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EXACT TRAVELING SOLUTIONS FOR SOME NONLINEAR EVOLUTION EQUATIONS WITH NONLINEAR TERMS OF ANY ORDER

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In this paper, we improved the tanh method by means of a proper transformation and general ansätze. Using the improved method, with the aid of *Mathematica*™, we consider some nonlinear evolution equations with nonlinear terms of any order. As a result, rich explicit exact traveling solutions for these equations, which contain kink profile solitary wave solutions, bell profile solitary wave solutions, rational solutions, periodic solutions, and combined formal solutions, are obtained.

Keywords: Nonlinear evolution equations; exact traveling solution; symbolic computation.

1. Introduction

In recent years, important progress has been made in understanding nonlinear evolution equations (NEEs). The finding of explicit exact solutions, in particular, solitary wave solutions, of NEEs in mathematical physics plays an important role in soliton theory.[1–14] Various powerful methods have been presented, such as Bäcklund transformation, Darboux transformation, Cole–Hopf transformation, tanh method, sine-cosine method, Painlevé method, Hirota method,[1,2] rank analysis method,[3] homogeneous balance method (HBM),[9–12] variable-coefficient balancing act method.[13,14] In particular, some general ansätzes were proposed in order to obtain new formal solutions for given NEEs. The direct search for exact solutions of NEEs has become more attractive, partly due to the availability of applications like *Maple* or *Mathematica*™, which allow one to perform complicated and tedious algebraic calculations on the computer, as well as in finding the new exact solution of NEES.

Based on the well-known Riccati equation, Fan[15,16] has presented a useful extended tanh method to find the exact solutions of given NEEs. More recently, Fan[17] and Yan[18] further developed this idea and made it much more lucid and straightforward for a class of NEEs.

This paper aims to further improve the extended tanh method by introducing a more general proper transformation. Using the improved method, with the aid of *Mathematica*™, we consider some NEEs with nonlinear terms of any order: the generalized Fitzhugh–Nagumo equation,[3,7] the generalized Burgers–Fisher equation,[3–6] and the generalized Burgers–Huxley equation.[3,4]

In the next section, we describe the improved tanh method and provide some examples to illustrate the efficiency of this approach. We end the discussion with some conclusions.

2. Method

In this section, we describe the extended tanh method developed by some authors[15–18] and improve on it by introducing some transformations for given NEEs in two variables, x and t:

$$F(u, u_t, u_x, u_{xt}, u_{tt}, \ldots) = 0\,. \tag{2.1}$$

First, we make the transformation into a traveling solution:

$$u(x,t) = u(\xi)\,, \quad \xi = x - \beta t\,, \tag{2.2}$$

where β is a constant to be determined later. Equation (2.1) then reduces to a nonlinear ordinary differential equation (ODE):

$$G(u, u', u'', u''', \ldots) = 0\,, \tag{2.3}$$

which is integrated if all the terms contain derivatives, and where the prime denotes $d/d\xi$. The next crucial step is to express the solution of the resulting ODE by the more general ansatz

$$u(\xi) = \sum_{i=1}^{m} \omega^{i-1}(\xi)[A_i\omega(\xi) + B_i\sqrt{R + \omega^2(\xi)}\,] + A_0\,, \tag{2.4}$$

with the new variable $\omega = \omega(\xi)$ satisfying

$$\omega' - (R + \omega^2) = \frac{d\omega}{d\xi} - (R + \omega^2) = 0\,, \tag{2.5}$$

where A_0, A_i, $B_i\,(i = 1, 2, \ldots, m)$, and R are constants to be determined later, and m is a positive integer. However, when we balance the highest order derivative term with the nonlinear term in Eq. (2.3), we find that the constant m need not be restricted to a positive integer. In order to apply the extended tanh method described in Refs. 20 to 23 when m is equal to a fraction or a negative integer, we make the following transformation:

(1) When $m = q/p$ is a low fraction, we substitute

$$u(\xi) = \varphi^{q/p}(\xi) \tag{2.6}$$

into Eq. (2.3) and return to determine the value of m by balancing the highest order derivative term with the nonlinear term in the new Eq. (2.3).

(2) When m is a negative integer, we substitute

$$u(\xi) = \varphi^m(\xi) \tag{2.7}$$

into Eq. (2.3) and return to determine the value of m as before.

In general, the constant m can be changed into a positive integer by means of the above transformation. Otherwise, we have to seek another proper transformation.

We summarize the extended tanh method as follows:

Step 1. Determine the values of m in Eq. (2.4) by balancing the highest order derivative term with the nonlinear term in Eq. (2.3):

(1) If m is a positive integer, proceed to Step 2.
(2) If $m = q/p$, we make the transformation, Eq. (2.6), and return to Step 1.
(3) If m is a negative integer, we make the transformation, Eq. (2.7), and return to Step 1.

Step 2. With the aid of *Mathematica*™, substituting Eq. (2.4), along with the condition (2.5), into Eq. (2.3) yields a system of algebraic equations with respect to $\omega^k(\sqrt{R+\omega^2}\,)^j$ $(j = 0, 1; k = 0, 1, 2, \ldots)$, where ω^k denotes k power of ω and $(\sqrt{R+\omega^2}\,)^j$ denotes j power of $\sqrt{R+\omega^2}$.

