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On Lins, A., W. de Melo and Pugh C.C.'s Conjecture (Part II) *

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Abstract: In this paper, we give a affirmative answer to Lins.A., W.de Melo and pugh c.c.'s conjecture[1] for $F(x) = -F(-x)$.

Key words: limit cycle; state function.

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For Lienard's equation $\ddot{x} + f(x)\dot{x} + x = 0$ or the equivalent system

$$\dot{x} = y - F(x), \quad \dot{y} = -x \quad (F(x) = \int_0^x f(\xi)\mathrm{d}\xi), \tag{*}$$

there were a lot of papers[2,3] which studied (*) to have some limit cycles. In 1977, Lins. A., W. de Melo and Pugh C. C. put forward the conjecture as follows:

When $F(x) = \sum_{i=1}^N a_i x^i (N = 2n+1, 2n+2)$, system (*) has at most n limit cycles.

In this paper, we proved that the conjecture for $F(x) = -F(-x)$ is an affirmative proposition. The analogy between the research methods of zero points of functions in calculus and the closed orbit of planar system is well-known to us. For example, the Poincare-Bendixson's annular region theorem corresponds to the following conclusion: if a continuous function $\Phi(x)$ satisfies $\Phi(a)\Phi(b) < 0$, then $\Phi(x) = 0$ has at least one real root in (a, b). By the point transformation method the conclusion that system (*) has at most n limit cycles corresponds to conclusion that: $\Phi(x) = 0$ has at most n positive real roots.

We always suppose that $f(x)$ is continuous even function on $(-\infty, +\infty)$, and $f(x)$ satisfies the conditions of the existence and uniqueness theorem of the solutions for (*). For system (*), we make the Filippov's transformation $z = \int_0^x f(\xi)\mathrm{d}\xi = \frac{x^2}{2}$. Thus, the trajectories of (*) on the region $0 \le x \le +\infty$ of right-half plane (x, y) and $-\infty \le x \le 0$ of left half plant (x, y) are transformed into integral curves of equation

$$\frac{\mathrm{d}z}{\mathrm{d}y} = F_i(z) - y. \tag{1}$$

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On $0 \le z \le +\infty$ of plane (z,y), where $F_1(z) = F(\sqrt{2z}), F_2(z) = F(-\sqrt{2z})$.

Unless special explanation, we always suppose that the function $F_i(z)(i = 1,2)$ are continuously differentiable with respect to z and are equal to zero at $z = 0$.

Obviously, we have[2,3]; the trajectory of $(i)(i = 1,2)$ passing through point $((z_0, F_i(z_0))$ on the curve $y = F_i(z)$ must intersect y-axis at two points A and B. Where either, $y_A \le 0, y_B > 0$ or $y_A < 0, y_B \ge 0$ and $F_1(z) = -F_2(z)$. Writing $F(z) = F_1(z)$, we present a state function as follows. Making energy function $\lambda = (z,y) = \frac{(y-F(z))^2}{2} + z$. On the trajectory $\widehat{ACB}$ of (*) passing $(z_0, F(z_0))$ (see figure 1)[5], we can easily obtain

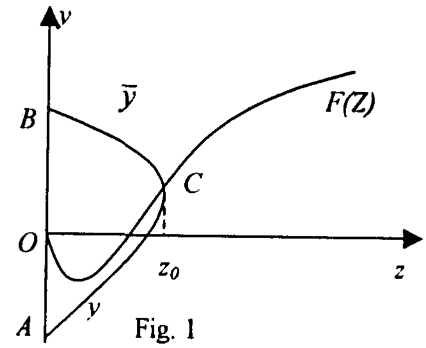

Fig. 1

$$\frac{d\lambda}{dz} = (y - F(z)(\frac{dy}{dz} - F'(z)) + 1 = (y - F(z))\frac{dy}{dz} - F'(z)(y - F(z)) + 1$$
$$= (y - F(z))/(F(z) - y) - F'(z)(y - F(z)) + 1 = -F'(z)(y - F(z)). \quad (2)$$

