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GENERALIZED EXTENDED TANH-FUNCTION METHOD TO CONSTRUCT NEW EXPLICIT EXACT SOLUTIONS FOR THE APPROXIMATE EQUATIONS FOR LONG WATER WAVES

YONG CHEN

Department of Applied Mathematics, Dalian University of Technology
Dalian 116024, P. R. China
chenyong@dlut.edu.cn

ZHENG YU

Mathematics Department of East China Normal University
Australian National University

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In this paper, a generalized extended tanh-function method for constructing exact solutions of nonlinear evolution equations (NEEs) is presented and implemented in a computer algebraic system. Applying the generalized method, with the aid of *Maple*, we consider the system of the approximate equations for long water waves and find the new more general explicit exact solutions. The properties of the new solitary wave solutions are shown by some figures.

Keywords: Nonlinear evolution equations; exact solutions; symbolic computation; Riccati equation.

1. Introduction

In recent years, the nonlinear evolution equations (NEEs) and solitons are a major subject in many fields of physics and direct searching for explicit exact solutions of NEEs has become more and more attractive, partly due to many effectively straightforward methods to construct explicit exact solutions of NEEs.[1−16,20−21,24−26] Various versions of tanh method have recently been proposed. In Refs. 3–6, Hereman,[3] Parks and Duffy[4−6] presented tanh method to find the tanh-styled solitary-wave solutions. Recently, based on the well-known Riccati equation, Fan[12] has proposed an extended tanh-function method. More recently, Fan[13,14] and Yan[15] further developed this idea and made it much more lucid and straightforward for a class of NEEs. Most recently, Elwakil[16] modified the extended tanh-function method and obtained some new exact solution. On the other hand, it is necessary to point out that Gao and Tian proposed a generalized hyperbolic-function method and its algorithm to find some nontraveling soliton-like solutions for certain NEEs.[8−11] In this paper, we proceed along the same direction, but consider only the traveling wave solutions for NEEs. Based on the above works[3–6,12–16,24–26] by introducing a new more general ansätz than the ones in above methods,[3–6,12–16,24–26] a generalized extended tanh-function method for constructing exact solutions of (NEEs) is presented and implemented in a computer algebraic system.

Applying the generalized extended tanh-function method, with the help of *Maple*, we consider the following system of the approximate equations for long water waves,

$$w_t - ww_x - H_x + \frac{1}{2}w_{xx} = 0\,, \tag{1}$$

$$H_t - (wH)_x - \frac{1}{2}H_{xx} = 0\,, \tag{2}$$

which was found by Whitham[17] and Broer.[18] The symmetries and conservation laws of Eqs. (1) and (2) were discussed by Kuperschmidt.[19] By using the homogeneous balance method, Zhang[20] obtained multiple soliton solutions of the system. Yan[21] used sine-cosine method to obtain three families of soliton solutions. As a result, we can successfully recover the previously known solitary wave solutions that have been found by the extended tanh-function method and other more sophisticated method. More importantly, for the system of the approximate equation for long water waves, we also obtain other new and more general solutions at the same time. The results include kink-profile solitary-wave solutions, periodic wave solutions, singular solutions and new formal solutions. The properties of new soliton solutions for the system are shown by some figures.

2. Summary of Method

Now, we simply describe the generalized extended tanh-function method. Consider a given system of NEEs, say, in two variables, x, t

$$H(u, v, u_t, v_t, u_x, v_x, u_{xt}, v_{xt}, \ldots) = 0\,, \tag{3}$$

$$G(u, v, u_t, v_t, u_x, v_x, u_{xt}, v_{xt}, \ldots) = 0\,. \tag{4}$$

We first consider the following formal traveling wave solutions $u(x,t) = \phi(\xi)$, $v(x,t) = \theta(\xi)$, $\xi = x - \lambda t$, where λ is a constant to be determined later. Then Eqs. (3) and (4) become a system of nonlinear ordinary differential equations (ODEs)

$$H(\phi, \theta, \phi', \theta', \phi'', \theta'', \ldots) = 0\,, \tag{5}$$

$$G(\phi, \theta, \phi', \theta', \phi'', \theta'', \ldots) = 0\,, \tag{6}$$

where "$'$" denotes $d/d\xi$. In order to seek the traveling wave solutions of Eqs. (3) and (4), we introduce the following new ansätz