Step 3. Collect all terms with the same power in $\omega^k(\sqrt{R+\omega^2}\,)^j$ $(j = 0, 1; k = 0, 1, 2, \ldots)$ and set the coefficients of the terms $\omega^k(\sqrt{R+\omega^2}\,)^j$ $(j = 0, 1; k = 0, 1, 2, \ldots)$ to zero to get an over-determined system of nonlinear algebraic equations with respect to the unknown variables λ, R, A_0, A_i, B_i $(i = 1, 2, \ldots, m)$.

Step 4. With the aid of *Mathematica*™, solving the above over-determined system of nonlinear algebraic equations obtained in Step 3 yields the values λ, R, A_0, A_i, B_i $(i = 1, 2, \ldots, m)$.

Step 5. It is well known that the general solutions of Eq. (2.5) are:

(1) When $R < 0$:

$$\omega(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi)\,, \quad \omega(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi)\,. \tag{2.8}$$

(2) When $R = 0$:

$$\omega(\xi) = -\frac{1}{\xi}\,. \tag{2.9}$$

(3) When $R > 0$:

$$\omega(\xi) = \sqrt{R}\tan(\sqrt{R}\xi)\,, \quad \omega(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi)\,. \tag{2.10}$$

Thus according to Eqs. (2.2), (2.4), and (2.6), or Eqs. (2.7), (2.8), (2.9), and (2.10), and the conclusions in Step 4, we can obtain many traveling wave solutions of Eq. (2.1).

3. Examples

The traveling wave solutions of the form $u = u(\xi)$, $\xi = x - \lambda t$ are considered in all the following examples.

Example 1. The generalized Fitzhugh–Nagumo equation[3,7] is

$$u_t - \alpha u_{xx} = \beta u(1 - u^\delta)(u^\delta - r)\,, \tag{3.1}$$

where α, β, $\delta \geq 0$ and $r \in [-1, 1)$. Considering the traveling wave solution to the above equation, we have

$$-\lambda u' - \alpha u'' = \beta u(1 - u^\delta)(u^\delta - r)\,. \tag{3.2}$$

According to Step 1 in Sec. 2, by balancing u'' and $u^{2\delta+1}$ in Eq. (3.2), we get $m = 1/\delta$. Therefore, we make the transformation

$$u(\xi) = \varphi^{1/\delta}(\xi)\,. \tag{3.3}$$

Substituting Eq. (3.3) into Eq. (3.2) yields

$$\gamma\beta\delta^2 v^2 - (1 + r)\beta\delta^2 v^3 + \beta\delta^2 v^4 + \alpha(-1 + \delta)v'^2 - \delta v(\lambda v' + \alpha v'') = 0\,. \tag{3.4}$$

Balancing vv'' and v^4, we get $m = 1$. According to Step 1 in Sec. 2, we suppose that Eq. (3.4) has the formal solutions

$$\varphi(\xi) = A_0 + A_1\omega + B_1\sqrt{R + \omega^2} \tag{3.5}$$

and $\omega = \omega(\xi)$ satisfies Eq. (2.5), where A_0, A_1, and B_1 are constants to be determined later.

With the aid of *Mathematica*™, substituting Eq. (3.5) into Eq. (3.4), along with Eq. (2.5) and collecting all terms with the same power in $\omega^k(\sqrt{R + \omega^2})^j$ $(j = 0, 1; k = 0, 1, 2, 3, 4)$, yields a system of equations with respect to $\omega^k(\sqrt{R + \omega^2})^j$. Setting the coefficients of $\omega^k(\sqrt{R + \omega^2})^j$ $(j = 0, 1; k = 0, 1, 2, 3, 4)$ in the obtained system of equations to zero, we deduce the following set of over-determined algebraic polynomials with respect to the unknowns A_0, A_1, B_1, R, β:

$$\begin{aligned} &A_1^2 R^2\alpha(-1 + \delta) + \delta((-1 + A_0)A_0^2(A_0 - r)\beta\delta + B_1^4 R^2\beta\delta \\ &\quad + B_1^2 R(-R\alpha + (6A_0^2 + r - 3A_0(1 + r))\beta\delta)) - A_0 A_1 R\delta\lambda = 0\,, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &B_1\delta(4A_0^3\beta\delta - 3A_0^2(1 + r)\beta\delta + A_0(-R\alpha + 2r\beta\delta + 4B_1^2 R\beta\delta) \\ &\quad - R(B_1^2(1 + r)\beta\delta + A_1\lambda)) = 0\,, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\delta(4A_0^3 A_1\beta\delta - 3A_0^2 A_1(1 + r)\beta\delta + 2A_0 A_1(-R\alpha + r\beta\delta + 6B_1^2 R\beta\delta) \\ &\quad - R(3A_1 B_1^2(1 + r)\beta\delta + A_1^2\lambda + B_1^2\lambda)) = 0\,, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &B_1(-A_1(-2(6A_0^2 + r - 3A_0(1 + r))\beta\delta^2 \\ &\quad + R(-4B_1^2\beta\delta^2 + \alpha(2 + \delta))) - A_0\delta\lambda) = 0\,, \end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&A_1^2((6A_0^2 + r - 3A_0(1+r))\beta\delta^2 - 2R(\alpha - 3B_1^2\beta\delta^2)) \\
&\quad + B_1^2((6A_0^2 + r - 3A_0(1+r))\beta\delta^2 - R(\alpha + 2\alpha\delta - 2B_1^2\beta\delta^2)) \\
&\quad - A_0A_1\delta\lambda = 0\,,
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
&B_1\delta(-3A_1^2(1+r)\beta\delta - B_1^2(1+r)\beta\delta - 2A_0(\alpha - 2(3A_1^2 + B_1^2)\beta\delta) - 2A_1\lambda) \\
&\quad = 0\,,
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
&\delta(-A_1^3(1+r)\beta\delta - 3A_1B_1^2(1+r)\beta\delta + 2A_0A_1(-\alpha + 2(A_1^2 + 3B_1^2)\beta\delta) \\
&\quad - A_1^2\lambda - B_1^2\lambda) = 0\,,
\end{aligned} \tag{3.12}
$$

