ \rho\text{ is nonsingular}:L^\dagger(\rho)=0\},$$

and define $\tilde{S},\ \tilde{S}^\dagger$ for operator $\tilde{L},\ \tilde{L}^\dagger$ etc. Thus, this elementary DT (8) defines the mapping:
$G_\theta:S\to\tilde{S}$.

For the Darboux transformation, as we known, if we pose some restriction to the potential (e.g. this $v(x,y,t)=\epsilon\bar{u}(-x,-y,t)$ in the partially $\mathcal{PT}$-symmetric nonlocal DS equations), then the transforation does not naturally preserve the conditions. Therefore, in this case, we need more restrictions on the choices of solution matrix θ.

Let $\sigma=\begin{pmatrix}0 & -\epsilon\\ 1 & 0\end{pmatrix}$, then potential matrix P and V_2 in (5)-(6) satisfy the following symmetric reduction

$$\sigma P(x,y,t)\sigma^{-1}=\overline{P}(-x,y,t),\ \sigma V_2(x,y,t)\sigma^{-1}=\overline{V}_2(-x,y,t), \tag{9}$$

here we need the property $\omega_1(x,y,t)=-\overline{\omega_2}(-x,y,t)$, which can be derived form the integrability condition.

This give rise to the symmetry constraint in L, M:

$$\sigma L\sigma^{-1}=\overline{L}_{(x\to-x)},\ \sigma M\sigma^{-1}=\overline{M}_{(x\to-x)}. \tag{10}$$

Suppose $\begin{pmatrix}\xi(x,y,t)\\ \eta(x,y,t)\end{pmatrix}$ is a vector solution of Eqs. (5)-(6), it is inferred from symmetry (10) that $\begin{pmatrix}-\epsilon\bar{\eta}(-x,y,t)\\ \bar{\xi}(-x,y,t)\end{pmatrix}$ is also a solution. Hence we can choose the matrix θ as:

$$\theta=\begin{pmatrix}\xi(x,y,t) & -\epsilon\bar{\eta}(-x,y,t)\\ \eta(x,y,t) & \bar{\xi}(-x,y,t)\end{pmatrix}, \tag{11}$$

and θ also admits the symmetry

$$\overline{\theta}(x,y,t)=\sigma\theta(-x,y,t)\sigma^{-1}. \tag{12}$$

Since the n-fold DT is nothing but a n-times iteration of the one-fold DT, we merely consider the one-fold DT. With the action of elementary DT, we obtain the relation between potential matrices:

$$\widetilde{P}=P+[J,S],\ S=\theta_x\theta^{-1}, \tag{13}$$

$$\widetilde{V}_2 = V_2 + V_{1,x} + 2V_0 S_x + [V_0, S]S + [V_1, S]. \tag{14}$$

Moreover, it can be verified that transformation G_θ keep the reduction relation (9) and (10) invariant, i.e:

$$\sigma \widetilde{P}(x, y, t)\sigma^{-1} = \overline{\widetilde{P}}(-x, y, t), \quad \sigma \widetilde{L}\sigma^{-1} = \overline{\widetilde{L}}_{(x\to -x)},$$

$$\sigma \widetilde{V}_2(x, y, t)\sigma^{-1} = \overline{\widetilde{V}}_2(-x, y, t), \quad \sigma \widetilde{M}\sigma^{-1} = \overline{\widetilde{M}}_{(x\to -x)},$$

which implies the solution for partially $\mathcal{PT}$-symmetric nonlocal DS-I equation:

$$\widetilde{u} = u + 2\alpha^{-1}S_{1,2}, \quad \overline{\widetilde{u}} = \overline{u} + 2\epsilon\alpha^{-1}S_{2,1}, \tag{15}$$

$$\widetilde{w} = w - 2\alpha^2[\mathbf{tr}(S)]_x = w - 2\alpha^2[\ln(\det(\theta))]_{xx}. \tag{16}$$

In general, the N-fold Darboux matrix for partially $\mathcal{PT}$-symmetric nonlocal DS-I equation has the form:

$$T_N = \partial^N - \sum_{k=1}^{N} s_k \partial^{N-k}. \tag{17}$$

Transformation (17) maps: $L \to \tilde{L} = T_N L T_N^{-1}$, with $\tilde{L} = \partial_y - J\partial_x - P_{[N]}$, and the potential matrix has the relation:

$$P_{[N]} = P + [J, s_1], \tag{18}$$

$$V_{2,[N]} = V_2 + V_{1,x} + 2V_0 s_{1,x} + [V_0, s_1]s_1 + [V_1, s_1]. \tag{19}$$

The coefficients matrices $s_1, s_2, \ldots, s_N$ are determined by the system of linear algebraic equations:

$$T_N(\Psi_k) = 0, \;\; \Psi_k = \begin{pmatrix} \xi_k & -\epsilon\bar{\eta}_k \\ \eta_k & \bar{\xi}_k \end{pmatrix}, \;\; k = 1, 2, \ldots, N. \tag{20}$$

Furthermore, the N-th order potential function for partially $\mathcal{PT}$-symmetric nonlocal DS-I equation solved from (18)-(19) can be represented in a determinant form:

$$u_{[N]} = u + 2\alpha^{-1}(s_1)_{1,2}, \quad \overline{u}_{[N]} = \overline{u} + 2\epsilon\alpha^{-1}(s_1)_{2,1}, \tag{21}$$

$$w_{[N]} = w - 2\alpha^{-2}[\mathbf{tr}(s_1)]_x, \tag{22}$$

where

$$(s_1)_{1,2} = \det\Sigma^{1,2}\det\Sigma^{-1}, \;\; (s_1)_{2,1} = \det\Sigma^{2,1}\det\Sigma^{-1}, \;\; \Sigma = \begin{pmatrix} \partial^{N-1}\Psi_1 & \cdots & \partial^{N-1}\Psi_N \\ \cdots & \cdots & \cdots \\ \Psi_1 & \cdots & \Psi_N \end{pmatrix}, \tag{23}$$

$\Sigma^{j,k}$ is the matrix which derived by replacing the$k$-th row of Σ with the$j$-th row of $\left(\partial^N\Psi_1, \cdots, \partial^N\Psi_N\right)$, $(s_1)_{k,j}$stands the entry in the k-th row and the j-th column of matrixs_1.

Furthermore, (22) can be further simplified into another form:

$$w_{[N]} = w - 2\alpha^{-2}[\ln(\det(\Sigma))]_{xx}, \tag{24}$$

and this can be verified via a direct calculation.

2.2. Binary darboux transformation for the partially $\mathcal{PT}$-symmetric nonlocal DS-II

As we known, the local DS-I equation does not possess a Darboux transformation in differential form. Instead, it has a binary Darboux transformation in integral form. As it has been shown for this partially $\mathcal{PT}$-symmetric nonlocal DS-I equation, we can construct an elementary DT in differential form, which has the same form with the DT reported in [15] where the DT is used to derive several types of bounded global explicit soliton solutions. However, for this partially $\mathcal{PT}$-symmetric nonlocal DS-II equation, the elementary DT is not enough. In the following, we are going to construct a binary DT in integral form for this equation.

Firstly, we recall some important properties for quasi-determinants which are introduced in Refs [33–37]. It is a generalization of the determinant to matrices with noncommutative entries. For a $n \times n$ matrix $M = (m_{i,j})$ over an, in general, non-commutative ring $\mathcal{R}$, the quasi-determinant for M is defined by

$$|M|_{i,j} = m_{i,j} - r_i^j (M^{i,j})^{-1} c_j^i, \tag{25}$$

where r_i^j represents the i-th row of M with the j-th element removed, c_j^i is the j-th column of M with the i-th element removed, and $M^{i,j}$ is a $(n-1)\times(n-1)$ minor obtained by deleting the i-th row and the j-th column in M. Usually, as what is shown below, quasi-determinants can be denoted by boxing the entry about which the expansion is made

$$|M|_{i,j} = \begin{vmatrix} M^{i,j} & c_j^i \\ r_i^j & \boxed{m_{i,j}} \end{vmatrix}. \tag{26}$$

In this paper, we consider the quasi-determinants that are only expanded about a term in the last entry. Taking a block matrix $M = \begin{pmatrix} A & B \\ C & d \end{pmatrix}$ for example, where $d \in \mathcal{R}$, A is a square matrix over $\mathcal{R}$ of arbitrary size, B, C are column and row vectors over $\mathcal{R}$ with compatible lengths, respectively, then the quasi-determinant of M expended about d is

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = d - CA^{-1}B.$$

Moreover, as a quasi-determinant version of Jacobis identity for determinants, the noncommutative Sylvesters theorem was established in [33], and a simple version of this theorem is given by

$$\begin{vmatrix} E & F & G \\ H & A & B \\ J & C & \boxed{D} \end{vmatrix} = \begin{vmatrix} E & F \\ J & \boxed{D} \end{vmatrix} - \begin{vmatrix} E & F \\ J & \boxed{C} \end{vmatrix} \begin{vmatrix} E & F \\ J & \boxed{A} \end{vmatrix}^{-1} \begin{vmatrix} E & G \\ H & \boxed{B} \end{vmatrix}. \tag{27}$$

Next, we give a brief derivation of the DT for the partially $\mathcal{PT}$-symmetric nonlocal DS-II equation. For this equation, the operator L has the constraint

$$-\kappa L^{\dagger}\kappa^{-1} = L_{(x\to -x)}, \ \ \kappa = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}. \tag{28}$$

Here the denotation $L_{(x\to -x)}$ means changing all the variables x in L to $-x$. However, for this operator, one can not find a suitable matrix solution θ to construct the DT to preserve the constraint (28). In order to overcome this problem one need to use the binary Darboux transformation (BDT). The standard BDT scheme has been introduced and developed in ref [31]. Several different forms of Darboux transformations for the DS equations have been studied in Refs [38,39]. In this work, we inherit the idea from [31,32] and construct a corresponding BDT for this partially $\mathcal{PT}$-symmetric nonlocal DS-II equation.

