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Generalized Riccati equation expansion method and its application to the Bogoyavlenskii's generalized breaking soliton equation*

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Based on the computerized symbolic system Maple and a Riccati equation, a Riccati equation expansion method is presented by a general ansatz. Compared with most of the existing tanh methods, the extended tanh-function method, the modified extended tanh-function method and generalized hyperbolic-function method, the proposed method is more powerful. By use of the method, we not only can successfully recover the previously known formal solutions but also construct new and more general formal solutions for some nonlinear differential equations. Making use of the method, we study the Bogoyavlenskii's generalized breaking soliton equation and obtain rich new families of the exact solutions, including the non-travelling wave and coefficient functions' soliton-like solutions, singular soliton-like solutions, periodic form solutions.

Keywords: generalized Riccati equation expansion, Bogoyavlenskii's generalized breaking soliton equation, soliton-like solution, periodic form solution

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1. Introduction

Since the soliton phenomenon was first observed by Scott Russell in 1834 and KdV equation was solved by the inverse scattering method by Gardner *et al* in 1967,[1] the study of solutions and the related issue of the construction of solution to a wide range of nonlinear evolution equations (NEEs) has become one of the most exciting and extremely active areas of research. Many effective methods have been presented, such as, Darboux transformation, Cole–Hopf transformation, Hirota method[2], Bäcklund transformation[2,3], Painlevé method,[2,4,5] homogeneous balance method,[6−13] tanh method,[14,15] and the generalized hyperbolic-function method.[16,17] Direct searching for exact solutions of NEEs has become more and more attractive,[4−30] on the one hand, due to their occurrence in many fields of science, in physics as well as in chemistry or biology and the interesting features and rich variety of their solutions, on the other hand, due to the availability of computer systems like Maple or Mathematica which allow us to perform some complicated and tedious algebraic calculation and differential calculation on a computer, at the same time, help us to find new exact solutions of NEEs.

One of the most effective straightforward methods is to construct exact solutions of NEEs is tanh method.[14,15] Recently, Fan[18] has proposed an extended tanh-function method. More recently, Fan,[19,20] Yan[21] and Chen *et al*[22−26] further developed this idea and made it much more lucid and straightforward for a class of NEEs. Most recently, Elwakil *et al*[27] modified the extended tanh-function method and obtained some new exact solutions. On the other hand, Gao and Tian[16] presented a generalized hyperbolic-function method by introducing coefficient functions. As we know, when using the direct method, the choice of an appropriate ansatz is of great importance. In this paper, based on the above works,[14−27] by introducing a more general ansatz than the ansatz in the above methods, we present a

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generalized Riccati equation expansion method. Then we choose the Bogoyavlenskii's generalized breaking soliton equation to illustrate our algorithm and obtain new families of exact solutions, including the non-travelling wave and the soliton-like solutions of the coefficient functions, singular soliton-like solutions, as well as the periodic form solutions.

2. The generalized Riccati equation expansion method

We now establish the generalized Riccati equation expansion method as follows

(A) For a given nonlinear partial differential equation (NPDE) with one physical field u in three variables (x, y, t), we have

$$H(u, u_t, u_x, u_y, u_{xx}, u_{xt}, u_{xy}, u_{yt}, \cdots) = 0. \quad (1)$$

We express the solutions of the NPDE by the new more general ansatz

$$u(x,y,t) = f + \sum_{i=1}^{m}[g_i\phi^i(\xi) + h_i\phi^{i-1}(\xi)\sqrt{R+\phi^2(\xi)} + k_i\phi^{-i}(\xi)], \quad (2)$$

where m is an integer to be determined by balancing the highest-order derivative terms with the nonlinear terms in Eq.(1), R is a real constant, while $f = f(x,y,t)$, $g_i = g_i(x,y,t)$, $h_i = h_i(x,y,t)$, $k_i = k_i(x,y,t)$ $(i = 1, \cdots, m)$, $\xi = \xi(x,y,t)$ are all differentiable functions, and $\phi(\xi)$ satisfies