$$
2A_1B_1(2(A_1^2 + B_1^2)\beta\delta^2 - \alpha(1+\delta)) = 0\,, \tag{3.13}
$$

$$
A_1^4\beta\delta^2 - A_1^2(\alpha + \alpha\delta - 6B_1^2\beta\delta^2) + B_1^2(B_1^2\beta\delta^2 - \alpha(1+\delta)) = 0\,. \tag{3.14}
$$

From Eqs. (3.6) to (3.14), with the aid of *Mathematica*™, we have:

Case 1.

$$
\begin{aligned}
&A_0 = \frac{1}{2}\,, \quad A_1^2 = \frac{\alpha(1+\delta)}{\beta\delta^2}\,, \quad B_1 = 0\,, \\
&R = -\frac{\beta\delta^2}{4\alpha(1+\delta)}\,, \quad \lambda = \pm\sqrt{\frac{\alpha\beta(-1+r+r\delta)^2}{1+\delta}}\,.
\end{aligned} \tag{3.15}
$$

Case 2.

$$
\begin{aligned}
&A_0 = \frac{r}{2}\,, \quad A_1^2 = \frac{\alpha(1+\delta)}{\beta\delta^2}\,, \quad B_1 = 0\,, \\
&R = -\frac{r^2\beta\delta^2}{4\alpha(1+\delta)}\,, \quad \lambda = \pm\sqrt{\frac{\alpha\beta(1-r+\delta)^2}{1+\delta}}\,.
\end{aligned} \tag{3.16}
$$

Case 3.

$$
\begin{aligned}
&A_0 = \frac{1+r}{2}\,, \quad A_1^2 = \frac{\alpha(1+\delta)}{\beta\delta^2}\,, \quad B_1 = 0\,, \\
&R = -\frac{(-1+r)^2\beta\delta^2}{4\alpha(1+\delta)}\,, \quad \lambda^2 = \frac{\alpha\beta(1+r^2+2r\delta)}{1+\delta}\,.
\end{aligned} \tag{3.17}
$$

Case 4.

$$
\begin{aligned}
&A_0 = \frac{1}{2}\,, \quad A_1 = \pm B_1\,, \quad B_1^2 = \frac{\alpha(1+\delta)}{4\beta\delta^2}\,, \\
&R = -\frac{\beta\delta^2}{\alpha(1+\delta)}\,, \quad \lambda^2 = \frac{\alpha\beta(-1+r+r\delta)^2}{1+\delta}\,.
\end{aligned} \tag{3.18}
$$

Case 5.

$$
\begin{aligned}
&A_0 = \frac{r}{2}\,, \quad A_1 = \pm B_1\,, \quad B_1^2 = \pm\frac{\alpha(1+\delta)}{4\beta\delta^2}\,,\\
&R = -\frac{r^2\beta\delta^2}{\alpha(1+\delta)}\,, \quad \lambda^2 = \frac{\alpha\beta(1-r+\delta)^2}{1+\delta}\,.
\end{aligned} \tag{3.19}
$$

Case 6.

$$
R = A_0 = r = 0\,, \quad A_1^2 = B_1^2 = \frac{(1+\alpha)\delta}{4\beta\delta^2}\,, \quad \lambda^2 = \alpha\beta(1+\delta)\,. \tag{3.20}
$$

Case 7.

$$
A_0 = A_1 = \lambda = 0\,, \quad B_1^2 = \frac{\alpha(1+\delta)}{\beta\delta^2}\,, \quad R = \frac{\beta\delta^2}{\alpha}\,. \tag{3.21}
$$

Therefore, combining Eqs. (2.8), (2.9), (2.10), (3.3), and (3.5), along with Cases 1 to 7, we obtain the traveling wave solutions of the generalized Fitzhugh–Nagumo equation as follows:

Case 1.

$$
u_{11} = \left\{\frac{1}{2} \pm \frac{1}{2}\tanh\left[\sqrt{\frac{\beta\delta^2}{4\alpha(1+\delta)}}\left(x \pm \sqrt{\frac{\alpha\beta(-1+r+r\delta)^2}{1+\delta}}\,t\right)\right]\right\}^{1/\delta}\,,
$$

$$
R < 0\,, \quad \text{i.e., } \alpha\beta(1+\delta) > 0\,, \tag{3.22}
$$

$$
u_{12} = \left\{\frac{1}{2} \pm \frac{1}{2}\coth\left[\sqrt{\frac{\beta\delta^2}{4\alpha(1+\delta)}}\left(x \pm \sqrt{\frac{\alpha\beta(-1+r+r\delta)^2}{1+\delta}}\,t\right)\right]\right\}^{1/\delta}\,,
$$

$$
\alpha\beta(1+\delta) > 0\,, \tag{3.23}
$$

$$
u_{13} = \left\{\frac{1}{2} \pm \frac{i}{2}\tan\left[\sqrt{-\frac{\beta\delta^2}{4\alpha(1+\delta)}}\left(x \pm \sqrt{\frac{\alpha\beta(-1+r+r\delta)^2}{1+\delta}}\,t\right)\right]\right\}^{1/\delta}\,,
$$

$$
\alpha\beta(1+\delta) < 0\,, \tag{3.24}
$$

$$
u_{14} = \left\{\frac{1}{2} \pm \frac{i}{2}\cot\left[\sqrt{-\frac{\beta\delta^2}{4\alpha(1+\delta)}}\left(x \pm \sqrt{\frac{\alpha\beta(-1+r+r\delta)^2}{1+\delta}}\,t\right)\right]\right\}^{1/\delta}\,,
$$

$$
\alpha\beta(1+\delta) < 0\,. \tag{3.25}
$$

For simplification, the rest cases of the periodic solutions are omitted.