Considering operators $\hat{L}$, which is another copy of L with new coefficients. Define the corresponding sets of non-singular solutions $\hat{S}$, let $\hat{\theta} \in \hat{S}$ s.t $G_{\hat{\theta}} : \hat{S} \to \tilde{S}$. Thus, we can get the following mapping:

$$S \xrightarrow{G_\theta} \tilde{S} \xrightarrow{G_{\hat{\theta}}^{-1}} \hat{S}$$

$$S^{\dagger} \xrightarrow{G_\theta^{\dagger -1}} \tilde{S}^{\dagger} \xrightarrow{G_{\hat{\theta}}^{\dagger}} \hat{S}^{\dagger}$$

For a given $\phi \in S\dagger$, $G_\theta^{\dagger -1}(\phi) \in \tilde{S}^{\dagger}$, by determine the kernel of $G_{\hat{\theta}}^{\dagger}$ we can obtain some nontrivial solutions in $\tilde{S}^{\dagger}$. Thus, one can further define a solution $\hat{\theta} = \left(G_\theta^{\dagger -1}(\phi)\right)^{\dagger -1} = -\theta\Omega^{-1}(\theta,\phi)$, and the BDT for operator L is:

$$G_{\theta,\phi} = G_{\hat{\theta}}^{-1} G_\theta = I - \theta\Omega^{-1}(\theta,\phi)\partial^{-1}\phi^{\dagger}, \quad \Omega(\theta,\phi) = \partial^{-1}(\phi^{\dagger}\theta). \tag{29}$$

To proceed the iteration of DT we also need

$$G_{\theta,\phi}^{\dagger -1} = G_{\hat{\theta}}^{\dagger} G_\theta^{\dagger -1} = I - \phi\Omega^{\dagger -1}(\theta,\phi)\partial^{-1}\theta^{\dagger}. \tag{30}$$

This Darboux transformation makes sense for any $m \times k$ matrices θ and ϕ, and we only need $\Omega(\theta, \phi)$ to be an invertible square matrix. To reduce (29) to the BDT for the partially $\mathcal{PT}$-symmetric nonlocal DS-II equation, we have to take the choice according to symmetry (28) that: $\phi(x,y,t) = R^{\dagger}(\theta(-x,y,t))$, $R = -i\kappa$. Then the potential solutions in this equation can be constructed by the combination of an elementary DT with its inverse:

$$\hat{P} = P + [J, \theta\Omega^{-1}(\theta,\phi)\phi^{\dagger}], \tag{31}$$

$$\hat{w} = w - 2[\mathbf{tr}(\theta\Omega^{-1}(\theta,\phi)\phi^{\dagger})]_x. \tag{32}$$

The above Binary DT is iterated as following:

$$\Phi_{[n+1]} = G_{\theta_{[n]},\phi_{[n]}}\left(\Phi_{[n]}\right) = \Phi_{[n]} - \theta_{[n]}\Omega^{-1}(\theta_{[n]},\phi_[n])\Omega(\Phi_{[n]},\phi_[n]), \tag{33}$$

$$\Psi_{[n+1]} = G_{\theta_{[n]},\phi_{[n]}}^{\dagger -1}\left(\Psi_{[n]}\right) = \Psi_{[n]} - \phi_{[n]}\Omega^{\dagger -1}(\theta_{[n]},\phi_[n])\Omega(\Psi_{[n]},\theta_{[n]}), \tag{34}$$

$$\theta_{[n]}=\lim_{\Phi\to\theta_n}\Phi_{[n]},\quad \phi_{[n]}=\lim_{\Psi\to\phi_n}\Psi_{[n]}. \tag{35}$$

For the potential matrix, introducing a 2×2 matrix Q s.t $P=[Q,J]$, which of the form:

$$Q=-\frac{\alpha}{2}\begin{pmatrix} * & u(x,y,t)\\ \epsilon\bar{u}(-x,y,t) & * \end{pmatrix}, \tag{36}$$

while the entries $*$ are arbitrary and do not contribute to P. Then it follows from (31) that:

$$\hat{P}=[\hat{Q},J], \tag{37}$$

$$\hat{Q}=Q-\theta\Omega^{-1}(\theta,\phi)\phi^{\dagger}. \tag{38}$$

After n times applications of the BDT we obtain:

$$Q_{[n+1]}=Q_{[n]}-\theta_{[n]}\Omega^{-1}(\theta_{[n]},\phi_{[n]})\phi_{[n]}^{\dagger}, \tag{39}$$

$$w_{[n+1]}=w_{[n]}-2[\mathbf{tr}(\theta_{[n]}\Omega^{-1}(\theta_{[n]},\phi_{[n]})\phi_{[n]}^{\dagger})]_x. \tag{40}$$

Denoting

$$\Theta=(\theta_1,\cdots,\theta_n),\ \mathbf{P}=(\phi_1,\cdots,\phi_n),\ W=\begin{pmatrix}\frac{\partial^{-1}(w)}{2} & 0\\ 0 & 0\end{pmatrix}.$$

By using the noncommutative Jacobi identity (27), one can express the above results on n-th order BDT in terms of quasi-determinants:

$$\Phi_{[n+1]}=\begin{vmatrix}\Omega(\Theta,\mathbf{P}) & \Omega(\Phi_{[1]},\mathbf{P})\\ \Theta & \boxed{\Phi_{[1]}}\end{vmatrix},\quad \Psi_{[n+1]}=\begin{vmatrix}\Omega^{\dagger}(\Theta,\mathbf{P}) & \Omega^{\dagger}(\Theta,\Psi_{[1]})\\ \mathbf{P} & \boxed{\Psi_{[1]}}\end{vmatrix}, \tag{41}$$

$$Q_{[n+1]}=\begin{vmatrix}\Omega(\Theta,\mathbf{P}) & \mathbf{P}^{\dagger}\\ \Theta & \boxed{Q_{[1]}}\end{vmatrix},\ w_{[n+1]}=2\partial_x[\mathbf{tr}\left(\begin{vmatrix}\Omega(\Theta,\mathbf{P}) & \mathbf{P}^{\dagger}\\ \Theta & \boxed{W}\end{vmatrix}\right)]. \tag{42}$$

For convenience, introducing vectors ψ_i and φ_i, $i=1,2$, which satisfy:

$$\Theta=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad \mathbf{P}=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},$$

we further obtain :

$$Q_{[n+1]}=Q_{[1]}+\begin{pmatrix}\begin{vmatrix}\Omega(\Theta,\mathbf{P}) & \varphi_1^{\dagger}\\ \psi_1 & \boxed{0}\end{vmatrix} & \begin{vmatrix}\Omega(\Theta,\mathbf{P}) & \varphi_2^{\dagger}\\ \psi_1 & \boxed{0}\end{vmatrix}\\ \begin{vmatrix}\Omega(\Theta,\mathbf{P}) & \varphi_1^{\dagger}\\ \psi_2 & \boxed{0}\end{vmatrix} & \begin{vmatrix}\Omega(\Theta,\mathbf{P}) & \varphi_2^{\dagger}\\ \psi_2 & \boxed{0}\end{vmatrix}\end{pmatrix}. \tag{43}$$

Then, combination of (39)–(40) and (43) leads to the transformations between potential functions in terms of quasi-Grammian expressions:

$$u_{[n+1]}(x,y,t)=u(x,y,t)-\frac{2}{\alpha}\begin{vmatrix}\Omega(\Theta,\mathbf{P}) & \varphi_2^{\dagger}\\ \psi_1 & \boxed{0}\end{vmatrix}, \tag{44}$$

$$w_{[n+1]}(x,y,t)=w(x,y,t)+2\partial_x[\mathbf{tr}\left(\begin{vmatrix}\Omega(\Theta,\mathbf{P}) & \mathbf{P}^{\dagger}\\ \Theta & \boxed{\mathbf{0}}\end{vmatrix}\right)]. \tag{45}$$

Here $\mathbf{0}$ represents a 2×2 zero matrix. Noting that in this expression one has to calculate the inverse of matrix $\Omega(\Theta,\mathbf{P})$. To overcome this problem, we can reformulate the expression into the quotient of determinants instead of using quasi determinants, that is

$$\begin{vmatrix}\Omega(\Theta,\mathbf{P}) & \varphi_j^{\dagger}\\ \psi_k & \boxed{0}\end{vmatrix}=\frac{\det\left(M^{(k,j)}\right)}{\det(\Omega(\Theta,\mathbf{P}))},\ \text{where } M^{(k,j)}=\begin{pmatrix}\Omega(\Theta,\mathbf{P}) & \varphi_j^{\dagger}\\ \psi_k & 0\end{pmatrix}, 1\le k,j\le 2. \tag{46}$$

Remark 2.2.1. *Moreover, formula (45) can be further transformed into a more compact form:*

$$w_{[n+1]}(x,y,t) = w(x,y,t) - 2\partial_x^2\{\log[\det(\Omega(\Theta,\mathbf{P}))]\}. \tag{47}$$

Proof. In fact, via the Laplace expansion into the **trace** in (35), we can show that

$$\mathbf{tr}\left(\begin{vmatrix} \Omega(\Theta,\mathbf{P}) & \mathbf{P}^\dagger \\ \Theta & \boxed{\mathbf{0}} \end{vmatrix}\right) = \frac{\sum_{k=1}^n \sum_{j=1}^n (-1)^{j+k-1}\left(\phi_j^\dagger \theta_k\right) M_{j,k}}{\det(\Omega(\Theta,\mathbf{P}))},$$

where $M_{j,\,k}$ is the minor matrix of $\Omega(\Theta,\mathbf{P})$, i.e., the determinant of a $(n-1)\times(n-1)$ matrix that results from the j-th row and the k-th column of $(\Omega(\Theta,\mathbf{P}))$. On the other hand, it can be easily verified that

$$\partial_x\{\det(\Omega(\Theta,\mathbf{P}))\} = \sum_{k=1}^n \sum_{j=1}^n (-1)^{j+k}\left(\phi_j^\dagger \theta_k\right) M_{j,k}.$$