$$\phi(\xi) = a_{10} + \sum_{j=1}^{m_1} \left\{ a_{1j}\omega^j + b_{1j}\omega^{-j} + c_{1j}\omega^{j-1}\sqrt{R+\omega^2} + d_{1j}\frac{\sqrt{R+\omega^2}}{\omega^j} \right\}\,, \tag{7}$$

$$\theta(\xi) = a_{20} + \sum_{j=1}^{m_2} \left\{ a_{2j}\omega^j + b_{2j}\omega^{-j} + c_{2j}\omega^{j-1}\sqrt{R+\omega^2} + d_{2j}\frac{\sqrt{R+\omega^2}}{\omega^j} \right\}, \quad (8)$$

where $a_{i0},\ a_{ij},\ b_{ij},\ c_{ij},\ d_{ij}\,(j = 1, 2, \ldots, m_i;\ i = 1, 2)$ and R are constants to be determined later. The value of m_i in Eqs. (7) and (8) can be determined by balancing the highest-order derivative term with the nonlinear term[9,10] in Eqs. (5) and (6). The new variable $\omega = \omega(\xi)$ satisfying

$$\omega' = \frac{d\omega}{d\xi} = R + \omega^2 \,. \quad (9)$$

It is easy to see that the ansätz, Eqs. (7) and (8), are more general than the ansätz in the typical tanh method,[3–6] the extended tanh-function method[12–15] and the modified extended tanh-function method.[16] When $b_{ij} = 0$, $c_{ij} = 0$, $d_{ij} = 0\,(j = 1, 2, \ldots, m_i;\ i = 1, 2)$ in Eqs. (7) and (8), Eqs. (7) and (8) become the transformation proposed by Fan.[12–14] But as $b_{ij} = 0$, $d_{ij} = 0\,(j = 1, 2, \ldots, m_i;\ i = 1, 2)$ in Eqs. (7) and (8), Eqs. (7) and (8) become the transformation proposed by Yan.[15] But as $c_{ij} = 0$, $d_{ij} = 0\,(j = 1, 2, \ldots, m_i;\ i = 1, 2)$ in Eqs. (7) and (8), Eqs. (7) and (8) become the transformation proposed by Elwakil.[16] So we can find many new exact solutions of systems (3) and (4) by the generalized extended tanh-function method.

There exist the following steps to be considered further:

Step 1. Determine the m_1 and m_2 of Eqs. (7) and (8) by respectively balancing the highest order partial differential terms and the nonlinear terms in Eqs. (5) and (6).

Step 2. Substituting Eqs. (7) and (8) into Eqs. (5) and (6), the corresponding ODEs, and then let all coefficients of $\omega^p(\sqrt{R+\omega^2})^q\ (q = 0, 1;\ p = 0, 1, 2, \ldots)$ to be zero to get an over-determined system of nonlinear algebraic equations with respect to $a_{i0},\ a_{ij},\ b_{ij},\ c_{ij},\ d_{ij},\ R,\ \lambda\,(j = 1, 2, \ldots, m_i;\ i = 1, 2)$.

Step 3. With the aid of *Maple*, we apply Wu-elimination method[22,23] to solve the above-mentioned over-determined system of nonlinear algebraic equations to yield the values of $a_{i0},\ a_{ij},\ b_{ij},\ c_{ij},\ d_{ij},\ R,\ \lambda\,(j = 1, 2, \ldots, m_i;\ i = 1, 2)$.

Step 4. It is well-known that the Riccati equation, Eq. (9), has the following general solutions

(i) If $R < 0$,

$$\omega(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi)\,, \quad (10)$$

$$\omega(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi)\,. \quad (11)$$

(ii) If $R = 0$,

$$\omega(\xi) = -\frac{1}{\xi}\,. \quad (12)$$

(iii) If $R > 0$,

$$\omega(\xi) = \sqrt{R}\tan(\sqrt{R}\xi)\,, \tag{13}$$

$$\omega(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi)\,. \tag{14}$$

Then combined the values of a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}, R, $\lambda\,(j = 1, 2, \ldots, m_i;\ i = 1, 2)$ with Eqs. (7), (8) and (10)–(14), more traveling wave solutions of Eqs. (3) and (4) can be obtained.

In what follows we would like to apply the generalized method to the approximate equations for long water waves to illustrate our algorithm, which is more powerful than the typical tanh method, the extended tanh-function method and the modified extended tanh-function method.

Note: Since tan- and cot-type solutions appear in pairs with tanh- and coth-type solutions, respectively, we omit them in the rest of this paper. In addition, some rational solutions are also omitted.

