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ORDERED ANALYTIC REPRESENTATION OF PDES BY HAMILTONIAN CANONICAL SYSTEM

Zheng Yu Chen Yong

Abstract. Based on the method of symplectic geometry and variational calculation, the method for some PDEs to be ordered and analytically represented by Hamiltonian canonical system is discussed. Meanwhile some related necessary and sufficient conditions are obtained.

§1 Introduction

The study of Hamiltonian system is very important in mathematics, mechanics and physics, and Hamiltonian canonical system is one of the most simple and important systems. Meanwhile, the problem that which equations can be represented by this system is also very important. It has been found that there are many important equations which have their Hamiltonian structure, such as KdV, wave equation and so on[1~3]. But it's natural that among all the method for PDEs to be represented by any Hamiltonian system the ordinary one is to find if the discussed PDEs can be represented by Hamiltonian canonical system after some transformations of the original equations and their unknown functions, which we will call here as the ordered analytic representation of Hamiltonian canonical system. In this paper, some discussions of this case will be carried out on the theory of symplectic geometry and a necessary and sufficient condition is obtained. In this field [4] has discussed the similar problem for Lagrangian and Hamiltonian system.

§2 Basic Concept and Results

Consider a $2n$-dimensional differential manifold M. Let w^2 be a 2-form on M.

Definition 2.1. A 2-form w^2 is called to be a symplectic form on M, if w^2 is a nondegenerate

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and closed 2-form on M.

Here the nondegeneracy means that $w(m)$ is a nondegenerate bilinear functional of $T_mM \times T_mM$ for any $m \in M$, and the closed form means that $dw^2 = 0$. For any symplectic form w, we have the following canonical form.

Theorem 2.1 (**Darboux**). For any symplectic form w^2 in a neighborhood U of a point $m \in M$, one can choose a coordinate system $(p_1, \ldots, p_n, q_1, \ldots, q_n)^T$ of U such that w^2 has the standard form:

$$w^2 = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

For the proof see[5].

Now let´s consider the following PDEs: $Eu = 0$, that is

$$\sum_{j=1}^{2N_0} a_{ij}(u)\dot{u}_j + c_i[x, u] = 0, \quad i = 1, \ldots, 2N_0, \tag{2.1}$$

where $x \in \Omega \subset \mathbf{R}^m$, "$\cdot$" denotes the partial derivative with respect to some fixed variable x_{k_0} and $c_i[x, u]$ is a smooth function of x, u and u´s partial derivative with respect to x except $x_{k_0}(i = 1, \ldots, 2N_0)$.

Definition 2.2. PEDs (2.1) is said to be ordered and analytically represented by Hamiltonian canonical system, if there exist diffeomorphisms A and B : $\mathbf{R}^{2N_0} \to \mathbf{R}^{2N_0}$ such that Eq. (2.2) has the following form of Hamiltonian canonical system(*), where

$$A \circ E \circ B^{-1}\tilde{u} = 0, \tag{2.2}$$

$$J^{-1}\dot{u} - \delta\mathscr{H}/\delta u = 0, \tag{*}$$

where "$\cdot$" denotes the partial derivative with respect to t,

$$\mathscr{H} = \int_\Omega H[t, x, u]\, dt dx, \quad J^{-1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Specially, if A and B are nondegenerate linear maps, it is called the ordered linear analytic representation; if A and B are locally defined, it is called the locally one.

§ 3 The Main Theorem

Before discussing the condition of the representation above, let´s first introduce the following 2-form w^2 corresponding to PDEs(2.1).

Definition 3.1. The following 2-form w^2 is called the 2-form of the PDEs (2.1), if

$$w^2 = \sum_{i<j} a_{ij}(u)\, du_i \wedge du_j,$$

where $(u_1, \ldots, u_{2N_0})^T \in \mathbf{R}^{2N_0}$.

Then we have first the following result.

Theorem 3.1. For Eq. (2.1) with smooth coefficients $a_{ij}(1 \le i \le 2N_0, 1 \le j \le 2N_0)$, if and only if the following three conditions(a) (b) and (c) are satisfied, so that Equations(2.1) can be viewed as the Euler-Lagrange equations of the following functional $\mathscr{F}[u]$, where

$$\mathscr{F}[u] = \int_\Omega \left(\sum_{i=1}^{2N_0} p_i(u) q_i(u) - H[x, P, Q] \right) dx,$$

where $P = (p_1, \ldots, p_{N_0})$, $Q = (q_1, \ldots, q_{N_0})$ are diffeomorphisms from $\mathbf{R}^{2N_0}$ to itself, $H[x, P, Q]$ is a smooth function of x, P, Q and their partial derivative with respect to x except x_{k_0}. The three conditions are

(a) $a_{ij} = -a_{ji}$, a_{ij} is smooth, $1 \le i, j \le 2N_0$;

(b) The 2-form w^2 of Eq. (2.1) is a symplectic form;

(c) The operator C: $c[x, u] = (c_1[x, u], \ldots, c_{2N_0}[x, u])^T$ is a potential operator with respect to u.