And this completes the proof. □

2.3. High-order Darboux transformation for the partially $\mathcal{PT}$-symmetric nonlocal DS system

To construct the high-order solution, the high-order Darboux transformation are needed. It is assumed by introducing a parameter k_i in the fundamental matrix solution $\theta_i(k_i)$. As it was pointed in Ref [31], a generalized DT does exist. Through a limiting process, the general high-order DT for nonlocal DS-I equation is constructed in the following forms:

Theorem 1. (*Theorem 2*, [40]) *Assuming $\Psi_i(k_i),\ i=1,2,\ldots,n$ (which are given in (21)) are n distinct matrix solutions of the linear problem (5)–(6), and their Taylor expansions are*

$$\Psi_i(k_i+\delta) = \Psi_i(k_i) + \Psi_i^{[1]}\delta + \cdots + \Psi_i^{[m_i]}\delta^{m_i} + \cdots,\ \ i=1,2,\ldots,n,$$
$$\Psi_i^{[j]} = \frac{1}{j!}\frac{\partial^j}{\partial k^j}\Psi_i(k)|_{k=k_i},\ \ j=1,2,\cdots.$$

Then the N-fold generalized Darboux transformation is defined as

$$T = G_n G_{n-1}\cdots G_0,$$

where,

$$\begin{aligned} G_i =& G_i[m_i]\cdots G_i[1]\ (i\geq 1),\ \ G_0 = I,\ \ n+\sum_{i=1}^n m_i = N,\\ G_i[j] =& \partial_x - \Psi_{i,x}[j-1]\Psi_i[j-1]^{-1},\quad 1\leq j\leq m_i,\\ \Psi_i[k] =& \lim_{\delta\to 0}\frac{[G_i[k]\cdots G_i[1]G_{i-1}\cdots G_0]}{\delta^k}\Psi_i(k_i+\delta),\\ &= G_i[k]\cdots G_i[1]G_{i-1}\cdots G_0[\Psi_i^{[k]}(k_i)]. \end{aligned}$$

By performing the above limit process on the determinant form (21), we get the formula for high-order solutions for the partially $\mathcal{PT}$-symmetric nonlocal DS-I equation:

$$u_{[N]} = u + 2\alpha^{-1}\det\Sigma_0^{1,2}(\det\Sigma_0)^{-1}, \tag{48}$$

$$w_{[N]} = w - 2\alpha^{-2}[\ln(\det(\Sigma_0))]_{xx}, \tag{49}$$

where,

$$\Sigma_0 = [\Sigma_1\ \ldots\ \Sigma_n],\quad \Sigma_0^{1,2} = [\Sigma_1^{1,2}\ \ldots\ \Sigma_n^{1,2}],\ \ \Sigma_i = \begin{pmatrix} \partial^{N-1}\Psi_i & \ldots & \partial^{N-1}\Psi_i^{[m_i]} \\ \cdots & \cdots & \cdots \\ \Psi_i & \ldots & \Psi_i^{[m_i]} \end{pmatrix},$$

and

$\Sigma_i^{j,k}$ is the matrix which derived by replacing the k-th row of Σ_i with the j-th row of $\left(\partial^N\Psi_i,\cdots,\partial^N\Psi_i^{[m_i]}\right)$.

Next, following the idea proposed for the nonlinear Schrödinger equation in [40], we construct the corresponding high-order DT in the partially $\mathcal{PT}$-symmetric nonlocal DS-II equation. Indeed, the binary DT considered above are degenerate in the sense that $G_{\theta_1,\phi_1}(\theta_1)=0$ and $G^{\dagger^{-1}}_{\theta_1,\phi_1}(\phi_1)=0$, thus we may work with

$$\theta_1[1] = \lim_{\delta\to 0}\frac{G_{\theta_1,\phi_1}(\theta_1(k_1+\delta))}{\delta} = G_{\theta_1,\phi_1}\frac{d\theta_1}{dk}|_{k=k_1},\quad \phi_1[1] = \lim_{\tilde{\delta}\to 0}\frac{G^{\dagger^{-1}}_{\theta_1,\phi_1}(\phi_1(\tilde{k}_1+\tilde{\delta}))}{\tilde{\delta}} = G^{\dagger^{-1}}_{\theta_1,\phi_1}\frac{d\phi_1}{d\tilde{k}}|_{\tilde{k}=\tilde{k}_1}.$$

This serves the seed solution for proceeding the next step binary Darboux transformation. Generally, we assume that solutions $\theta_i = (\xi_i,\ \eta_i)^T$ $(i = 1 \dots s)$ are given for the Lax operator L and solutions $\rho_i = (\mu_i,\ \nu_i)^T$ $(i = 1 \dots s)$ are given for its adjoint operator L†, then we have the following generalized Binary DT.

Theorem 2. *Let solutions $(\xi_i, \eta_i)^T \in S$, and $(\mu_i,\ \nu_i)^T \in S^\dagger$ $(i = 1 \dots s)$, so the high-order Binary DT is constructed in the form as*

$$G_N = G_{\theta_s^{[m_s-1]},\ \rho_s^{[m_s-1]}} \cdots G_{\theta_s,\ \rho_s} \cdots G_{\theta_1^{[m_1-1]},\ \rho_1^{[m_1-1]}} \cdots G_{\theta_1,\ \rho_1},$$

where $N = \sum_{i=1}^{s} m_i$, and

$$G_{\theta_i^{[j]},\ \rho_i^{[j]}} = I - \theta_i^{[j]} \Omega^{-1}\left(\theta_i^{[j]},\ \rho_i^{[j]}\right) \partial^{-1} \rho_i^{[j]\dagger},$$
$$G^{\dagger -1}_{\theta_i^{[j]},\ \rho_i^{[j]}} = I - \rho_i^{[j]} \Omega^{-1}\left(\rho_i^{[j]},\ \theta_i^{[j]}\right) \partial^{-1} \theta_i^{[j]\dagger},$$

here $\theta_i^{[j]}$ and $\rho_i^{[j]}$ are derived by performing the limit on the fundamental eigenfunctions with perturbation parameters δ and $\tilde{\delta}$:

$$\theta_i^{[j]} = \lim_{\delta \to 0} \frac{\left[G_{\theta_i^{[j-1]},\ \rho_i^{[j-1]}} \cdots G_{\theta_i,\ \rho_i} \cdots G_{\theta_1^{[m_1-1]},\ \rho_1^{[m_1-1]}} \cdots G_{\theta_1,\ \rho_1}\right]_{k=k_i+\delta} \theta_i(k_i + \delta)}{\delta^j},$$
$$\rho_i^{[j]} = \lim_{\tilde{\delta} \to 0} \frac{\left[G^{\dagger -1}_{\theta_i^{[j-1]},\ \rho_i^{[j-1]}} \cdots G^{\dagger -1}_{\theta_i,\ \rho_i} \cdots G^{\dagger -1}_{\theta_1^{[m_1-1]},\ \rho_1^{[m_1-1]}} \cdots G^{\dagger -1}_{\theta_1,\ \rho_1}\right]_{k=\tilde{k}_i+\tilde{\delta}} \rho_i(\tilde{k}_i + \tilde{\delta})}{\tilde{\delta}^j}.$$

By taking above limitation directly on (44)–(45), the transformations between potential matrices can be represented in a form of quasi-gram determinant.

Theorem 3. *The above generalized binary Darboux matrix and the corresponding transformation between the potential matrices can be represented as the following forms:*

$$Q_{[N]} = Q_{[1]} + \begin{vmatrix} \Omega(\Theta, \mathbf{P}) & \mathbf{P}^\dagger \\ \Theta & \boxed{0} \end{vmatrix} = Q_{[1]} - \Theta \Omega^{-1}(\Theta, \mathbf{P}) \mathbf{P}^\dagger, \tag{50}$$

$$w_{[N]}(x, y, t) = w_{[1]}(x, y, t) - 2\partial_x^2 \{\log [\det (\Omega(\Theta, \mathbf{P}))]\}, \tag{51}$$

where,

$$\theta_i = \theta_i(k_i + \delta),\ \ \rho_j = \rho_j(\tilde{k}_i + \tilde{\delta}),$$
$$\Theta = (\Theta_1, \Theta_2, \dots, \Theta_s),\ \ \Theta_i = \left(\theta_i, \frac{d\theta_i}{d\delta}, \dots, \frac{1}{(r_i - 1)!} \frac{d^{r_i-1}\theta_i}{d\delta^{r_i-1}}\right)|_{\delta \to 0},$$
$$\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_s),\ \ \mathbf{P}_j = \left(\rho_j, \frac{d\rho_j}{d\tilde{\delta}}, \dots, \frac{1}{(r_j - 1)!} \frac{d^{r_j-1}\rho_j}{d\tilde{\delta}^{r_j-1}}\right)|_{\tilde{\delta} \to 0},$$
$$\Omega(\Theta, \mathbf{P}) = \left(\Omega^{[ij]}\right)_{1 \le i, j \le s},\ \ \Omega^{[ij]} = \left(\Omega^{[ij]_{m,n}}\right)_{r_i \times r_j},$$
$$\Omega^{[ij]_{m,n}} = \lim_{\delta,\ \tilde{\delta} \to 0} \frac{1}{(m-1)!(n-1)!} \frac{\partial^{m+n-2}}{\partial \delta^{n-1} \partial \tilde{\delta}^{m-1}} \Omega(\theta_j, \rho_i).$$

Proof. The above results can be obtained by directly taking limits in formula (42) with property (46).

One can further derive the Bucklünd transformation of solution $u_{[N]}(x,\ y,\ t)$ from (50), which is taken from the 1-st row and the 2-nd column element in potential matrix $Q_{[N]}$. □

3. General rational solution in partially $\mathcal{PT}$-symmetric nonlocal DS-I system

It is shown in ref [23,24]. that with the bilinear method, a family of rational solutions lead to the rogue waves for the local DS equations. In this work, the rogue wave solution for nonlocal DS equations was derived via a generalized version of Darboux transformation.