3. Application

Example 1. Consider the following approximate equations for long water waves[17–21]

$$w_t - ww_x - H_x + \frac{1}{2}w_{xx} = 0\,, \tag{15}$$

$$H_t - (wH)_x - \frac{1}{2}H_{xx} = 0\,. \tag{16}$$

First, we let

$$w(x,t) = \phi(\xi)\,, \quad H(x,t) = \theta(\xi)\,, \quad \xi = x - \lambda t\,, \tag{17}$$

where λ is constant to be determined later. Then Eqs. (15) and (16) reduce to a system of nonlinear ordinary differential equations

$$-\lambda\phi - \frac{1}{2}\phi^2 - \theta + \frac{1}{2}\phi' = 0\,, \tag{18}$$

$$\lambda\theta + \phi\theta + \frac{1}{2}\theta' = 0\,. \tag{19}$$

According to description in Sec. 2, by balancing the highest-order derivative term with the nonlinear term in Eqs. (18) and (19), we support that Eqs. (18) and (19) have the following formal solutions:

$$\phi = a_{10} + a_{11}\omega + \frac{b_{11}}{\omega} + c_{11}\sqrt{R+\omega^2} + \frac{d_{11}\sqrt{R+\omega^2}}{\omega}\,, \tag{20}$$

$$\theta = a_{20} + a_{21}\omega + \frac{b_{21}}{\omega} + c_{21}\sqrt{R+\omega^2} + \frac{d_{21}\sqrt{R+\omega^2}}{\omega} + a_{22}\omega^2 + \frac{b_{22}}{\omega^2} + c_{22}\omega\sqrt{R+\omega^2} + \frac{d_{22}\sqrt{R+\omega^2}}{\omega^2}\,, \tag{21}$$

and $\omega = \omega(\xi)$ satisfying Eq. (9), where λ, R, a_{10}, a_{11}, b_{11}, c_{11}, d_{11}, a_{20}, a_{21}, b_{21}, c_{21}, d_{21}, a_{22}, b_{22}, c_{22}, d_{22}, are constants to be determined later. With the aid of *Maple*, substituting Eqs. (20) and (21) into Eqs. (18) ane (19) along with Eq. (9), and let the coefficients to be zero of $\omega^p(\sqrt{R+\omega^2})^q$ and ($q = 0, 1;\ p = 0, 1, 2, \ldots$) with the same power, we get the following over-determined algebraic equations system:

$$-b_{22} - \frac{1}{2}d_{11}^2 R - \frac{1}{2}b_{11}^2 - \frac{1}{2}b_{11}R = 0\,, \tag{22}$$

$$-\frac{1}{2}a_{11}^2 - a_{22} + \frac{1}{2}a_{11} - \frac{1}{2}c_{11}^2 = 0\,, \tag{23}$$

$$-\frac{1}{2}a_{10}^2 - \frac{1}{2}d_{11}^2 - a_{11}b_{11} - \frac{1}{2}b_{11} + \frac{1}{2}a_{11}R - a_{20} - \frac{1}{2}c_{11}^2 R - \lambda a_{10} = 0\,, \tag{24}$$

$$-c_{22} + \frac{1}{2}c_{11} - a_{11}c_{11} = 0\,, \tag{25}$$

$$-c_{11}d_{11} - a_{10}a_{11} - a_{21} - \lambda a_{11} = 0\,, \tag{26}$$

$$-a_{10}d_{11} - d_{21} - b_{11}c_{11} - \lambda d_{11} = 0\,, \tag{27}$$

$$-c_{21} - a_{10}c_{11} - a_{11}d_{11} - \lambda c_{11} = 0\,, \tag{28}$$

$$-b_{11}d_{11} - d_{22} - \frac{1}{2}d_{11}R = 0\,, \tag{29}$$

$$-a_{10}b_{11} - b_{21} - \lambda b_{11} - c_{11}d_{11}R = 0\,, \tag{30}$$

$$-b_{22}R + d_{11}d_{22}R + b_{11}b_{22} = 0\,, \tag{31}$$

$$\lambda a_{21} + a_{11}a_{20} + a_{22}R + c_{11}c_{22}R + c_{11}d_{21} + a_{10}a_{21} + d_{11}c_{21} + b_{11}a_{22} = 0\,, \tag{32}$$

$$c_{11}a_{22} + a_{11}c_{22} + c_{22} = 0\,, \tag{33}$$

$$a_{11}a_{21} + c_{11}c_{21} + \frac{1}{2}a_{21} + a_{10}a_{22} + d_{11}c_{22} + \lambda a_{22} = 0\,, \tag{34}$$