$$\frac{\mathrm{d}\phi(\xi)}{\mathrm{d}\xi} = R + \phi^2(\xi). \quad (3)$$

(B) Substituting Eq.(2) along with Eq.(3) into Eq.(1), multiplying the most simple common denominator in the obtained system, and setting the coefficients of $\phi^j(\xi)(\sqrt{R+\phi^2(\xi)})^n$ $(j = 0, 1, \cdots;\ n = 0, 1)$ (**Note:** here $\phi^i(\xi)$ denotes i power of $\phi(\xi)$ and $(\sqrt{R+\phi^2(\xi)})^n$ denotes n power of $\sqrt{R+\phi^2(\xi)}$) to zero, we obtain a set of over-determined partial differential equations with regard to differentiable functions f, g_i, h_i, k_i $(i = 1, \cdots, m)$ and ξ.

(C) Solving the over-determined partial differential equations, we would end up with the explicit expressions for f, g_i, h_i, k_i $(i = 1, \cdots, m)$ and ξ or the constrains among them.

(D) It is well known that the general solutions of Riccati Eq. (3) are

a) When $R < 0$,

$$\phi(\xi) = \begin{cases} -\sqrt{-R}\tanh(\sqrt{-R}\xi), \\ -\sqrt{-R}\coth(\sqrt{-R}\xi); \end{cases} \quad (4)$$

b) When $R = 0$,

$$\phi(\xi) = -\frac{1}{\xi}; \quad (5)$$

c) When $R > 0$,

$$\phi(\xi) = \begin{cases} \sqrt{R}\tan(\sqrt{R}\xi), \\ -\sqrt{R}\cot(\sqrt{R}\xi). \end{cases} \quad (6)$$

Thus, according to Eqs.(2), (4)–(6) and the conclusions in (C), we obtain many general solutions of Eq.(1).

Remarks

(1) Generalization

The method proposed here is more general than the generalized hyperbolic-function method,[16] tanh method,[14,15] extended tanh-function method,[18] modified extended tanh-function method.[27] Firstly, compared with the extended tanh-function by Fan,[18] the improved tanh-function method by Yan[21] as well as the modified extended tanh-function method by Elwakil,[27] the restriction on $\xi(x,y,t)$ as merely a linear function of x, y, t and the restriction on the coefficients f, g_i, h_i, $k_i (i = 1, \cdots, m)$ as constants are removed. Secondly, compared with the generalized hyperbolic-function method,[16] we cannot only recovered the exact solutions for a given NPDE which are the superposition of different power of the sechξ function, tanhξ function or their combinations, but also we can, with no extra effort, find other new and more general types of solutions, such as singular soliton-like solutions, coth-type solutions and periodic form solutions, tan-type solutions, and combinations of these types of function, even rational solutions, etc. More importantly, we can add terms $k_i\phi^{-i}(\xi)$ in the new ansatz (2), so more types of solutions would be expected for some equations.

(2) Feasibility

For the generalization of the ansatz, naturally more complicated computation is expected than ever before. Even if the availability of computer symbolic systems like Maple or Mathematica allow us to perform the complicated and tedious algebraic calculation and differentiation on a computer, in general it is very difficult, sometimes impossible, to solve the set of over-determined partial differential equations in (B). As the calculation goes on, in order to greatly simplify the work or make the work feasible, we often choose special function forms for f, g_i, h_i, k_i $(i=1,\cdots,m)$ and ξ, on a trial-and-error basis.