The general form of eigenfunctions are solved from the system (5)–(6) when the initial potential solution u is taken as a real constant ρ, which of the form:

$$\xi_i(x, y, t) = \rho_i \exp[\omega_i(x, y, t)],$$
$$\eta_i(x, y, t) = \frac{\lambda_i \rho_i}{\rho} \exp[\omega_i(x, y, t)],$$
$$\omega_i(x, y, t) = \alpha_i x + \beta_i y + \gamma_i t,$$

$$\alpha_i = -\frac{1}{2}\alpha\left(\lambda_i + \frac{\epsilon\rho^2}{\lambda_i}\right),\ \beta_i = \frac{1}{2}\left(\lambda_i - \frac{\epsilon\rho^2}{\lambda_i}\right),\ \gamma_i = \mathrm{i}\alpha^{-1}\alpha_i\beta_i,$$

where $\lambda_i = r_i \exp(\mathrm{i}\varphi_i)$, r_i, φ_i and ρ are free real parameters, ρ_i is set to be complex.

Generally, to derive rational type solutions, we choose the eigenfunction via superposition principle, which can be written in the form as:

$$\{\mathcal{F}_k + \partial_{\varphi_k}\}(\xi_k, \eta_k)^T := \begin{pmatrix} P_k(x,y,t)\xi_k \\ Q_k(x,y,t)\eta_k \end{pmatrix},\ \mathcal{F}_k = e_k + \mathrm{i}f_k,\ \ (e_k, f_k \in \mathbb{R}), \tag{52}$$

where,

$$P_k(x,y,t) = \mathcal{F}_k + \rho_k^{-1}\rho_{k,\varphi_k} + (-\mathrm{i}\alpha\beta_k)x + \left(-\mathrm{i}\alpha^{-1}\alpha_k\right)y + \frac{1}{2}\left(\lambda_k^2 + \frac{\rho^4}{\lambda_k^2}\right)t;$$

$$Q_k(x,y,t) = \mathcal{F}_k + \lambda_k\rho^{-1}\left(\mathrm{i}\rho_k + \rho_{k,\varphi_k}\right) + (-\mathrm{i}\alpha\beta_k)x + \left(-\mathrm{i}\alpha^{-1}\alpha_k\right)y + \frac{1}{2}\left(\lambda_k^2 + \frac{\rho^4}{\lambda_k^2}\right)t;$$

3.1. Fundamental rogue-wave in partially $\mathcal{PT}$-symmetric nonlocal DS-I

To derive the first order rational solution, we set $N=1$, $\rho = 1$ with $\rho_1 = \exp\left(\frac{-\mathrm{i}\varphi_1}{2}\right)$ in formula (21)–(22). Then the first-order rational solution is

$$u_1(x,y,t) = 1 - \frac{2iF_1(x,y,t)+1}{F(x,y,t)}, \tag{53}$$

$$w_1(x,y,t) = \epsilon - 2[\ln(F(x,y,t))]_{xx}, \tag{54}$$

where,

$$F(x,y,t) = F_1^2(x,y,t) + F_2^2(x,y,t) + \frac{\epsilon r_1^2}{(\epsilon + r_1^2)^2},\ p_1 = \frac{r_1 - \epsilon r_1^{-1}}{2},\ q_1 = \frac{r_1 + \epsilon r_1^{-1}}{2},$$

$$F_1(x,y,t) = -ip_1x\cos\varphi_1 - p_1y\sin\varphi_1 + (p_1^2+q_1^2)t\cos 2\varphi_1 + e_1,$$

$$F_2(x,y,t) = iq_1x\sin\varphi_1 - q_1y\cos\varphi_1 - 2p_1q_1t\sin 2\varphi_1 + \frac{p_1}{2q_1} + f_1.$$

By analysing the denominator in solution (53), it is shown that this rational solution has different dynamical patterns according to the parameter values of r_1 and φ_1.

(i). If $\varphi_1 = k\pi$ $(k = 0, \pm1, \pm2, \ldots)$, then $\lambda_1 = (-1)^n r_1$ is a real number. In this case, it is a rogue wave which approaches a constant background, i.e., $u_1 \to 1$, $w_1 \to \epsilon$ as $t \to -\infty$. And the function F in solution (53) becomes

$$F(x,y,t) = \left[\mathrm{i}p_1(-1)^{n+1}x + (p_1^2+q_1^2)t + e_1\right]^2 + \left[(-1)^{n+1}q_1y + \frac{p_1}{2q_1} + f_1\right]^2 + \frac{\epsilon r_1^2}{(\epsilon + r_1^2)^2}.$$

This function becomes zero at a critical time $t_{@c@} = \frac{-2e_1r_1^2}{1+r_1^4}$, and it occurs on the (x, y) plane when $r_1^2 \neq 1$:

$$-p_1^2x^2 + \left[(-1)^{n+1}q_1y + \frac{p_1}{2q_1} + f_1\right]^2 + \frac{\epsilon r_1^2}{(\epsilon + r_1^2)^2} = 0.$$

Thus, this rogue wave arises from a constant background and develops finite-time singularity on a hyperbola at $t_{@c@} = \frac{-2e_1r_1^2}{1+r_1^4}$, and it shows some cross-shape properties at some time points. For example, if we take $\varphi_1 = 2\pi$, $r_1 = 2$ with $f_1 = 1$, $e_1 = 0$, the singularity of this solution occurs when $t = 0$. Here we only plot solutions up to time $t = -0.03$ in Fig. 1, shortly before the exploding time, where the amplitude of rogue wave could attain very high. Moreover, in the de-focusing case $\epsilon = -1$, for any $t_c \in I_c$, where

$$I_{@c@} = \left[\left(-\frac{|r_1|}{|r_1^2-1|} - e_1\right)\frac{2}{(r_1^2 - r_1^{-2})},\ \left(\frac{|r_1|}{|r_1^2-1|} - e_1\right)\frac{2}{(r_1^2 - r_1^{-2})}\right],\ r_1^2 \neq 1,$$

function $F(x, y, t)$ becomes zero at the spatial locations $x = 0, y = y_{\pm c}$, and $y_{\pm c}$ are solved from the following quadratic equation:

$$\left[(-1)^{n+1}q_1y + \frac{p_1}{2q_1} + f_1\right]^2 = \frac{r_1^2}{(r_1^2-1)^2} - \left[\left(\frac{r_1^2 + r_1^{-2}}{2}\right)t_{@c@} + e_1\right]^2.$$

Therefore, when $\epsilon = -1$, this rogue wave develops extra singularity on a finite-time interval.

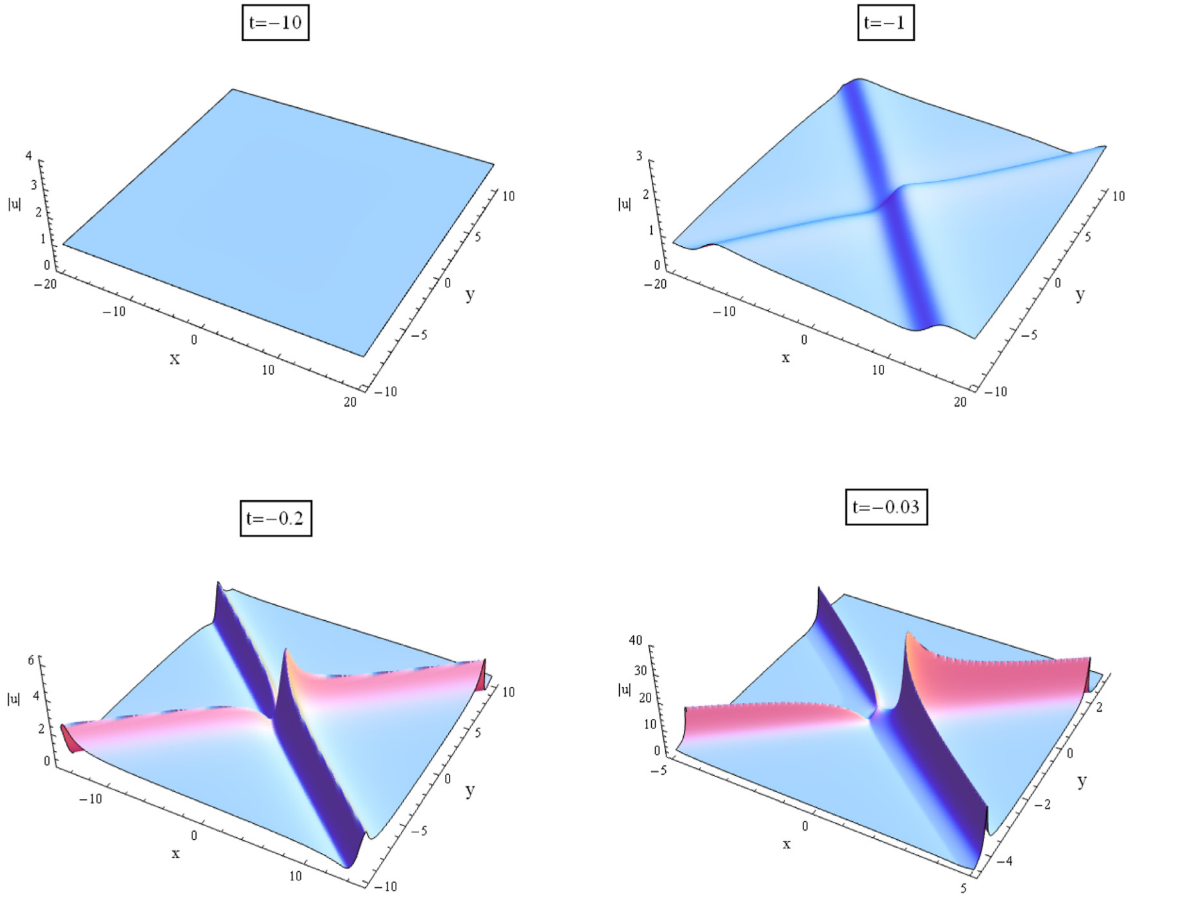

Fig. 1. Time evolution for the 1-st order cross-shape exploding fundamental rogue wave solution in the nonlocal DS-I equation, with parameters: $\varphi_1 = 2\pi$, $r_1 = 2$, $e_1 = 0$, $f_1 = 1$.