$$a_{22} + c_{11}c_{22} + a_{11}a_{22} = 0\,, \tag{35}$$

$$c_{11}a_{21} + \lambda c_{22} + a_{11}c_{21} + a_{10}c_{22} + \frac{1}{2}c_{21} + d_{11}a_{22} = 0\,, \tag{36}$$

$$-b_{22} + d_{11}c_{21}R + b_{11}a_{20} + \lambda b_{21} + d_{11}d_{22} + a_{11}b_{22} + a_{10}b_{21} + c_{11}d_{21}R = 0\,, \tag{37}$$

$$d_{11}a_{21}+c_{11}a_{20}+a_{10}c_{21}+\frac{1}{2}c_{22}R+a_{11}d_{21}+\lambda c_{21}+b_{11}c_{22}=0\,, \tag{38}$$

$$d_{11}c_{22}R+a_{11}b_{21}+\frac{1}{2}a_{21}R+\lambda a_{20}+d_{11}d_{21}+a_{10}a_{20}+b_{11}a_{21}+c_{11}d_{22} -\frac{1}{2}b_{21}+c_{11}c_{21}R=0\,, \tag{39}$$

$$c_{11}b_{22}+\lambda d_{22}+d_{11}b_{21}+b_{11}d_{21}+a_{10}d_{22}-\frac{1}{2}d_{21}R=0\,, \tag{40}$$

$$b_{11}c_{21}+a_{11}d_{22}-\frac{1}{2}d_{22}+d_{11}a_{20}+c_{11}b_{21}+a_{10}d_{21}+\lambda d_{21}=0\,, \tag{41}$$

$$-d_{22}R+b_{11}d_{22}+d_{11}b_{22}=0\,, \tag{42}$$

$$d_{11}d_{21}R+\lambda b_{22}-\frac{1}{2}b_{21}R+c_{11}d_{22}R+a_{10}b_{22}+b_{11}b_{21}=0\,. \tag{43}$$

With the aid of *Maple* and applying Wu-elimination method,[22,23] we have

Case 1. $a_{11}=-\frac{1}{2}$, $b_{11}=0$, $c_{11}=-\frac{1}{2}$, $d_{11}=0$, $a_{20}=2a_{10}^2$, $a_{21}=0$, $b_{21}=0$, $c_{21}=0$, $d_{21}=0$, $a_{22}=-\frac{1}{2}$, $b_{22}=0$, $c_{22}=-\frac{1}{2}$, $d_{22}=0$, $R=-4a_{10}^2$, $\lambda=-a_{10}$.

Case 2. $a_{10}=-2\sqrt{-b_{11}}$, $a_{11}=-1$, $c_{11}=0$, $d_{11}=0$, $a_{20}=-2b_{11}$, $a_{21}=0$, $b_{21}=0$, $c_{21}=0$, $d_{21}=0$, $a_{22}=-1$, $b_{22}=-b_{11}^2$, $c_{22}=0$, $d_{22}=0$, $R=b_{11}$, $\lambda=2\sqrt{-b_{11}}$.

Case 3. $a_{10}=\sqrt{-2b_{11}}$, $a_{11}=-1$, $c_{11}=0$, $d_{11}=0$, $a_{20}=0$, $a_{21}=0$, $b_{21}=0$, $c_{21}=0$, $d_{21}=0$, $a_{22}=-1$, $b_{22}=0$, $c_{22}=0$, $d_{22}=0$, $R=-b_{11}$, $\lambda=-\sqrt{-2b_{11}}$.

Case 4. $a_{10}=\sqrt{2}\sqrt{b_{11}}$, $a_{11}=1$, $c_{11}=0$, $d_{11}=0$, $a_{20}=0$, $a_{21}=0$, $b_{21}=0$, $c_{21}=0$, $d_{21}=0$, $a_{22}=0$, $b_{22}=-b_{11}^2$, $c_{22}=0$, $d_{22}=0$, $R=b_{11}$, $\lambda=-\sqrt{2}\sqrt{b_{11}}$.

Case 5. $a_{10}=-\sqrt{-b_{11}}$, $a_{11}=0$, $c_{11}=0$, $d_{11}=0$, $a_{20}=-b_{11}$, $a_{21}=0$, $b_{21}=0$, $c_{21}=0$, $d_{21}=0$, $a_{22}=0$, $b_{22}=-b_{11}^2$, $c_{22}=0$, $d_{22}=0$, $R=b_{11}$, $\lambda=\sqrt{-b_{11}}$.