3. Bogoyavlenskii's generalized breaking soliton equation

In this section, by use of the generalized Riccati equation expansion method, we investigate the Bogoyavlenskii's generalized breaking soliton equation,[16,31,32]

$$(u_{xt}-4u_xu_{xy}-2u_yu_{xx}+u_{xxxy})_x=-\alpha^2u_{yyy}, \quad (7)$$

where the Bogoyavlenskii parameter α^2 is real. The Bogoyavlenskii's generalized breaking soliton equation describes the (2+1)-dimensional interaction of Riemann wave propagation with the long-wave propagation.[28] By balancing the highest-order contributions from both the linear and nonlinear terms in Eq.(7), we obtain $m=1$ in Eq.(2). Therefore, we assume the solutions of Eq.(7) in the following form

$$u(x,y,t)=f+g\phi(\xi)+h\sqrt{R+\phi^2(\xi)}+k\phi^{-1}(\xi), \quad (8)$$

where $f=f(y,t)$, $g=g(y,t)$, $h=h(y,t)$, $k=k(y,t)$ and $\xi=xp+q$ (p is a constant and $q=q(y,t)$) are all differential functions, and $\phi(\xi)$ satisfies Eq.(3).

Substituting Eq.(8) along with Eq.(3) into Eq.(7), multiplying $\phi(\xi)^6\sqrt{R+\phi(\xi)^2}$ in the obtained system, then setting the coefficients of $\phi(\xi)^n(R+\phi(\xi)^2)^{j/2}$ $(j=0,1;\ n=0,1,2,\cdots)$ in the obtained system of partial differential equation to zero, we can deduce the following set of over-determined partial differential equations with respect to the unknown derivative functions f, g, h, k and q (**Note:** in this paper, $g_y=\dfrac{\partial g(y,t)}{\partial y}$, and so on.)

$$\begin{aligned}&-R(8p^3gR^2h_y-5p^4h_yR^2+6p^3g_yhR^2\\&-3\alpha^2q_yq_{yy}hR-12p^3kRh_y\\&-p^2h_tR+2Rp^3k_yh-3\alpha^2q_y^2h_yR-\alpha^2h_{yyy})=0, \quad (9)\end{aligned}$$

$$\begin{aligned}&241p^4q_yhR^2+11p^2q_thR+11\alpha^2q_y^3hR\\&+\alpha^2q_{yyy}h+3\alpha^2q_yh_{yy}+3\alpha^2q_{yy}h_y\\&-22p^3f_yhR-252p^3gR^2q_yh+66p^3kRq_yh=0, \quad (10)\end{aligned}$$

$$\begin{aligned}&52p^4h_yR+8p^3kh_y+2p^2h_t+6\alpha^2q_y^2h_y\\&-68p^3g_yhR-72p^3gRh_y+6\alpha^2q_yq_{yy}h=0, \quad (11)\end{aligned}$$

$$\begin{aligned}&-6h(2p^3f_y-50p^4q_yR-p^2q_t\\&+51p^3gRq_y-\alpha^2q_y^3-6p^3kq_y)=0, \quad (12)\end{aligned}$$

$$\begin{aligned}&12p^3kRh_y+9\alpha^2q_yq_{yy}hR-42p^3g_yhR^2\\&+\alpha^2h_{yyy}+3p^2h_tR+9\alpha^2q_y^2h_yR\\&+33p^4h_yR^2-48p^3gR^2h_y=0, \quad (13)\end{aligned}$$

$$\begin{aligned}&48p^3gq_ykR+136p^4q_ygR^2+8p^2q_tgR+3\alpha^2q_yg_{yy}\\&-84p^3g^2R^2q_y+8\alpha^2q_y^3gR-16p^3f_ygR\\&+\alpha^2q_{yyy}g-60p^3h^2q_yR^2+3\alpha^2q_{yy}g_y=0, \quad (14)\end{aligned}$$

$$\begin{aligned}&-2R^2(4Rp^3gk_y-20p^4k_yR-3kg_{yy}q_y\alpha^2\\&-3\alpha^2q_y^2k_y-p^2k_t-28p^3kk_y)=0, \quad (15)\end{aligned}$$