Case 2.

$$
u_{21} = \left\{\frac{r}{2} \pm \frac{r}{2}\tanh\left[\sqrt{\frac{r^2\beta\delta^2}{4\alpha(1+\delta)}}\left(x \pm \sqrt{\frac{\alpha\beta(1-r+\delta)^2}{1+\delta}}\,t\right)\right]\right\}^{1/\delta}\,,
$$

$$
\alpha\beta(1+\delta) > 0\,, \tag{3.26}
$$

$$u_{22} = \left\{ \frac{r}{2} \pm \frac{r}{2} \coth \left[\sqrt{\frac{r^2\beta\delta^2}{4\alpha(1+\delta)}} \left(x \pm \sqrt{\frac{\alpha\beta(1-r+\delta)^2}{1+\delta}}\, t \right) \right] \right\}^{1/\delta},$$

$$\alpha\beta(1+\delta) > 0\,. \tag{3.27}$$

Case 3.

$$u_{31} = \left\{ \frac{1+r}{2} \pm \frac{r-1}{2} \tanh \left[\sqrt{\frac{(r-1)^2\beta\delta^2}{4\alpha(1+\delta)}} \left(x \pm \sqrt{\frac{\alpha\beta(1+r^2+2r\delta)}{1+\delta}}\, t \right) \right] \right\}^{1/\delta},$$

$$\alpha\beta(1+\delta) > 0\,, \tag{3.28}$$

$$u_{32} = \left\{ \frac{1+r}{2} \pm \frac{r-1}{2} \coth \left[\sqrt{\frac{(r-1)^2\beta\delta^2}{4\alpha(1+\delta)}} \left(x \pm \sqrt{\frac{\alpha\beta(1+r^2+2r\delta)}{1+\delta}}\, t \right) \right] \right\}^{1/\delta},$$

$$\alpha\beta(1+\delta) > 0\,. \tag{3.29}$$

Case 4.

$$u_{41} = \left\{ \frac{1}{2} [1 \pm \tanh[\sqrt{-R}(x-\lambda t)] \pm i\,\mathrm{sech}[\sqrt{-R}(x-\lambda t)]] \right\}^{1/\delta}, \tag{3.30}$$

$$u_{42} = \left\{ \frac{1}{2} [1 \pm \coth[\sqrt{-R}(x-\lambda t)] \pm \mathrm{csch}[\sqrt{-R}(x-\lambda t)]] \right\}^{1/\delta}, \tag{3.31}$$

where $R = -\frac{\beta\delta^2}{\alpha(1+\delta)} < 0$, $\lambda^2 = \frac{\alpha\beta(-1+r+r\delta)^2}{1+\delta}$.

Case 5.

$$u_{51} = \left\{ \frac{r}{2} [1 \pm \tanh[\sqrt{-R}(x-\lambda t)] \pm i\,\mathrm{sech}[\sqrt{-R}(x-\lambda t)]] \right\}^{1/\delta}, \tag{3.32}$$

$$u_{52} = \left\{ \frac{r}{2} [1 \pm \coth[\sqrt{-R}(x-\lambda t)] \pm \mathrm{csch}[\sqrt{-R}(x-\lambda t)]] \right\}^{1/\delta}, \tag{3.33}$$

where $R = -(r^2\beta\delta^2)/\alpha(1+\delta) < 0$, $\lambda^2 = (\alpha\beta(1-r+\delta)^2)/1+\delta$.

Case 6. From Eq. (3.20), the generalized Fitzhugh–Nagumo equation $u_t - \alpha u_{xx} = \beta u(1-u^\delta)u^\delta$ has the following rational solutions:

$$u_6 = \left\{ \pm\sqrt{\frac{(1+\alpha)\delta}{\beta\delta^2}} \frac{1}{x \pm \sqrt{\alpha\beta(1+\delta)}t} \right\}^{1/\delta}. \tag{3.34}$$

Case 7.

$$u_{71} = \left\{ \pm\sqrt{-(1+\delta)}\,\mathrm{sech}\left(\sqrt{-\frac{\beta\delta^2}{\alpha}}\,x\right)\right\}^{1/\delta}, \quad \alpha\beta < 0\,, \tag{3.35}$$

$$u_{72} = \left\{ \pm\sqrt{(1+\delta)}\,\mathrm{csch}\left(\sqrt{-\frac{\beta\delta^2}{\alpha}}\,x\right)\right\}^{1/\delta}, \quad \alpha\beta < 0\,. \tag{3.36}$$

Remark 1. From Eqs. (3.22), (3.23), (3.26), and (3.27), the solutions obtained in Refs. 3 to 7 are recovered. But, to our knowledge, the other solutions obtained here were not found earlier.