Case 6. $a_{11}=0$, $b_{11}=0$, $c_{11}=1$, $d_{11}=0$, $a_{20}=-\frac{1}{2}a_{10}^2$, $a_{21}=0$, $b_{21}=0$, $c_{21}=0$, $d_{21}=0$, $a_{22}=-\frac{1}{2}$, $b_{22}=0$, $c_{22}=\frac{1}{2}$, $R=2a_{10}^2$, $d_{22}=0$, $\lambda=-a_{10}$.

Case 7. $a_{11}=-1$, $b_{11}=0$, $c_{11}=0$, $d_{11}=0$, $a_{20}=a_{10}^2$, $a_{21}=0$, $b_{21}=0$, $c_{21}=0$, $d_{21}=0$, $a_{22}=-1$, $b_{22}=0$, $c_{22}=0$, $d_{22}=0$, $R=-a_{10}^2$, $\lambda=-a_{10}$.

Case 8. $a_{10}=id_{11}$, $a_{11}=0$, $b_{11}=2d_{11}^2$, $c_{11}=0$, $a_{20}=-2d_{11}^2$, $a_{21}=0$, $b_{21}=0$, $c_{21}=0$, $d_{21}=0$, $a_{22}=0$, $b_{22}=-8d_{11}^4$, $c_{22}=0$, $d_{22}=-4d_{11}^3$, $R=4d_{11}^2$, $\lambda=-id_{11}$.

Case 9. $a_{10}=-\frac{1}{2}\sqrt{2}d_{11}$, $a_{11}=0$, $b_{11}=0$, $c_{11}=0$, $a_{20}=-\frac{1}{4}d_{11}^2$, $a_{21}=0$, $b_{21}=0$, $c_{21}=0$, $d_{21}=0$, $a_{22}=0$, $b_{22}=-\frac{1}{2}d_{11}^4$, $c_{22}=0$, $d_{22}=-\frac{1}{2}d_{11}^3$, $R=d_{11}^2$, $\lambda=\frac{1}{2}\sqrt{2}d_{11}$.

Then, combining Eqs. (20) and (21) along with Cases 1–9, we obtain the traveling wave solutions of approximate equations for long water waves as follows:

Case 1

$$w_{11} = a_{10} + \sqrt{a_{10}^2}\tanh\tilde{\xi} \mp i\sqrt{a_{10}^2}\,\mathrm{sech}\,\tilde{\xi}\,, \tag{44}$$

$$H_{11} = 2a_{10}^2 - 2a_{10}^2(\tanh\tilde{\xi})^2 \pm 2ia_{10}^2\tanh\tilde{\xi}\,\mathrm{sech}\,\tilde{\xi}\,, \tag{45}$$

$$w_{12} = a_{10} + \sqrt{a_{10}^2}\coth\tilde{\xi} \mp \sqrt{a_{10}^2}\,\mathrm{csch}\,\tilde{\xi}\,, \tag{46}$$

$$H_{12} = 2a_{10}^2 - 2a_{10}^2(\coth\tilde{\xi})^2 \pm 2a_{10}^2\coth\tilde{\xi}\,\mathrm{csch}\,\tilde{\xi}\,, \tag{47}$$

where $\tilde{\xi} = 2\sqrt{a_{10}^2}\,(x + a_{10}t)$.

Case 2

$$w_{21} = \mp 2\sqrt{-b_{11}} + \sqrt{-b_{11}}\tanh\tilde{\xi} - \frac{b_{11}\coth\tilde{\xi}}{\sqrt{-b_{11}}}\,, \tag{48}$$

$$H_{21} = -2b_{11} + b_{11}(\tanh\tilde{\xi})^2 + b_{11}(\coth\tilde{\xi})^2\,, \tag{49}$$

where $\tilde{\xi} = \sqrt{-b_{11}}(x \mp 2\sqrt{-b_{11}}t)$.

Case 3

$$w_{31} = \pm\sqrt{-2b_{11}} + \sqrt{b_{11}}\tanh\tilde{\xi} - \sqrt{b_{11}}\coth\tilde{\xi}\,, \tag{50}$$

$$H_{31} = -b_{11}(\tanh\tilde{\xi})^2\,, \tag{51}$$

$$w_{32} = \pm\sqrt{-2b_{11}} + \sqrt{b_{11}}\coth\tilde{\xi} - \sqrt{b_{11}}\tanh\tilde{\xi}\,, \tag{52}$$

$$H_{32} = -b_{11}(\coth\tilde{\xi})^2\,, \tag{53}$$

where $\tilde{\xi} = \sqrt{b_{11}}(x \pm \sqrt{-2b_{11}}t)$.