$$\begin{aligned}&40p^4g_yR+2p^2g_t+6\alpha^2q_yq_{yy}g-52p^3h_yhR\\&+6\alpha^2q_y^2g_y+8p^3kg_y-56p^3gRg_y=0, \quad (16)\end{aligned}$$

$$\begin{aligned}&R(48p^3gR^2q_yk-136p^4q_ykR^2+16p^3f_ykR\\&-84p^3k^2q_yR-8p^2q_tkR-8\alpha^2q_y^3kR\\&-3\alpha^2q_yk_{yy}-3\alpha^2g_{yy}k_y-\alpha^2q_{yyy}k)=0, \quad (17)\end{aligned}$$

$$\begin{aligned}&6kR^3(-22p^3kq_y+6p^3gRq_y-40p^4q_yR\\&-\alpha^2q_y^3+2p^3f_y-p^2q_t)=0, \quad (18)\end{aligned}$$

$$\begin{aligned}&24p^3kRk_y+16p^4k_yR^2+6\alpha^2q_y^2k_yR+\alpha^2k_{yyy}\\&-8p^3gR^2k_y+2p^2k_tR+6\alpha^2q_yg_{yy}kR=0, \quad (19)\end{aligned}$$

$$\begin{aligned}&-6p^3h^2q_yR^3-2\alpha^2q_y^3kR-12p^3g^2R^3q_y\\&+\alpha^2f_{yyy}+2\alpha^2q_y^3gR^2+24p^3gR^2q_yk\\&+2p^2q_tgR^2-2p^2q_tkR+3\alpha^2q_yg_{yy}R\\&-3\alpha^2q_yk_{yy}-3\alpha^2g_{yy}k_y-12p^3k^2q_yR\\&-\alpha^2q_{yyy}k+\alpha^2q_{yyy}gR-4p^3f_ygR^2\\&-16p^4q_ykR^2+16p^4q_ygR^3\\&+3\alpha^2g_{yy}g_yR+4p^3f_ykR=0, \quad (20)\end{aligned}$$

$$\begin{aligned}&16p^4g_yR^2-24p^3gR^2g_y+6\alpha^2q_yq_{yy}gR-20p^3h_yhR^2\\&+\alpha^2g_{yyy}+6\alpha^2q_y^2g_yR+8p^3kRg_y+2p^2g_tR=0, \quad (21)\end{aligned}$$

$$\begin{aligned}&240p^4q_ygR-12p^3f_yg-114p^3h^2q_yR+6\alpha^2q_y^3g\\&+6p^2q_tg+36p^3kq_yg-132p^3g^2Rq_y=0, \quad (22)\end{aligned}$$

$$\begin{aligned}&R(-66p^3gR^2q_yh+61p^4q_yhR^2+36p^3kRq_yh\\&+5p^2q_thR+5\alpha^2q_y^3hR-10p^3f_yhR\\&+\alpha^2q_{yyy}h+3\alpha^2g_{yy}h_y+3\alpha^2q_yh_{yy})=0, \quad (23)\end{aligned}$$

$$8p^3R^3k_y(3Rp+4k)=0, \tag{24}$$

$$18p^3kR^3q_yh=0, \tag{25}$$

$$2p^3R^3(-k_yh+10h_yk)=0, \tag{26}$$

$$-8p^3(-3g_yp+4h_yh+4g_yg)=0, \tag{27}$$

$$-60p^3q_y(g^2-2pg+h^2)=0, \tag{28}$$

$$-120q_yhp^3(-p+g)=0, \tag{29}$$

$$-60kp^3R^4q_y(2Rp+k)=0, \tag{30}$$

$$-8p^3(-3h_yp+4h_yg+4g_yh)=0, \tag{31}$$

$$12p^3hq_ykR^4=0, \tag{32}$$

$$12p^3h_ykR^4=0. \tag{33}$$

Using the powerful PDE tools package of Maple, solving the set of partial differential Eqs. (9)–(33), we can obtain the following results. (**Note**: in the following $F_i(t)$ $(i=1,\cdots,5)$ denote arbitrary differentiable functions with respect to t and $F_i'(t)=\dfrac{\partial F_i(t)}{\partial t}$.)