Example 2. The generalized Burgers–Fisher equation[3–6] is

$$u_t + \alpha u^\delta u_x - \frac{m}{u}\,u_x^2 - u_{xx} = \beta u(1-u^\delta) \quad (\delta > 0)\,. \tag{3.37}$$

Considering its traveling wave solution, we have

$$-\lambda u u' + \alpha u^{\delta+1}u' - mu'^2 - uu'' - \beta u^2(1-u^\delta) = 0\,. \tag{3.38}$$

Balancing uu'' with $u^{\delta+1}u'$ or $u^{\delta+2}$ gives $m = 1/\delta$. Thus, we make the transformation

$$u(\xi) = \varphi^{1/\delta}(\xi)\,. \tag{3.39}$$

Substituting Eq. (3.39) into Eq. (3.38) yields

$$\beta\delta^2(-1+\varphi)\varphi^2 + (-1-m+\delta)\varphi'^2 - \delta\varphi[(\lambda-\alpha\varphi)\varphi' + \varphi''] = 0\,. \tag{3.40}$$

Balancing $\varphi\varphi''$ with $\varphi^2\varphi'$, we get $m = 1$. Hence, we may choose

$$\varphi(\xi) = A_0 + A_1\omega + B_1\sqrt{R+\omega^2}\,, \tag{3.41}$$

and $\omega = \omega(\xi)$ satisfies Eq. (2.5).

Substituting Eq. (3.41) into Eq. (3.40), and using *Mathematica*™, engenders the following set of algebraic equations for A_0, A_1, B_1, R, and λ:

$$\begin{aligned} &-A_1^2R^2(1+m-\delta) + \delta((-1+A_0)A_0^2\beta\delta - B_1^2R(R+\beta\delta-3A_0\beta\delta)) \\ &\quad + A_1R\delta(\alpha(A_0^2+B_1^2R) - A_0\lambda) = 0\,, \end{aligned} \tag{3.42}$$

$$B_1\delta(-A_0R + 2\alpha A_0A_1R - 2A_0\beta\delta + 3A_0^2\beta\delta + B_1^2R\beta\delta - A_1R\lambda) = 0\,, \tag{3.43}$$

$$\begin{aligned} &\delta(3A_0^2A_1\beta\delta + 2A_0(\alpha A_1^2R + \alpha B_1^2R - A_1(R+\beta\delta)) \\ &\quad - R(-3A_1B_1^2\beta\delta + A_1^2\lambda + B_1^2\lambda)) = 0\,, \end{aligned} \tag{3.44}$$

$$\begin{aligned} &B_1(2\alpha A_1^2R\delta - A_1(2(1-3A_0)\beta\delta^2 + R(2+2m+\delta)) \\ &\quad + \delta(\alpha(A_0^2+B_1^2R) - A_0\lambda)) = 0\,, \end{aligned} \tag{3.45}$$

$$\alpha A_1^3 R\delta - A_1^2(2(1+m)R + (1-3A_0)\beta\delta^2) - B_1^2((1-3A_0)\beta\delta^2 + R(1+m+2\delta)) + A_1\delta(\alpha(A_0^2+4B_1^2R) - A_0\lambda) = 0\,, \tag{3.46}$$

$$B_1\delta(A_0(-2+4\alpha A_1) + 3A_1^2\beta\delta + B_1^2\beta\delta - 2A_1\lambda) = 0\,, \tag{3.47}$$

$$\delta(2A_0(-A_1+\alpha A_1^2+\alpha B_1^2) + A_1^3\beta\delta + 3A_1B_1^2\beta\delta - A_1^2\lambda - B_1^2\lambda) = 0\,, \tag{3.48}$$

$$B_1(3\alpha A_1^2\delta + \alpha B_1^2\delta - 2A_1(1+m+\delta)) = 0\,, \tag{3.49}$$

$$\alpha A_1^3\delta + 3\alpha A_1B_1^2\delta - A_1^2(1+m+\delta) - B_1^2(1+m+\delta) = 0\,. \tag{3.50}$$

Solving the algebraic equations (3.42) to (3.50) by means of *Mathematica*$^{\text{TM}}$ gives:

Case 1.

$$A_0 = \frac{1}{2}, \quad A_1 = \frac{1+m+\delta}{\alpha\delta}, \quad B_0 = 0\,, \quad R = -\frac{\alpha^2\delta^2}{4(1+m+\delta)^2}\,, \quad \lambda = \frac{\alpha^2(1+m)+\beta(1+m+\delta)^2}{\alpha(1+m+\delta)}\,. \tag{3.51}$$

Case 2.

$$A_0 = \frac{1}{2}, \quad A_1 = \frac{1+m+\delta}{2\alpha\delta}, \quad B_1 = \pm\frac{1+m+\delta}{2\alpha\delta}\,, \quad R = -\frac{\alpha^2\delta^2}{(1+m+\delta)^2}\,, \quad \lambda = \frac{\alpha^2(1+m)+\beta(1+m+\delta)^2}{\alpha(1+m+\delta)}\,. \tag{3.52}$$