Case 4

$$w_{41} = \pm\sqrt{2}\sqrt{b_{11}} - \sqrt{-b_{11}}\tanh\tilde{\xi} - \frac{b_{11}\coth\tilde{\xi}}{\sqrt{-b_{11}}}\,, \tag{54}$$

$$H_{41} = b_{11}(\coth\tilde{\xi})^2\,, \tag{55}$$

$$w_{42} = \pm\sqrt{2}\sqrt{b_{11}} - \sqrt{-b_{11}}\coth\tilde{\xi} - \frac{b_{11}\tanh\tilde{\xi}}{\sqrt{-b_{11}}}\,, \tag{56}$$

$$H_{42} = b_{11}(\tanh\tilde{\xi})^2\,, \tag{57}$$

where $\tilde{\xi} = \sqrt{-b_{11}}(x \pm \sqrt{2}\sqrt{b_{11}}t)$.

Case 5

$$w_{51} = \mp\sqrt{-b_{11}} - \frac{b_{11}\coth\tilde{\xi}}{\sqrt{-b_{11}}}\,, \tag{58}$$

$$H_{51} = -b_{11} + b_{11}(\coth\tilde{\xi})^2\,, \tag{59}$$

$$w_{52} = \mp\sqrt{-b_{11}} - \frac{b_{11}\tanh\tilde{\xi}}{\sqrt{-b_{11}}}\,, \tag{60}$$

$$H_{52} = -b_{11} + b_{11}(\tanh\tilde{\xi})^2\,, \tag{61}$$

where $\tilde{\xi} = \sqrt{-b_{11}}(x \mp \sqrt{-b_{11}}t)$.

Case 6

$$w_{61} = a_{10} \pm i\sqrt{-2a_{10}^2}\,\mathrm{sech}\,\tilde{\xi}\,, \tag{62}$$

$$H_{61} = -\frac{1}{2}a_{10}^2 + a_{10}^2(\tanh\tilde{\xi})^2 \pm ia_{10}^2\tanh\tilde{\xi}\,\mathrm{sech}\,\tilde{\xi}\,, \tag{63}$$

$$w_{62} = a_{10} \pm \sqrt{-2a_{10}^2}\,\mathrm{csch}\,\tilde{\xi}\,, \tag{64}$$

$$H_{62} = -\frac{1}{2}a_{10}^2 + a_{10}^2(\coth\tilde{\xi})^2 \pm a_{10}^2\coth\tilde{\xi}\,\mathrm{csch}\,\tilde{\xi}\,, \tag{65}$$

where $\tilde{\xi} = \sqrt{-2a_{10}^2}(x + a_{10}t)$.

Case 7

$$w_{81} = a_{10} + \sqrt{a_{10}^2}\tanh\tilde{\xi}\,, \tag{66}$$

$$H_{81} = a_{10}^2 - a_{10}^2(\tanh\tilde{\xi})^2\,, \tag{67}$$

$$w_{82} = a_{10} + \sqrt{a_{10}^2}\coth\tilde{\xi}\,, \tag{68}$$

$$H_{82} = a_{10}^2 - a_{10}^2(\coth\tilde{\xi})^2\,, \tag{69}$$

where $\tilde{\xi} = \sqrt{a_{10}^2}(x + a_{10}t)$.

Case 8

$$w_{71} = \pm id_{11} - 2\frac{d_{11}^2\coth\tilde{\xi}}{\sqrt{-4d_{11}^2}} - id_{11}\mathrm{csch}\,\tilde{\xi}\,, \tag{70}$$

$$H_{71} = -2d_{11}^2 + 2d_{11}^2(\coth\tilde{\xi})^2 + 2id_{11}\sqrt{-d_{11}^2}\cosh\tilde{\xi}(\mathrm{csch}\,\tilde{\xi})^2\,, \tag{71}$$

$$w_{72} = \pm id_{11} - 2\frac{d_{11}^2\tanh\tilde{\xi}}{\sqrt{-4d_{11}^2}} - d_{11}\mathrm{sech}\,\tilde{\xi}\,, \tag{72}$$

$$H_{72} = -2d_{11}^2 + 2d_{11}^2(\tanh\tilde{\xi})^2 + 2d_{11}\sqrt{-d_{11}^2}\sinh\tilde{\xi}(\operatorname{sech}\tilde{\xi})^2\,, \tag{73}$$

where $\tilde{\xi} = 2\sqrt{-d_{11}^2}(x \pm id_{11}t)$.