Thus, we have

$$\begin{aligned}\int_{\widehat{ACB}} d\lambda = &\int_{\widehat{AC}} d\lambda + \int_{\widehat{CB}} d\lambda \\ =&\{[(F(z_0) - F(z_0))^2/2 + z_0] - [(y_A - 0)^2/2 + 0]\}+ \\ &\{[(y_B - 0)^2)/2 + 0] - [(F(z_0) - F(z_0))^2/2 + z_0]\} \\ =&(y_B^2 - y_A^2)/2 \end{aligned} \quad (3)$$

$$\begin{aligned}\int_{\widehat{AC}} d\lambda + \int_{\widehat{CB}} d\lambda &= \int_0^{z_0} F'(z)(y - F(z))dz - \int_{z_0}^0 F'(z)(\bar{y} - F(z))dz \\ &= \int_0^{z_0} F'(z)(\bar{y} - y)dz. \end{aligned} \quad (4)$$

By (3) and (4), we define the state function

$$\Phi_1(z_0) = \int_{\widehat{ACB}} d\lambda = \int_0^{z_0} F'(z)(\bar{y} - y)dz = (y_B^2 - y_A^2)/2.$$

By $\Phi_1(z_0)$, we obtain

Property 1[4] *The State function $\Phi_1(z_0)$ is continuous; if there exist numbers $0 < z_1 < z_2$ such that $\Phi_1(z_1)\Phi_1(z_2) < 0$, then $\Phi_1(z_0) = 0$ has at least one real root in (z_1, z_2), that is system (*) has at least one limit cycle.*

Property 2[4] *If $F'(z) \equiv 0$, then (0,0) is a center; if $F'(z) \ge 0, (F'(z) \le 0), F'(z) \not\equiv 0$, then (0,0) is a stable (unstable) focus; it is a necessary condition for ensuring system (*) to have limit cycle, in which $F(z)$ is the change sign.*

Theorem *If $\Phi_1(z_0) = 0$ has at most n positive real roots on $(0, +\infty)$, then system (*)*

has at most n limit cycles.

Proof For $\Phi(z_0)=\int_{\widehat{ACB}}\mathrm{d}\lambda=\int_0^{z_0}F'(z)(\bar{y}-y)\mathrm{d}z$. Since $\bar{y}-y>0$ on $(0,z)$

$$F(z)=\sum_{i=1}^{N}a_i(\sqrt{2z})^i,\ \text{on the other hand } F(x)=-F(-x), a_{2i}=0\quad (i=1,2,\cdots,n+1)$$

$$F'(z)=(\sum_{i=0}^{n}a_{2i+1}(\sqrt{2z})^{2i+1})'=\frac{1}{\sqrt{2z}}\sum_{i=0}^{n}2^i(2i+1)a_{2i+1}z^i.$$

Therefore, by integral mean value theorem there is ξ such that

$$\Phi_1(z)=\frac{1}{\sqrt{2\xi}}(\bar{y}(\xi)-y(\xi))\sum_{i=0}^{n}2^i(2i+1)a_{2i+1}z^i=\frac{1}{\sqrt{2\xi}}(\bar{y}(\xi)-y(\xi))\int_0^{z_0}p_n(z)\mathrm{d}z,$$

where $p_n=\sum_{i=0}^{n}2^i(2i+1)a_{2i+1}z^i$, it has at most n positive real roots on $(0,\infty)$ by Gauss theorem. We know that $\int_0^{z_0}F'(z)(\bar{y}-y)\mathrm{d}z$ changes sign at most n times, therefore system (*) has at most n limit cycles. □

In [6], we gave an example of the conjecture. In this paper, we continue to study the conjecture and give an affirmative answer of the conjecture for the case $F(x)=-F(-x)$. Next work we will study the conjecture in general case.

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关于 Lins, A., W. de Melo 和 Pugh C.C. 的猜想 (II)

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摘 要: 在本文中，我们证明了在 $F(x)=-F(x)$ 条件下，Lins, A., W. de Melo 和 Pugh C.C. 的猜想是成立的.