Hence, we obtain the following solutions:

$$u_{11} = \left\{\frac{1}{2} \pm \frac{1}{2}\tanh\left[\pm\frac{\alpha\delta}{2(1+m+\delta)}\left(x - \frac{\alpha^2(1+m)+\beta(1+m+\delta)^2}{\alpha(1+m+\delta)}\,t\right)\right]\right\}^{1/\delta}\,, \tag{3.53}$$

$$u_{12} = \left\{\frac{1}{2} \pm \frac{1}{2}\coth\left[\pm\frac{\alpha\delta}{2(1+m+\delta)}\left(x - \frac{\alpha^2(1+m)+\beta(1+m+\delta)^2}{\alpha(1+m+\delta)}\,t\right)\right]\right\}^{1/\delta}\,. \tag{3.54}$$

$$u_{21} = \left\{\frac{1}{2}\,[1 \pm \tanh[\sqrt{-R}(x-\lambda t)] \pm i\,\mathrm{sech}[\sqrt{-R}(x-\lambda t)]]\right\}^{1/\delta}\,, \tag{3.55}$$

$$u_{21} = \left\{\frac{1}{2}\,[1 \pm \coth[\sqrt{-R}(x-\lambda t)] \pm \mathrm{csch}[\sqrt{-R}(x-\lambda t)]]\right\}^{1/\delta}\,, \tag{3.56}$$

where $R = -(\alpha^2\delta^2)/(1+m+\delta)^2 < 0$, $\lambda = (\alpha^2(1+m)+\beta(1+m+\delta)^2)/\alpha(1+m+\delta)$.

Remark 2. It is easy to see that Eqs. (3.53) and (3.54) are the same as the solutions obtained in Refs. 3 to 6. But to our knowledge, the other solutions (including periodic solutions) obtained here were not found earlier.

Example 3. The generalized Burgers–Huxley equation[3,4] is

$$u_t + au^\delta u_x - \frac{m}{u}\, u_x^2 - Du_{xx} = \beta u(1-u^\delta)(u^\delta - r)\,. \tag{3.57}$$

Considering its traveling wave solutions, we have

$$-\lambda uu' + au^{\delta+1}u' - mu'^2 - Duu'' - \beta u^2(1-u^\delta)(u^\delta - r) = 0\,. \tag{3.58}$$

Balancing uu'' with $u^{2\delta+2}$ or $u^\delta u'$, we get $m = 1/\delta$. Hence, we make the transformation

$$u(\xi) = \varphi^{1/\delta}(\xi). \tag{3.59}$$

Substituting Eq. (3.59) into Eq. (3.58) yields

$$\begin{aligned} &\beta\delta^2(-1+\varphi)\varphi^2(-r+\varphi) + (-D-m+D\delta)\varphi'^2 \\ &\quad - \delta\varphi[(\lambda - a\varphi)\varphi' + D\varphi''] = 0\,. \end{aligned} \tag{3.60}$$

Balancing $\varphi\varphi''$ with φ^4 leads to the following ansatz:

$$\varphi(\xi) = A_0 + A_1\omega + B_1\sqrt{R+\omega^2} \tag{3.61}$$

and $\omega = \omega(\xi)$ satisfies Eq. (2.5). Substituting Eq. (3.61) into Eq. (3.60) leads to

$$\begin{aligned} &A_1^2R^2(-D-m+D\delta) + \delta((-1+A_0)A_0^2(A_0-r)\beta\delta + B_1^4R^2\beta\delta \\ &\quad + B_1^2R(-DR + (6A_0^2 + r - 3A_0(1+r))\beta\delta)) \\ &\quad + A_1R\delta(a(A_0^2 + B_1^2R) - A_0\lambda) = 0\,, \end{aligned} \tag{3.62}$$

$$\begin{aligned} &B_1\delta(2aA_0A_1R + 4A_0^3\beta\delta - 3A_0^2(1+r)\beta\delta + A_0(-DR + 2(r + 2B_1^2R)\beta\delta) \\ &\quad - R(B_1^2(1+r)\beta\delta + A_1\lambda)) = 0\,, \end{aligned} \tag{3.63}$$

$$\begin{aligned} &\delta(2aA_0(A_1^2 + B_1^2)R + 4A_0^3A_1\beta\delta - 3A_0^2A_1(1+r)\beta\delta \\ &\quad + 2A_0A_1(-DR + (r + 6B_1^2R)\beta\delta) - R(3A_1B_1^2(1+r)\beta\delta \\ &\quad + A_1^2\lambda + B_1^2\lambda)) = 0\,, \end{aligned} \tag{3.64}$$

$$\begin{aligned} &B_1(2aA_1^2R\delta - A_1(DR(2+\delta) + 2(mR - (6A_0^2 + r - 3A_0(1+r) \\ &\quad + 2B_1^2R)\beta\delta^2)) + \delta(a(A_0^2 + B_1^2R) - A_0\lambda)) = 0\,, \end{aligned} \tag{3.65}$$

$$\begin{aligned} &aA_1^3R\delta + A_1^2(-2DR - 2mR + (6A_0^2 + r - 3A_0(1+r) + 6B_1^2R)\beta\delta^2) \\ &\quad + B_1^2(-mR + (6A_0^2 + r - 3A_0(1+r) + 2B_1^2R)\beta\delta^2 - D(R + 2R\delta)) \\ &\quad + A_1\delta(a(A_0^2 + 4B_1^2R) - A_0\lambda) = 0\,, \end{aligned} \tag{3.66}$$

$$\begin{aligned} &B_1\delta(4aA_0A_1 - 2A_0D - 3A_1^2\beta\delta + 12A_0A_1^2\beta\delta - B_1^2\beta\delta \\ &\quad + 4A_0B_1^2\beta\delta - 3A_1^2r\beta\delta - B_1^2r\beta\delta - 2A_1\lambda) = 0\,, \end{aligned} \tag{3.67}$$