Case 9

$$w_{91} = \mp\frac{1}{2}\sqrt{2}d_{11} - id_{11}\operatorname{csch}\tilde{\xi}\,, \tag{74}$$

$$H_{91} = -\frac{1}{4}d_{11}^2 + \frac{1}{2}d_{11}^2(\coth\tilde{\xi})^2 + \frac{1}{2}id_{11}\sqrt{-d_{11}^2}\cosh\tilde{\xi}(\operatorname{csch}\tilde{\xi})^2\,, \tag{75}$$

$$w_{92} = \mp\frac{1}{2}\sqrt{2}d_{11} - d_{11}\operatorname{sech}\tilde{\xi}\,, \tag{76}$$

$$H_{92} = -\frac{1}{4}d_{11}^2 + \frac{1}{2}d_{11}^2(\tanh\tilde{\xi})^2 + \frac{1}{2}d_{11}\sqrt{-d_{11}^2}\sinh\tilde{\xi}(\operatorname{sech}\tilde{\xi})^2\,, \tag{77}$$

where $\tilde{\xi} = \sqrt{-d_{11}^2}(x \mp \frac{1}{2}\sqrt{2}d_{11}t)$.

Remark. From the families of solutions obtained, it is not difficult to see that we not only successfully recover the previously known solitary wave solutions found by the known tanh-function method, but also obtain new formal solutions, such as the families of solutions corresponding to the Cases 8 and 9, which cannot be found by the known tanh-function methods.

As an illustrative sample, the properties of the new formal solutions: singular solitary wave solutions, namely w_{91}, H_{91}, and nonsingular solitary wave solutions, namely w_{92}, H_{92}, are shown in Figs. 1–4, respectively.

FIGURE 1a

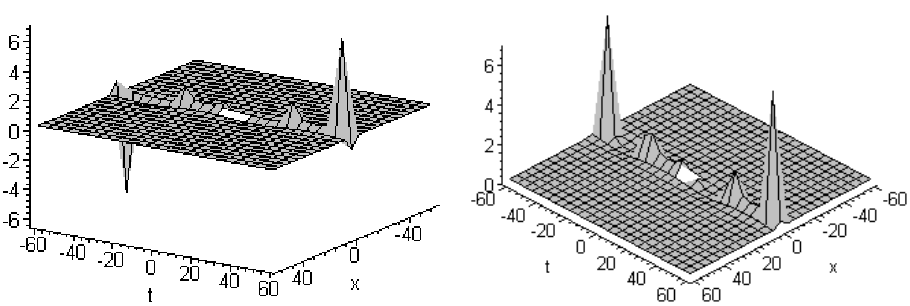

Fig. 1. The solution w_{91}, the real part, imaginary part and the modulus, where $d_{11} = 0.4i$.

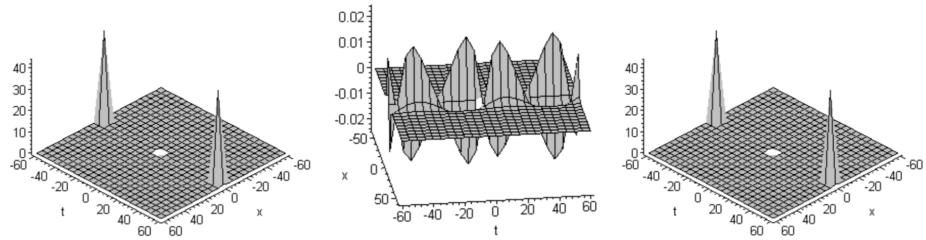

Fig. 2. The solution H_{91}, the real part, imaginary part and the modulus, where $d_{11} = 0.4i$.

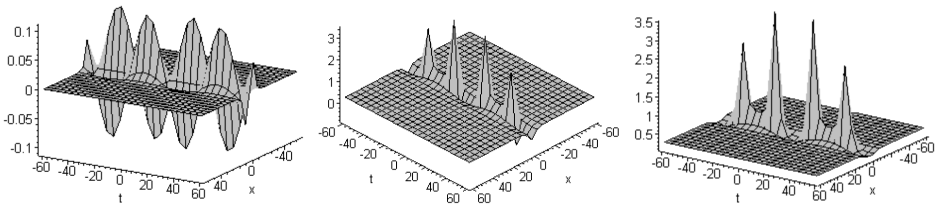

Fig. 3. The solution w_{92}, the real part, imaginary part and the modulus, where $d_{11} = 0.4i$.