Case 1

$$q=F_3(t)y+F_4(t),\quad g=p,\quad h=p,$$

$$f=\frac{1}{4p}F_3'(t)y^2+\Big[-\frac{1}{2}pF_3(t)R+\frac{1}{2p}F_4'(t)+\frac{1}{2p^3}\alpha^2F_3^3(t)\Big]y+F_5(t),\quad k=0. \tag{34}$$

$$g=p,\quad f=\frac{1}{4p}F_1'(t)y^2+\Big[-\frac{1}{2}pF_1(t)R+\frac{1}{2p}F_2'(t)+\frac{1}{2p^3}\alpha^2F_1^3(t)\Big]y+F_5(t),$$

$$h=-p,\quad q=F_1(t)y+F_2(t),\quad k=0. \tag{35}$$

Case 2

$$h=0,\quad k=-2Rp,\quad g=2p,\quad q=F_1(t)y+F_2(t),$$

$$f=\frac{1}{4p}F_1'(t)y^2+\Big[\frac{1}{2p^3}\alpha^2F_1^3(t)-8pF_1(t)R+\frac{1}{2p}F_2'(t)\Big]y+F_3(t). \tag{36}$$

Case 3

$$h=0,\quad k=-2Rp,\quad q=F_1(t)y+F_2(t),\quad g=0,$$

$$f=\frac{1}{4p}F_1'(t)y^2+\Big[-2pF_1(t)R+\frac{1}{2p^3}\alpha^2F_1(t)^3+\frac{1}{2p}F_2'(t)\Big]y+F_3(t). \tag{37}$$

$$f=\frac{1}{4p}F_1'(t)y^2+\Big[-2pF_1(t)R+\frac{1}{2p^3}\alpha^2F_1^3(t)+\frac{1}{2p}F_2'(t)\Big]y+F_3(t),$$

$$h=0,\quad g=2p,\quad q=F_1(t)y+F_2(t),\quad k=0. \tag{38}$$

Case 4

$$f=\frac{1}{2p}F_1'(t)y+F_2(t),\quad g=C_2,\quad h=C_1,$$

$$k=-\frac{3}{4}Rp,\quad q=F_1(t). \tag{39}$$

$$f=\frac{1}{2p}F_1'(t)y+F_2(t),\quad g=C_3,$$

$$k=C_1,\quad q=F_1(t),\quad h=C_2. \tag{40}$$

From Eqs.(8), (4), (6) and (34)–(40), we can obtain the following solutions for Bogoyavlenskii's generalized breaking soliton equation.

Case A

$$u_{11}=f+p\sqrt{-R}[\tanh(\sqrt{-R}\xi)\pm \mathrm{i}\,\mathrm{sech}(\sqrt{-R}\xi)], \tag{41}$$

$$u_{12}=f+p\sqrt{-R}[\coth(\sqrt{-R}\xi)\pm \mathrm{csch}(\sqrt{-R}\xi)], \tag{42}$$

$$u_{13}=f+p\sqrt{R}[\tan(\sqrt{R}\xi)\pm \sec(\sqrt{R}\xi)], \tag{43}$$

$$u_{14}=f+p\sqrt{R}[\cot(\sqrt{R}\xi)\pm \csc(\sqrt{R}\xi)], \tag{44}$$

where f is determined by Eqs.(34) and (35), and $\xi=xp+F_1(t)y+F_2(t)$.

Case B

$$u_{21}=f-2p\sqrt{-R}[\tanh(\sqrt{-R}\xi)+\coth(\sqrt{-R}\xi)],$$

$$R<0, \tag{45}$$

$$u_{22}=f-2p\sqrt{R}[\tan(\sqrt{R}\xi)+\cot(\sqrt{R}\xi)],\quad R>0, \tag{46}$$

where f is determined by Eq.(36), and $\xi=xp+F_1(t)y+F_2(t)$.