$$\delta(2aA_0(A_1^2+B_1^2)-2A_0A_1D-A_1^3\beta\delta+4A_0A_1^3\beta\delta-3A_1B_1^2\beta\delta + 12A_0A_1B_1^2\beta\delta-A_1^3r\beta\delta-3A_1B_1^2r\beta\delta-A_1^2\lambda-B_1^2\lambda)=0\,, \tag{3.68}$$

$$B_1(3aA_1^2\delta+aB_1^2\delta+4A_1^3\beta\delta^2-2A_1(D+m+D\delta-2B_1^2\beta\delta^2))=0\,, \tag{3.69}$$

$$aA_1^3\delta+3aA_1B_1^2\delta+A_1^4\beta\delta^2-A_1^2(D+m+D\delta-6B_1^2\beta\delta^2) + B_1^2(-D-m-D\delta+B_1^2\beta\delta^2)=0\,. \tag{3.70}$$

Solving them with the aid of *Mathematica*™, we find

Case 1.

$$A_0=\frac{1}{2}\,,\quad A_1=\frac{-a\pm\Delta}{2\beta\delta}\,,\quad B_1=0\,,\quad R=-\frac{\beta^2\delta^2}{(a\mp\Delta)^2}\,, \quad \lambda=\frac{-2(D+m)\beta+r[a^2+2\beta(D+m+D\delta)]\pm ar\Delta}{a\pm\Delta}\,. \tag{3.71}$$

Case 2.

$$A_0=\frac{r}{2}\,,\quad A_1=\frac{-a\pm\Delta}{2\beta\delta}\,,\quad B_1=0\,,\quad R=-\frac{r^2\beta^2\delta^2}{(a\mp\Delta)^2}\,, \quad \lambda=\frac{-2(D+m)(-1+r)\beta+2D\beta\delta+a(a\pm\Delta)}{a\pm\Delta}\,. \tag{3.72}$$

Case 3.

$$A_0=\frac{1+r}{2}\,,\quad A_1=\frac{-a\pm\Theta}{2\beta\delta}\,,\quad R=-\frac{(-1+r)^2\beta^2\delta^2}{a\mp\Theta}\,, \quad \lambda=-\frac{2D(1+r)\beta\delta}{a\pm\Theta}\,,\quad m=D(\delta-1)\,. \tag{3.73}$$

Case 4.

$$A_0=\frac{1}{2}\,,\quad A_1=\pm B_1=\frac{-a\pm\Delta}{4\beta\delta}\,,\quad R=-\frac{\delta^2[a^2+2\beta(D+m+D\delta)\pm a\Delta]}{2(D+m+D\delta)^2}\,, \quad \lambda=\frac{-2(D+m)\beta+r[2\beta(D+m+d\delta)+a^2\pm a\Delta]}{a\pm\Delta}\,. \tag{3.74}$$

Case 5.

$$A_0=\frac{r}{2}\,,\quad A_1=\pm B_1=\frac{-a\pm\Delta}{4\beta\delta}\,,\quad R=-\frac{r^2\delta^2[a^2+2\beta(D+m+D\delta)\pm a\Delta]}{2(D+m+D\delta)^2}\,, \quad \lambda=\frac{-2(D+m)(-1+r)\beta+2D\beta\delta+a^2\pm a\Delta}{a\pm\Delta}\,. \tag{3.75}$$

Case 6.

$$A_0 = \frac{1+r}{2}\,, \quad A_1 = \pm B_1 = \pm\frac{-a\pm\Theta}{4\beta\delta}\,, \quad R = -\frac{(-1+r)^2(a^2+4D\beta\delta\pm a\Theta)}{8D^2}\,,$$
$$\lambda = -\frac{2D(1+r)\beta\delta}{a\pm\Theta}\,, \quad m = D(\delta-1)\,, \tag{3.76}$$

where $\Delta = \sqrt{a^2+4\beta(D+m+D\delta)}$ and $\Theta = \sqrt{a^2+8D\beta\delta}$ (For the rest of this paper, Δ and Θ denote these two formulae). From Eqs. (3.71) to (3.76), it follows that:

Case 1.

$$u_{11} = \left\{\frac{1}{2}\pm\frac{1}{2}\tanh\left[\pm\frac{\beta\delta}{a\mp\Delta}(x-\lambda t)\right]\right\}^{1/\delta}\,, \tag{3.77}$$

$$u_{12} = \left\{\frac{1}{2}\pm\frac{1}{2}\coth\left[\pm\frac{\beta\delta}{a\mp\Delta}(x-\lambda t)\right]\right\}^{1/\delta}\,, \tag{3.78}$$

where $\lambda = (-2(D+m)\beta + r[a^2+2\beta(D+m+D\delta)] \pm ar\Delta)/a\pm\Delta$.

Case 2.

$$u_{21} = \left\{\frac{r}{2}\pm\frac{r}{2}\tanh\left[\pm\frac{r\beta\delta}{a\pm\Delta}(x-\lambda t)\right]\right\}^{1/\delta}\,, \tag{3.79}$$

$$u_{22} = \left\{\frac{r}{2}\pm\frac{r}{2}\coth\left[\pm\frac{r\beta\delta}{a\pm\Delta}(x-\lambda t)\right]\right\}^{1/\delta}\,, \tag{3.80}$$

where $\lambda = (-2(D+m)(-1+r)\beta + 2D\beta\delta + a(a\pm\Delta))/a\pm\Delta$.