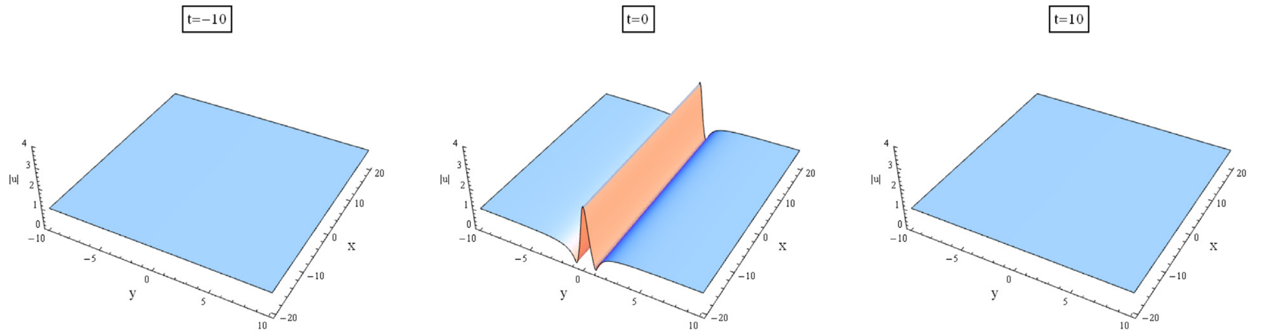

Fig. 2. Fundamental (1+1)-dimensional line rogue wave in the nonlocal DS-I equation, with parameters: $f_1 = 0$, $e_1 = 0$.

In addition, as a special case, when $r_1 = 1$, $\epsilon = 1$, this rogue wave is x-independent and degenerates into the following Peregrine soliton for the nonlocal NLS equation

$$u_1(x, y, t) = 1 - \frac{2it + 2ie_1 + 1}{(y \pm f_1)^2 + (t + e_1)^2 + \frac{1}{4}}, \tag{55}$$

where the parameters e_1 and f_1 can be moved by a shifting. Besides, this solution, in terms of the (1+2) dimensional space, is a (1+1) dimensional line rogue wave in this nonlocal DS-I equation, see Fig. 2.

As a trivial case, when $r_1 = 1$ and $\epsilon = -1$, then $u_1(x, y, t) \rightarrow 1$. (ii). In another case, when $\epsilon = 1$, $\varphi_1 = \frac{(2k-1)\pi}{2}$, i.e., λ is a purely imaginary. It can generate a two-dimensional non-singular rational travelling wave solution(while $\epsilon = -1$ may cause some singularities). The ridge of the solution lays approximately on the following $[x(t), y(t)]$ trajectory:

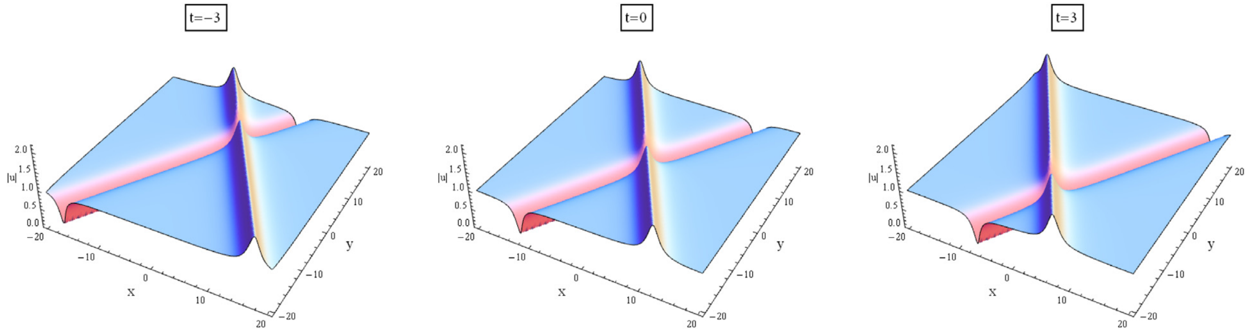

Fig. 3. The interactions of rational travelling wave solution for nonlocal DS-I equation at different time points, with parameters: $r_1 = 2,\ e_1 = 0,\ f_1 = 1$.

$(-1)^{k-1}\frac{(1+r_1^2)}{r_1}x + (-1)^{k-1}\frac{(1-r_1^2)}{r_1}y - \frac{1+r_1^4}{r_1^2}t + e_1 = 0,$
$(-1)^{k}\frac{(1+r_1^2)}{r_1}x + (-1)^{k-1}\frac{(1-r_1^2)}{r_1}y - \frac{1+r_1^4}{r_1^2}t + e_1 = 0.$

Although this solution is generated from the 1-st iteration of DT, it contains two rational travelling waves laying on different trajectories. For example, if one takes $r_1 = 2$, a time evolution process for this solution is displayed in Fig. 3. When $t \to \pm\infty$, two rational travelling waves move away from each other on a constant background, which behaviours like an interaction between a bright and dark soliton.

Especially, when $r_1 = 1$, the above solution is reduced to:

$$u_1(x, t) = 1 + \frac{4i(2t - 2e_1 + i)}{4(e_1 - t)^2 - 4(x + if_1)^2 + 1}.$$

This is an interesting one-dimension rational soliton solution for nonlocal NLS equation. Actually, under the variable transform $u \to \tilde{u} = ue^{-i\alpha^2 t}$, then $\tilde{u}$ satisfy the nonlocal NLS equation which are reduced from nonlocal system by removing the y-independence of the equation. Generally, utilizing this parameters choosing rules in nonlocal DS-I system, we may also derive multi-rogue waves which are just nonlinear combinations of these fundamental patterns.

3.2. Multi-rational solution in partially $\mathcal{PT}$-symmetric nonlocal DS-I equation

Normally, N-rational solutions are generated from N eigenfunctions with $4n$ parameters via Darboux transformation. With appropriate combinations of these parameters, it will present different dynamical patterns, including singular multi-rogue waves blow-up in the finite time interval and the nonsingular mixture of fundamental rogue wave and rational travelling wave solutions.

For instance, taking $N = 2$ in formula (21)–(22) and choose the special parameters as: $\varphi_1 = 2\pi,\ \varphi_2 = 2\pi,\ r_2 = 1/r_1,\ \mathcal{F}_1 = 0,\ \mathcal{F}_2 = 0$. It generates the two-rogue wave solution with particular singulary time points which are obtained by analysing its singularity. The imaginary part in the denominator is $16xyt[r_1^4(r_1^4+1)^2]$. Therefore, when $t = 0$, the imaginary part of the denominator vanishes while the real part becomes
$\Sigma_s(x, y) = \left[x^2r_1^2\left(r_1^2-1\right)^2 - y^2r_1^2\left(r_1^2+1\right)^2 + 3r_14\right]^2 + 24y^2r_1^8 - 12x^2r_1^6\left(r_1^4+1\right).$

Obviously, this part will give rise to the singularities on above surface $\Sigma_s(x, y) = 0$ at $t = 0$. Next, if $y = 0$, the singularity time for this solution will happen on a finite interval $[t_-, t_+]$, where $t_\pm = \pm\frac{|r_1|^3\sqrt{3(r_1^2-1)^2+4r_1^2}}{|r_1^2-1|(r_1^4+1)\sqrt{(r_1^4-r_1^2+1)}}$. Once t falls into the interval, there will be two pairs of singularity points distribute centered on x-axis. And these points are substantially the real roots of a quartic equation dependent on variable y. However, the number of the pairs down to one if t is locate on the edges of the interval. At last, the real part of the denominator is proved to be definite positive if $x = 0$. As an example, when $r_1 = 2$, we find that the solution rises from a nearly constant background at $t = -\infty$, and then it appears a cross-shape wave in a intermediate time near $t = t_-$. However, finally this wave explodes to infinity at $t = t_-$. Once the solution exploded, the evolution of the wave will cease. There are also similar phenomena which appeared in the second-order and two-rogue waves of local DS-II equation[23]. What is shown in Fig. 4 is the evolution of a two-rogue waves interaction together with the coming up of singularities. Especially, when $r_1 = 2$, the corresponding time interval is about $[-0.285287, 0.285287]$, so that the singular time $t = -0.2$ shown in Fig. 4 accurately falls into this interval. Another novel hybrid multi-rogue wave pattern is obtained by taking $N = 2$ in formula (21)-(22) with the parameters: $\varphi_1 = \pi,\ r_1 = 1,\ \varphi_2 = \frac{\pi}{2},\ r_2 = 1,\ \mathcal{F}_1 = 0,\ \mathcal{F}_2 = e_2 + if_2$, which leads a two-rational solution:

$$u_2(x, y, t) = \frac{G(x, y, t)}{F(x, y, t)}, \tag{56}$$

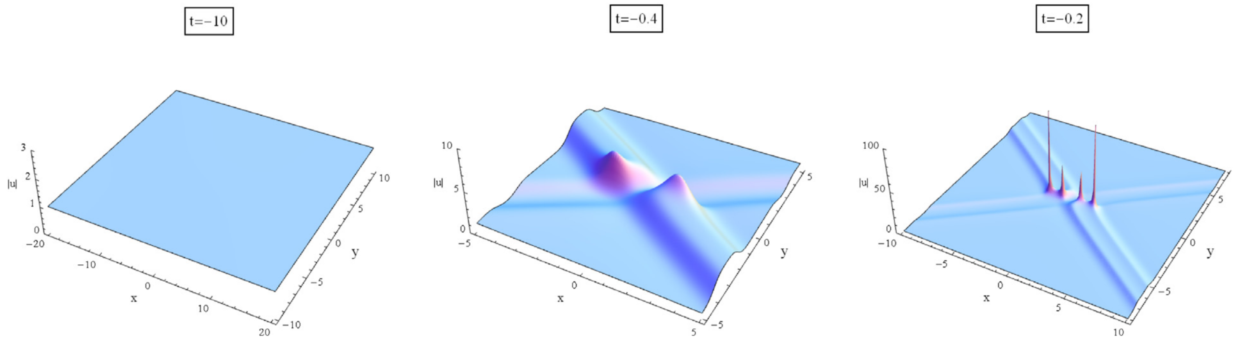

Fig. 4. Time evolution of a two-rogue wave solution for nonlocal DS-I equation with singularity points in pairs, with parameters: $\varphi_1 = 2\pi,\ \varphi_2 = 2\pi,\ r_1 = 2,\ r_2 = 1/r_1,\ \mathcal{F}_1 = 0,\ \mathcal{F}_2 = 0$.