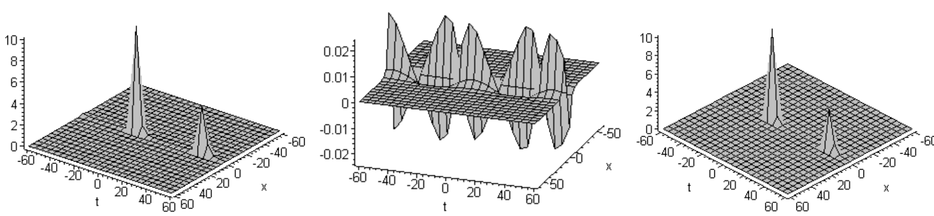

Fig. 4. The solution H_{92}, the real part, imaginary part and the modulus, where $d_{11} = 0.4i$.

4. Conclusion

In summary, based on computerized symbolic computation and by using generalized extended tanh-function method, we have obtained many new formal solitary wave solutions for the system of the approximate equations for long water waves, which include kink-profile solitary-wave solutions, periodic wave solutions, singular solutions and new formal solutions. We draw some figures that describe the characteristic features of every new formal solitary wave solution obtained by the generalized method. We can also see that some solutions develop singularity at a finite point, i.e., for any fixed $t = t_0$, there exist x_0 at which these solutions blow up. It appears that these singular solutions will model some physical phenomena. Although the method generalized here is more general than the known tanh method, it is very concise and basic, and it can be used to deal with other nonlinear evolution equations.

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References

1. M. J. Ablowitz and P. A. Clarkson, *Soliton, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, New York, 1991).
2. C. H. Gu *et al.*, *Soliton Theory ans Its Application* (Zhejiang Science and Technology Press, Zhejiang, 1990).
3. W. Hereman, *Comput. Phys. Commun.* **65**, 143 (1991).
4. E. J. Parkes and B. R. Duffy, *Comput. Phys. Commun.* **98**, 288 (1996).
5. B. R. Duffy and E. J. Parkes, *Phys. Lett. A* **214**, 171 (1996).
6. E. J. Parkes and B. R. Duffy, *Phys. Lett. A* **229**, 217 (1997).
7. Z. B. Li and S. Q. Zhang, *Acta Math. Sin.* **17**, 81 (1997).
8. Y. T. Gao and B. Tian, *Comput. Math. Appl.* **33**, 115 (1997).
9. Y. T. Gao and B. Tian, *Int. J. Mod. Phys. C* **12**, 1431 (2003).
10. B. Tian and Y. T. Gao, *Int. J. Mod. Phys. C* **12**, 361 (2001).
11. Y. T. Gao and B. Tian, *Comput. Phys. Commun.* **133**, 158 (2001).
12. E. G. Fan, *Phys. Lett. A* **277**, 212 (2000).
13. E. G. Fan, *Z. Naturforsch. A* **56**, 312 (2001).
14. E. G. Fan, J. Zhang, and B. Y. C. Hon, *Phys. Lett. A* **291**, 376 (2001).
15. Z. Y. Yan, *Phys. Lett. A* **292**, 100 (2001).
16. S. A. Elwakil, S. K. El-labany, M. A. Zahran, and R. Sabry, *Phys. Lett. A* **299**, 179 (2002).
17. G. B. Whitham, *Proc. R. Soc. A* **299**, 6 (1967).
18. L. T. F. Broer, *Appl. Sci. Res.* **31**, 377 (1957).
19. B. A. Kuperschmidt, *Commun. Math. Phys.* **99**, 51 (1985).
20. J. F. Zhang, *Acta Phys. Sin.* **47**, 1416 (1998), in Chinese.
21. Z. Y. Yan and H. Q. Zhang, *Acta Phys. Sin.* **48**, 1000 (1999), in Chinese.
22. W. Wu, *Kexue Tongbao* **31**, 1 (1986).
23. W. Wu, *Algorithms and Computation*, eds. D. Z. Du *et al.* (Springer, Berlin, 1994), p. 1.
24. Y. Chen, Z. Y. Yan, B. Li, and H. Q. Zhang, *Commun. Theor. Phys.* **38**, 261 (2002).
25. B. Li, Y. Chen, and H. Q. Zhang, *J. Phys. A: Math. Gen.* **35**, 8253 (2002).
26. B. Li, Y. Chen, and H. Q. Zhang, *Chaos, Soliton and Fractals* **15**, 647 (2003).