Case 3.

$$u_{31} = \left\{\frac{1+r}{2}\pm\frac{1+r}{2}\tanh\left[\pm\frac{(-1+r)\beta\delta}{a\pm\Theta}\left(x+\frac{2D(1+r)\beta\delta}{-a\pm\Theta}\,t\right)\right]\right\}^{1/\delta}\,, \tag{3.81}$$

$$u_{32} = \left\{\frac{1+r}{2}\pm\frac{1+r}{2}\coth\left[\pm\frac{(-1+r)\beta\delta}{a\pm\Theta}\left(x+\frac{2D(1+r)\beta\delta}{-a\pm\Theta}\,t\right)\right]\right\}^{1/\delta}\,, \tag{3.82}$$

where $\lambda = -(2D(1+r)\beta\delta)/a\pm\Theta$ and $m = D(\delta-1)$.

Case 4.

$$u_{41} = \left\{\frac{1}{2}\left[1\pm\tanh[\sqrt{-R}(x-\lambda t)]\pm i\,\mathrm{sech}[\sqrt{-R}(x-\lambda t)]\right]\right\}^{1/\delta}\,, \tag{3.83}$$

$$u_{42} = \left\{\frac{1}{2}\left[1\pm\coth[\sqrt{-R}(x-\lambda t)]\pm \mathrm{csch}[\sqrt{-R}(x-\lambda t)]\right]\right\}^{1/\delta}\,, \tag{3.84}$$

where R and λ are determined by Eq. (3.74).

Case 5.

$$u_{51} = \left\{\frac{r}{2}\,[1 \pm \tanh[\sqrt{-R}(x-\lambda t)] \pm i\,\mathrm{sech}[\sqrt{-R}(x-\lambda t)]]\right\}^{1/\delta}, \qquad (3.85)$$

$$u_{52} = \left\{\frac{r}{2}\,[1 \pm \coth[\sqrt{-R}(x-\lambda t)] \pm \mathrm{csch}[\sqrt{-R}(x-\lambda t)]]\right\}^{1/\delta}, \qquad (3.86)$$

where R and λ are determined by Eq. (3.75).

Case 6.

$$u_{61} = \left\{\frac{1+r}{2} \pm \frac{r-1}{2}\tanh[\sqrt{-R}(x-\lambda t)] \pm i\frac{r-1}{2}\,\mathrm{sech}[\sqrt{-R}(x-\lambda t)]\right\}^{1/\delta}, \qquad (3.87)$$

$$u_{62} = \left\{\frac{1+r}{2} \pm \frac{r-1}{2}\coth[\sqrt{-R}(x-\lambda t)] \pm \frac{r-1}{2}\,\mathrm{csch}[\sqrt{-R}(x-\lambda t)]\right\}^{1/\delta}, \qquad (3.88)$$

where $R = -((-1+r)^2(a^2 + 4D\beta\delta \pm a\Theta))/8D^2$, $\lambda = -(2D(1+r)\beta\delta)/a \pm \Theta$, and $m = D(\delta - 1)$.

Remark 3. The solutions, Eqs. (3.77) to (3.80), cover the solutions obtained in Refs. 3 and 4. But, to our knowledge, the other solutions obtained here were not found earlier.

4. Conclusions

In this paper, we improve the extended tanh method by introducing a more general ansatz, Eq. (2.4), and two transformation Eqs. (2.6) to (2.7). Using this improved method, we drive many traveling wave solutions for some NEEs. The method can be used on many other nonlinear equations or coupled ones. In addition, it can be computerized, which allows one to perform complicated and tedious algebraic calculations on a computer.

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References

1. M. J. Ablowitz and P. A. Clarkson, *Soliton, Nonlinear Evolution Equations and Inverse Scatting* (Cambridge University Press, New York, 1991).

2. C. H. Gu *et al.*, *Soliton Theory and Its Application* (Zhejiang Science and Technology Press, Zhejiang, 1990).
3. X. Feng, *Int. J. Theor. Phys.* **39**, 207 (2000).
4. B. L. Lu, B. Z. Liu, Z. L. Pang, and X. F. Jiang, *Phys. Lett. A* **113**, 175 (1993).
5. X. Y. Wang, *Phys. Lett. A* **131**, 277 (1988).
6. X. Y. Wang, Z. S. Chu, and Y. K. Lu, *J. Phys. A* **23**, 271 (1990).
7. Z. J. Yang, *J. Phys. A* **27**, 2837 (1994).
8. J. F. Zhang, *Int. J. Theor. Phys.* **38**, 1829 (1999).
9. M. L. Wang, *Phys. Lett. A* **213**, 279 (1996).
10. M. L. Wang, Y. B. Zhou, and Z. B. Li, *Phys. Lett. A* **216**, 67 (1996).
11. E. G. Fan and H. Q. Zhang, *Phys. Lett. A* **245**, 389 (1998).
12. E. G. Fan and H. Q. Zhang, *Phys. Lett. A* **246**, 403 (1998).
13. Y. T. Gao and B. Tian, *Internat. J. Modern Phys. C* **12**, 1431 (2001).
14. B. Tian and Y. T. Gao, *Internat. J. Modern Phys. C* **12**, 361 (2001).
15. E. Fan, *Phys. Lett. A* **277**, 212 (2000).
16. E. Fan, *Z. Naturforsch.* **56a**, 312 (2001).
17. E. Fan, J. Zhang, and B. Y. C. Hon, *Phys. Lett. A* **291**, 376 (2001).
18. Z. Y. Yan, *Phys. Lett. A* **292**, 100 (2001).