where,

$$\begin{aligned} F(x,y,t) &= 4y^2\left(1+4(e_2-t)^2\right) - \left(16y^2+16t^2+4\right)(x+if_2)^2 + \left(-4t^2+4te_2+3\right)^2 + 4e_2{}^2, \\ G(x,y,t) &= 4[(2t-2i)^2+1](x+if_2)^2 - [(2t-e_2)^2+(ie_2-1)^2-4][(2t-e_2)^2+(ie_2+3)^2-4] \\ &\quad +4[4(i(e_2-t)+1)^2-1]y^2. \end{aligned}$$

With two free parameters given in expression (56), it can produces an interesting novel hybrid pattern. This pattern is described by the interaction of line rogue wave with dark and anti-dark travelling wave solution. In other words, three different patterns appear at the same time in one solution. For any $f_2 \neq 0$, when $x = 0$, the imaginary part of the denominator in solution (56) becomes zero while the real part is positive definite. Therefore, this solution is nonsingular except for $f_2 = 0$. Furthermore, with adequate combinations of e_2 and f_2, solution (56) can generate several interesting structures.

For example, choosing $e_2 = 0$ with $f_2 = 1$, there is a fundamental line rogue wave and a travelling wave interaction at about $t = 0$. When $t \to \pm\infty$, it approaches two rational travelling waves which slowly move away from each other. Next, if one takes a larger value in f_2, i.e., $e_2 = 0$ and $f_2 = 10$. This solution behaviours more like a fundamental line rogue wave. This is because the amplitude of the travelling wave is much smaller than that of the rogue wave. However, as $t \to \pm\infty$, the rogue wave part decades very fast to a constant while the travelling wave portion continue moving apart.

Moreover, considering e_2 as a nonzero constant: $e_2 = 10,\ f_2 = 2$. As it is shown in Fig. 5 that when $t \to -\infty$, a dark and anti-dark rational travelling waves move away from each other. It is also shown that a hybrid pattern of line rogue wave with dark and anti-dark travelling waves interactions appear around $t = 0$. Afterwards, the line rogue wave soon disappears, then the dark and anti-dark rational travelling waves intersect and interact at about $t = 10$, then they separate and move away from each other in an opposite direction as $t \to +\infty$.

3.3. High-order rational solution in the partially $\mathcal{PT}$-symmetric nonlocal DS-I equation

The high-order rational solution is another subclass of rational solutions which exhibit different dynamics with multi-rational solution. And they can be obtained through the high-order Darboux transformation constructed in Theorem 1. Firstly, the second order rational solution is reduced from formula (48) by setting $N = 1$. Next, taking $\epsilon = 1,\ \alpha = 1,\ \varphi_1 = \pi/4,\ r_1 = 1,\ \mathcal{F}_1 = e_1$ for instance, here e_1 is a free real parameter, then solution $u_1^{[1]}(x,y,t)$ becomes

$$u_1^{[1]} = -1 + \frac{16(1+2ie_1)[(-ix+y)^2+4t] + 16i\left(2x^2+2y^2+4e_1^3+e\right) + 24\left(4e_1^2+1\right)}{\left[8t-2(-ix+y)^2+4e_1^2+3\right]^2 + 2[4e_1(-ix+y)-2(-ix-y)]^2 + 8(-ix+y)^2 + 16e_1^2}. \tag{57}$$

This solution is a very special case in the high-order solution for nonlocal DS-I equation. It is because solution (57) has the same form with the second-order rogue wave solution in local DS-II system except for a simple variable transformation $x \to ix,\ t \to -t$. However, we make this transformation at the price of causing complex singularities to solution (57). And these singularities are moving with the time. Furthermore, by another transformation $u_1^{[1]}(x,y,t) \to (-\sqrt{2})u_1^{[1]}(ix,y,t-\frac{3}{8})$, solution (57) becomes the second-order rogue wave solution for local DS-II equation which is derived in [23] via the bilinear method.

Moreover, the nonsingular solutions can be also reduced from the second order rational solution by taking $\varphi_1 = \pi/2$, $e_1 = 1$, and this yields a nonsingular high-order rational travelling wave solution but not a rogue wave.

4. General rational solution for the partially $\mathcal{PT}$-symmetric nonlocal DS-II equation

In this section, as what we have shown for nonlocal DS-I system, we construct the general rogue wave solution for nonlocal DS-II equation and analyze the dynamics of these rogue waves. In addition, we also exhibit other types of rational solutions which are reduced from the Darboux transformation.

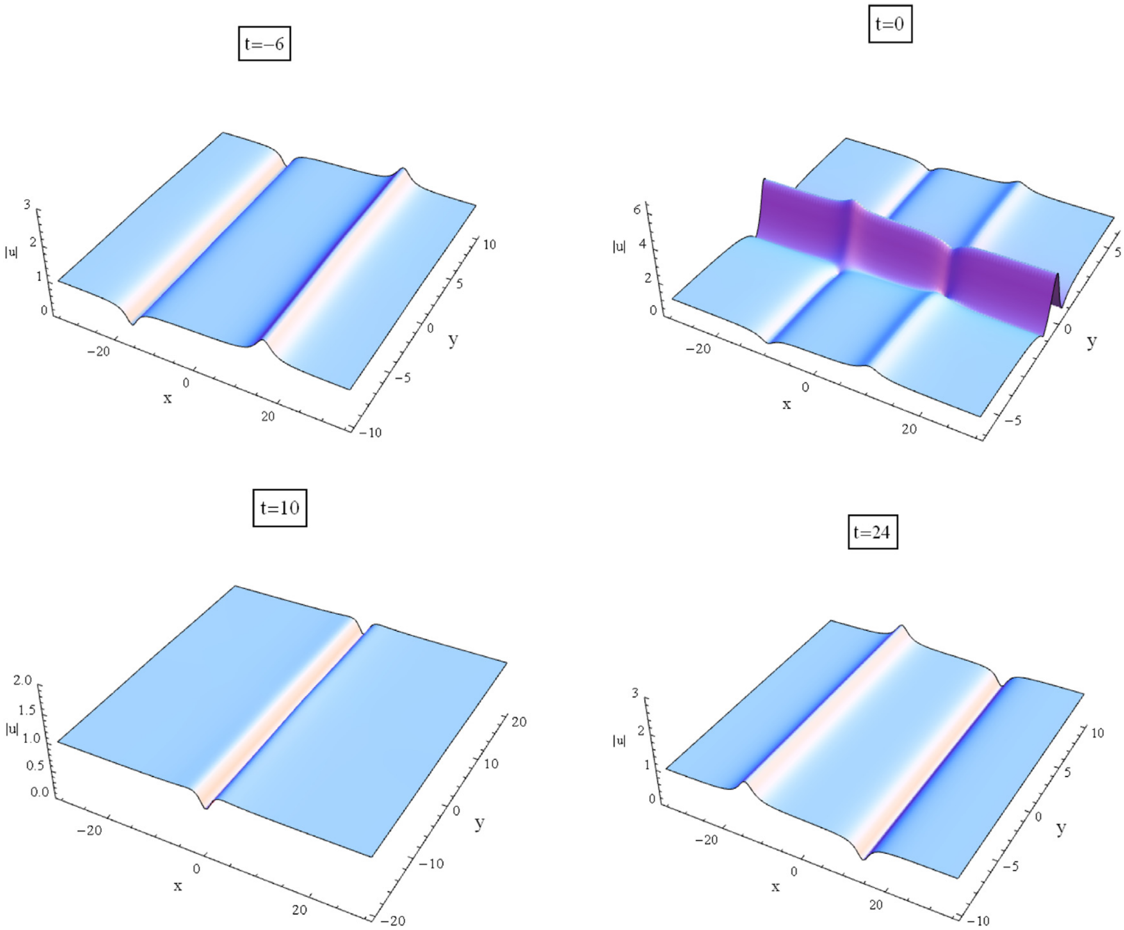

Fig. 5. The interactions of line rogue waves with dark and anti-dark rational travelling waves in the nonlocal DS-I equation, where the parameters are: $\varphi_1 = \pi,\ r_1 = 1,\ \varphi_2 = \frac{\pi}{2},\ r_2 = 1,\ \mathcal{F}_1 = 0,\ \mathcal{F}_2 = 10 + 2\mathrm{i}$.