Case C

$$u_{31}=f-2p\sqrt{-R}\tanh(\sqrt{-R}\xi),\quad R<0, \tag{47}$$

$$u_{32}=f-2p\sqrt{-R}\coth(\sqrt{-R}\xi),\quad R<0, \tag{48}$$

$$u_{33}=f-2p\sqrt{R}\tan(\sqrt{R}\xi),\quad R>0, \tag{49}$$

$$u_{34}=f-2p\sqrt{R}\tanh(\sqrt{R}\xi),\quad R>0, \tag{50}$$

where f is determined by Eqs.(27) and (28), and $\xi=xp+F_1(t)y+F_2(t)$.

Case D

$$u_{41} = f - g\sqrt{-R}\tanh(\sqrt{-R}\xi) - h\sqrt{R}\operatorname{sech}(\sqrt{-R}\xi) - \frac{k}{\sqrt{-R}}\coth(\sqrt{-R}\xi), \quad R<0, \tag{51}$$

$$u_{42} = f - g\sqrt{-R}\coth(\sqrt{-R}\xi) - h\sqrt{-R}\operatorname{csch}(\sqrt{-R}\xi) - \frac{k}{\sqrt{-R}}\tanh(\sqrt{-R}\xi), \quad R<0, \tag{52}$$

$$u_{43} = f - g\sqrt{R}\tan(\sqrt{R}\xi) - h\sqrt{R}\sec(\sqrt{R}\xi) - \frac{k}{\sqrt{R}}\cot(\sqrt{R}\xi), \quad R>0, \tag{53}$$

$$u_{44} = f - g\sqrt{-R}\cot(\sqrt{R}\xi) - h\sqrt{R}\csc(\sqrt{R}\xi) - \frac{k}{\sqrt{R}}\tan(\sqrt{R}\xi), \quad R>0, \tag{54}$$

where f, g, h, k are determined by Eqs.(39) and (40), and $\xi = xp + F_1(t)$.

The physical interest of the above solutions lies in the fact that they describe certain soliton-like surface waves. From the resulting Eqs. (41)–(44), we know that because of the entrance of the arbitrary functions, the Bogoyavlenskii's generalized breaking soliton equation possesses quite abundant localized soliton structures. The richness of the soliton structures caused by arbitrary functions may be a universal phenomenon in high dimensions. Because of the wide applications of the soliton theory in real physics, more about these high-dimensional localized solutions is worthy of studying further.

4. Conclusions

In summary, based on the computerized symbolic computation, by introducing a new more general ansatz than the ansatz in the extended tanh-function method, modified extended tanh-function method, and generalized hyperbolic-function method, we have proposed a generalized Riccati equation expansion method to search for exact solutions of NEEs and implemented in computerized symbolic system Maple. Making use of our method and with the aid of Maple, we study the Bogoyavlenskii's generalized breaking soliton equation and obtain new families of the exact solutions. In our obtained exact solutions the restriction on $\xi(x, y, t)$ as merely a linear function of x, y, t, and the restriction on the coefficients f, g_i, h_i, k_i $(i = 1, \cdots, m)$ as constants are removed and, with no extra effect, the singular solitonic solution and periodic form solutions, even rational solutions could be obtained. We think that the method presented in this paper is not just only the extension from the mathematical point of view, but hope that the solutions obtained by the method will lead to more deeper and more comprehensive understanding of the complex structures resulting from the NEEs. To make the work feasible, how to choose the forms for f, g_i, h_i, k_i $(i = 1, \cdots, m)$ and ξ in the ansatz would be the key step in the computation of our method. The method, proposed in this paper for single equation, may be extended to find exact solutions of coupled equations.

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