4.1. Fundamental rogue waves for nonlocal DS-II

To derive the fundamental rogue waves for nonlocal DS-II equation, we first need to present the general one-rational solution of the first order, which is obtained by taking $n = 1$ in formula (44)–(45):

$$u_1(x, y, t) = 1 - \frac{2iG(x, y, t) + 1}{F(x, y, t)}, \tag{58}$$

$$w_1(x, y, t) = \epsilon + 2[\ln(F(x, y, t))]_{xx}, \tag{59}$$

where,

$$F(x, y, t) = G^2(x, y, t) + H^2(x, y, t) + \frac{1}{4\cos^2\varphi_1},\ \ p_1 = \frac{r_1 + \epsilon r_1^{-1}}{2},\ \ q_1 = \frac{r_1 - \epsilon r_1^{-1}}{2},$$

$$G(x, y, t) = ip_1 x \sin\varphi_1 - q_1 y \sin\varphi_1 + (p_1^2 + q_1^2)t\cos 2\varphi_1 + (e_1 + \frac{1}{2}\tan\varphi_1),$$

$$H(x, y, t) = iq_1 x \cos\varphi_1 - p_1 y \cos\varphi_1 - 2p_1 q_1 t \sin 2\varphi_1 - (f_1 + \frac{1}{2}).$$

For this rational solution, different dynamics can be exhibited depending on the parameters ϵ, φ_1 and r_1. By performing solution analysis analogous to that in nonlocal DS-I equation, we find that:

(i).When $\epsilon = 1,\ r_1 = 1,$ then $|\lambda_1| = 1$. In this case, we obtain the fundamental rogue wave solution. And this rogue wave arises from a constant background as $t \to -\infty$ and develops finite-time singularity on a certain spatial location. To be specific, when $\varphi_1 = \frac{(2k-1)\pi}{2}$, we have $u(x, y, t) = 1$. For $\forall\varphi_1 \neq \frac{(2k-1)\pi}{2}$, the imaginary part of the denominator is $x\sin\varphi_1(\tan\varphi_1 + 2t\cos 2\varphi_1)$. If $x = 0,$ it can be shown the real part of the denominator is nonzero. Hence no singularities

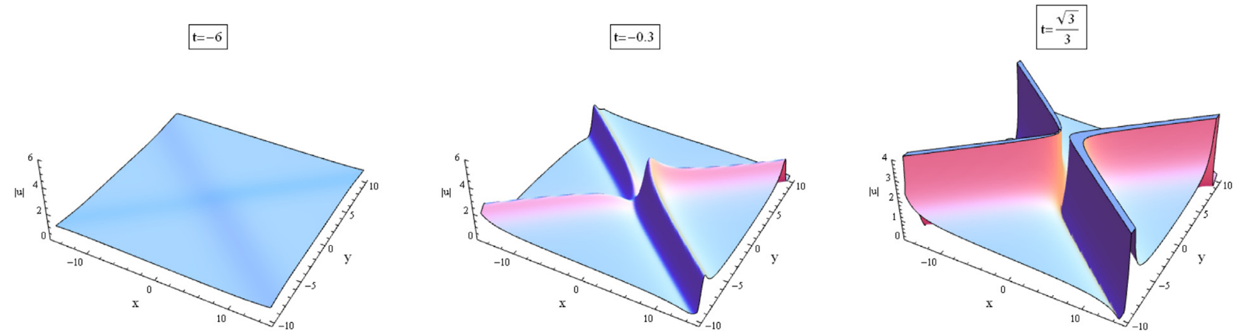

Fig. 6. Fundamental cross-shape rogue wave in the partially $\mathcal{PT}$-symmetric nonlocal DS-II equation, behaviours from the constant background to the exploding time, with parameters: $r_1 = 1,\ \varphi_1 = -\pi/6,\ e_1 = 0,\ f_1 = 0$.

will appear in this situation. However, if $t = \frac{2e\cos\varphi_1 + \sin\varphi_1}{-2\cos 2\varphi_1 \cos\varphi_1}$, the singularities will occurs at one time point and locates on the certain elliptic curve in the (x, y) plane:

$$(2y\cos^2\varphi_1 + \cos\varphi_1)^2 - (x\sin 2\varphi_1)^2 + 1 = 0.$$

For example, if we take $\varphi_1 = -\pi/6$ with $e_1 = 0$, then the singularity time occurs at $t_{@c@} = \sqrt{3}/3$. Before this point, the rogue wave is nonsingular and it shows some cross-shape or interaction phenomena, which are quite different from the solution dynamics exhibited by the rogue wave in the local DS systems. The dynamic evolution of this rogue wave is presented in Fig. 6, including its shape at singular time t_c, where a truncation surface is obvious to see from the figure. It is remarkable that this kind of exploding fundamental rogue waves in the partially $\mathcal{PT}$-symmetric nonlocal sytems were first obtained in [22] by a simple "transformation" method. Here, by using DT theory, one can easily generalize solution formulas and parameters choices to the multi-rogue waves.

In addition, if $r_1 = 1,\ \varphi_1 = k\pi$, as what we have found in the nonlocal DS-I system, this solution can also degenerate into a (1+1)-dimensional line rogue wave solution:

$$u_1(y, t) = 1 - \frac{4(1 + 2it)}{4t^2 + 4y^2 + 1}.$$

The graphs of this rogue wave solution are qualitatively similar to those in Fig. 2.

(ii). When $\epsilon = -1,\ r_1 = 1$, i.e., $|\lambda_1| = 1$. In generic case, if we require $f_1 \neq -1/2$, one can derive a nonsingular solution with three parameters, which is nothing but the rational travelling wave for the nonlocal DS-II equation. In fact, this solution can be seen as the corresponding counterpart of travelling wave solution for nonlocal DS-I system. Similarly, the ridge of this solution lays approximately on two lines with opposite slope, which is:

$$l_1:\ 2t\cos\varphi_1\cos 2\varphi_1 - y\sin 2\varphi_1 + 2x\cos^2\varphi_1 + \sin\varphi_1 + e_1 = 0,$$
$$l_2:\ 2t\cos\varphi_1\cos 2\varphi_1 - y\sin 2\varphi_1 - 2x\cos^2\varphi_1 + \sin\varphi_1 + e_1 = 0.$$

It is easy to see that the angle of l_1 and l_2 is $2\varphi_1$. And the solution are moving along these two center lines with the evolution of time.

Therefore, this lead us to make a summing up for the fundamental solutions of nonlocal DS systems. For the nonlocal DS-II equation, the value of ϵ determines the type of the fundamental rational solution under the unitary module reduction condition: $|\lambda_1| = 1$, which is quite different from the condition we previously discussed in the system of nonlocal DS-I equation, where the solution patterns are classified by the fact that whether λ_1 is a real or pure imaginary number. In the following, as what have already been shown in the nonlocal DS-I equation, these parameters conditions can also applied for every λ_i and it will produce several patterns of multi-rogue waves for the nonlocal DS-II system.

4.2. Multi-rational solution for nonlocal DS-II equation

To obtain the multi-rouge waves, one should make use of multi eigenfunctions with the form given in (52). And this will bring more free parameters. However, we have noted that for $\forall k, j$, the denominator in the integration $\Omega(\varphi_j, \phi_k)$ has the term: $r_k r_j \left(r_k + r_j e^{i\varphi_j + i\varphi_k}\right)^3$.

Therefore, one should choose parameters r_k and φ_k carefully because that may cause indeterminacy to the solution. Here we set all nonzero $r_k \in \mathbb{R}$, and parameters are limited to the condition: $r_k r_j \left(r_k + r_j e^{i\varphi_j + i\varphi_k}\right)^3 \neq 0$. More specifically:

(1). If $e^{i\theta_j + i\theta_k} = 1$, then $r_k + r_j \neq 0$; (2). If $e^{i\theta_j + i\theta_k} = -1$, then $r_k - r_j \neq 0$; (3). Once $r_k = r_j$, then $e^{i\theta_j + i\theta_k} \neq -1$.

These parameters can not be taken directly on the possible singular value points. However, the above restrictions might be removed through a limiting process. For example, taking $n = 2$ in formula (44)–(45), it generates a family of general two-rational solution for nonlocal DS-II equation. Firstly, if we choose the parameters $\epsilon = 1,\ \alpha = i,\ \varphi_1 = 2\pi,\ r_1 = 1,\ r_2 =$

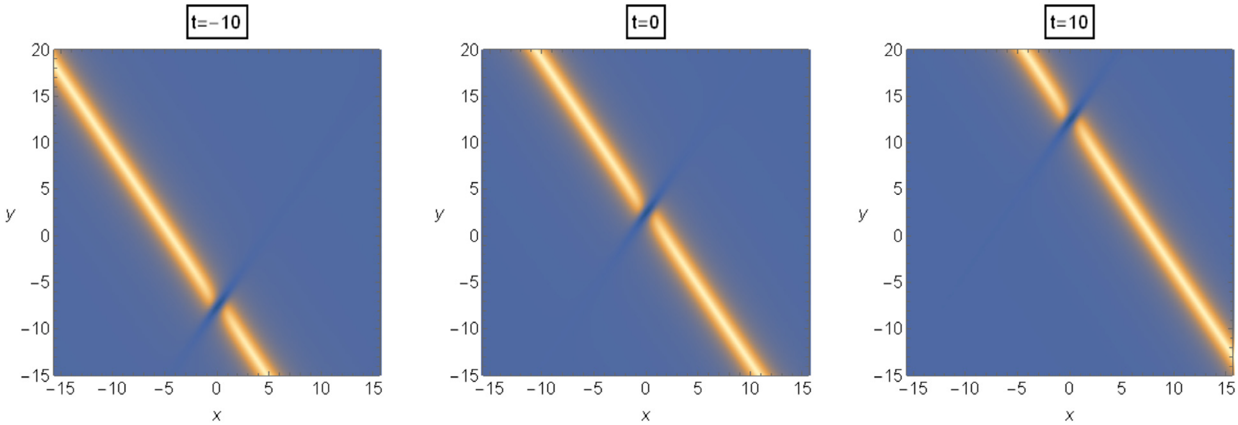

Fig. 7. Rational travelling waves interaction for the nonlocal DS-II equation, with parameters: $\epsilon = -1,\ \varphi_1 = \pi/6,\ r_1 = 1,\ e_1 = 1,\ f_1 = 0$.

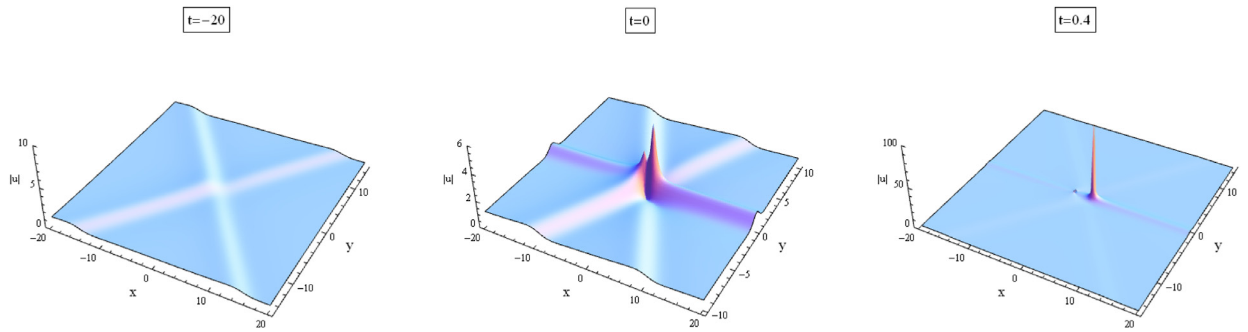

Fig. 8. Dynamics of exploding two-rogue wave solution in the nonlocal DS-II equation, with parameters: $\epsilon = 1,\ \alpha = i,\ \varphi_1 = 2\pi,\ r_1 = 1,\ \varphi_2 = \frac{\pi}{4},\ r_2 = 1,\ \mathcal{F}_1 = 0,\ \mathcal{F}_2 = 0$.

$1,\ \mathcal{F}_1 = 0,\ \mathcal{F}_2 = 0$ in (44)–(45) and take the limit $\varphi_2 \to \frac{\pi}{2}$, then this two-rational solution reduce to the one-dimensional fundamental rogue wave solution

$$u_2(y,t) = -1 + \frac{4 + 8it}{4t^2 + 4y^2 + 1}.$$

Next, when φ_2 continuously changes between 0 and 2π except for some particular values like 0, $\pi/2$, π and 2π. A family of rational solutions can be found with singularities existing on the corresponding time interval. However, usually it is a tedious process to determine the accurate interval values. Therefore, as a concrete example, we choose the special parameters $\epsilon = 1,\ \alpha = i,\ \varphi_1 = 2\pi,\ r_1 = 1,\ \varphi_2 = \frac{\pi}{4},\ r_2 = 1,\ \mathcal{F}_1 = 0,\ \mathcal{F}_2 = 0$ for the convenience in the following analysis. Then it becomes the two-rational solution with its singularity time t_0 occurs no more at one time point but on a finite time interval I_s. In this case, $t_0 \in I_s = [0.326232,\ 0.628852]$, while these two end points are the approximate values of the real roots satisfing the following quadratic equation:

$16\sqrt{2}@c@^2 - 20@c@^2 - 64\sqrt{2}@c@ + 88@c@ + 52\sqrt{2} - 73 = 0.$

In fact, this equation come from analysing the possible singular points from the denominator of the solution. First of all, the imaginary part of the this denominator is:

$$4x\left[4\left(4\sqrt{2} - 5\right)y^2 + \mathcal{P}_1(t)\right],\ \text{where}\ \ \mathcal{P}_1(t) = 4t\left(4\sqrt{2}t - 5t - 16\sqrt{2} + 22\right) + 52\sqrt{2} - 73.$$

If $x = 0$, it is verified that the real part of the denominator is positive definite. Hence $x = 0$ can not be the singularity point. Next, for $\left(4\sqrt{2} - 5\right)y^2 \geq 0$, if there exists t_0 such that $\mathcal{P}_1(t_0) \leq 0$, one can obtain a point y_0 s.t $4\left(4\sqrt{2} - 5\right)y_0^2 + \mathcal{P}_1(t_0) = 0$, thus the imaginary part becomes zero. Subsequently, put points $(y_0,\ t_0)$ to the real part, and then we can also solve out a real point x_0 to makes the real part be zero. However, if $\mathcal{P}_1(t_0) > 0$, then the imaginary part is proved to be nonzero. Hence the solution has no singularity under this condition.

The dynamics for this solution are shown in Fig. 7. We can see that as $t \to \pm\infty$, this solution approaches a "***X*** "-shape background wave with very small amplitude. While the solution could reach very high maximum amplitude near $t = 0$. Moreover, when $t = 0.4$, which belongs to the singular interval I_s, the solution exploding at this singular time point.

4.3. High-order rational solutions for nonlocal DS-II

As the case in the nonlocal DS-I quation, the high-order rational solution for nonlocal DS-II equation can be constructed via the generalized binary DT (50)–(51). For $N=2$, we have noted that the denominator in the integration $\Omega(\varphi_j,\ \phi_k)$ all contains the term: $1+e^{2i\varphi_1}$. Thus, one should choose parameter carefully with φ_1. For instance, choosing $\epsilon=1,\ \alpha=i$ with $\varphi_1=2\pi,\ r_1=1,\ \mathcal{F}_1=0$, we obtain the second-order rational solution

$$u_1^{[1]}=1+\frac{8(1+2it)[4it(1+it)+4ix-4y^2+1]}{16\left[t^4+t^2\left(-2ix+2y^2+1/2\right)+\left(ix+y^2\right)^2\right]+8ix+24y^2+5}, \tag{60}$$

$$w_1^{[1]}=-1+\frac{64\left(-16\left[t^4+t^2\left(-2ix-6y^2-3/2\right)+\left(ix+y^2\right)^2\right]-8ix+8y^2+3\right)}{\left(16\left[t^4+t^2\left(-2ix+2y^2+1/2\right)+\left(ix+y^2\right)^2\right]+8ix+24y^2+5\right)^2}. \tag{61}$$

For this solution, it can be shown that it is singular for almost full time points except for a transient time interval.

Moreover, if we take the variable transformation $x\to -ix,\ t\to -t$, as what we have done for the high-order rational solution in nonlocal DS-I system, then solution (60) becomes

$$u_1^{[1]}=1+\frac{8(1-2it)\left[-4it(1-it)+4x-4y^2+1\right]}{16\left[t^4+t^2\left(-2x+2y^2+1/2\right)+\left(x+y^2\right)^2\right]+8x+24y^2+5}. \tag{62}$$

And this is the high-order rogue wave in the local DS-I equation which has been derived in [24] through bilinear method. In this case, via a simple variable transformation, we derive this well-posed high-order solution in the local DS-I equation from a ill-posed one with full-time singularity in the nonlocal DS-II equation.

5. Summary and discussion

In summary, we have derived general rogue waves in the partially $\mathcal{PT}$-symmetric nonlocal DS-I and nonlocal DS-II equations. The tool we have used is the Darboux transformation method in soliton theory, and the solutions in these two equations are given in terms of determinants and quasi-determinants, separately. We have shown that the fundamental rogue waves in these two systems are rational solutions which arises from a constant background and then develops finite-time singularity on an entire hyperbola in the spatial plane at the critical time (or at certain time interval, as what is shown in the de-focusing nonlocal DS-I equation). We have also shown that multi rogue waves describes the interactions of several fundamental rogue waves. Especially, a novel hybrid-pattern rogue wave is found, which contains three different types of waves in one solution. It exhibits different dynamics and is generated from the interaction of line rogue waves with dark and anti-dark rational travelling waves. In addition, some high-order travelling waves can be reduced from the high-order rational solutions, and some singular solutions are also discovered, which can be transformed to the high-order rogue waves in the local DS systems through simple variables transformations.

Furthermore, it is interesting and meaningful to compare these rogue wave in the nonlocal DS equations with those in the local DS equations (see refs. [23,24]). Firstly, the parameter conditions for the generations of fundamental rogue waves are quite different between local and nonlocal DS equations. Secondly, we have known that for the local DS-II equation [23], rogue waves exist only when $\epsilon=1$, but in the nonlocal DS-I equation, we have shown that rogue waves exist for both signs of nonlinearity $\epsilon=\pm1$. Thirdly, in the local DS equations, fundamental rogue waves are line rogue waves which are never blow up in finite time; While in the nonlocal DS equations, fundamental rogue waves have richer structures, including (1+2)-dimensional exploding rogue waves and (1+1)-dimensional line rogue waves. Although some non-generic multi-rogue waves and higher-order rogue waves of the local DS-II equation in ref [23]. can also exploding in finite time, but the blowup only occurs at a single time point, unlike the fundamental rogue waves of the nonlocal DS equations where the blowup occurs on an entire hyperbola of the spatial plane.

Moreover, for this partially $\mathcal{PT}$-symmetric DS model, it is necessary to clarify the difference between some solutions obtained in Refs[15–17]. with those obtained in this paper. To be specific, in Ref [15]., soliton solutions with zero background are derive for the nonlocal DS-I equation, which can be bounded with n peaks. The "line dark soliton" with non-zero background are also derived in Ref [15]., which can be bounded with their norms changing fast along some straight lines. The analytic expressions for both these two types of solitons are constitute of pure hyperbolic or exponential functions. In Refs [16,17]., the (2+1)-dimensional breathers, lumps and periodic line waves are obtained for Eq. (2). The breathers are periodic in x direction and localized in y direction, which can be expressed in terms of hyperbolic and trigonometric functions. The localized lumps are pure rational solutions moving on the constant background in the $(x,\ y)$ plane. Moreover, there are also hybrid solutions between several lumps, breathers, and periodic line waves, which are semi-rational solutions combined of polynomial, exponential and/or trigonometric functions. In our paper, we mainly focus on the general rogue-wave solutions for Eq. (2). Mathematically, the analytic expressions for these solutions are reduced from a wider family of rational solutions in Eq. (2). In the view of dynamics, these rogue waves are only localized in time but not localized in space, and it normally presents cross-shape line pattern with possible finite-time blowing-ups. This is the main novelty for those rogue waves in comparison with other kinds of solutions discussed in [15–17]. Besides, the dark-anti-dark rational travelling waves are also shown in this paper, which are localized neither in time nor space, but moving away from each other along each trajectory on the constant background in the spatial plane over time (See Fig. 3).

Since partially $\mathcal{PT}$-symmetric physical systems has been shown possible applications in optics. We expect these rogue-wave solutions could have interesting implications for the partially $\mathcal{PT}$-symmetric in multi-dimensions. Moreover, we hope these solutions could play a role in the physical understanding of rogue water waves in the ocean.

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