Proof. (i) Suppose that Eq. (2.1) is the Euler-Lagrange equation of functional $\mathscr{F}$. Then by the calculation of $\delta\mathscr{F}$,

$$\delta\mathscr{F}[u] = \sum_{j=1}^{N_0} \int_\Omega \left(\sum_{i=1}^{2N_0} \left(\left(q_i - \frac{\delta\mathscr{H}}{\delta p_i} \right) \frac{\partial p_i}{\partial u_j} - \sum_{i=1}^{2N_0} \left(p_i + \frac{\delta\mathscr{H}}{\delta q_i} \right) \frac{\partial q_i}{\partial u_j} \right) \delta u_j dx$$

where $\mathscr{H} = \int_\Omega H\,dx$. From $\delta\mathscr{F}[u] = 0$, we get

$$\sum_{i=1}^{2N_0} \frac{\partial p_i}{\partial u_j} q_i - \frac{\partial q_i}{\partial u_j} p_i - \frac{\delta\mathscr{H}}{\delta p_i}\frac{\partial p_i}{\partial u_j} - \frac{\delta\mathscr{H}}{\delta q_i}\frac{\partial q_i}{\partial u_j} = 0, \quad j = 1, \ldots, 2N_0, \qquad (3.1)',$$

or equivalently

$$\sum_{k=1}^{2N_0} \left(\sum_{i=1}^{2N_0} \left(\frac{\partial p_i}{\partial u_j} \frac{\partial q_i}{\partial u_k} - \frac{\partial q_i}{\partial u_j} \frac{\partial p_i}{\partial u_k} \right) \right) u_k - \frac{\delta\mathscr{H}}{\delta u_j} = 0, \quad j = 1, \ldots, 2N_0. \qquad (3.1)$$

This is just the original Eq. (2.1). Comparing (2.1) with (3.1), we get

$$a_{jk}(u) = \sum_{i=1}^{N_0} \left(\frac{\partial p_i}{\partial u_j} \frac{\partial q_i}{\partial u_k} - \frac{\partial q_i}{\partial u_j} \frac{\partial p_i}{\partial u_k} \right), \quad j, k = 1, \ldots, 2N_0,$$

therefore $a_{jk} = -a_{kj}$, that is the condition (a). Meanwhile, the 2-form w^2 of Eq. (2.1) is

$$w^2 = \sum_{j<k} \sum_{i=1}^{N_0} \left(\frac{\partial p_i}{\partial u_j} \frac{\partial q_i}{\partial u_k} - \frac{\partial q_i}{\partial u_j} \frac{\partial p_i}{\partial u_k} \right) du_j \wedge du_k = \sum_{i=1}^{N_0} dp_i \wedge dq_i.$$

Therefore $dw^2 = 0$, that means w being a closed 2-form. Also, as mapping (P, Q) is a diffeomorphism, it becomes a new coordinate system instead of u, so that w^2 is nondegenerate, thus condition (b) is satisfied. Finally, as $C[u] = \frac{\delta\mathscr{H}}{\delta u}$, it´s easy to see that condition (c) is also satisfied by the result in [6].

(ii) Conversely, when the conditions (a)-(c) are satisfied, the 2-form

$$w^2 = \sum_{i<j} a_{ij}(u)\,du_i \wedge du_j$$

is a nondegenerate close 2-form, and according to Darboux´s Theorem 2.1, there exist coordinate functions $(p_1(u), \ldots, p_{N_0}(u), q_1(u), \ldots, q_{N_0}(u))^T$ which satisfies

(a) The Jacobi matrix of vector $(p_1, \ldots, p_{N_0}, q_1, \ldots, q_{N_0})^T$ with respect to u is nondegenerate;

(b) $w = \sum_{j<k} a_{jk}(u)\,\mathrm{d}u_j \wedge \mathrm{d}u_k = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$.

As

$$\mathrm{d}p_i \wedge \mathrm{d}q_i = \left(\sum_{j=1}^{2N_0} \frac{\partial p_i}{\partial u_j}\mathrm{d}u_j\right) \wedge \left(\sum_{k=1}^{2N_0} \frac{\partial q_i}{\partial u_k}\mathrm{d}u_k\right) = \sum_{j=1}^{2N_0}\sum_{k=1}^{2N_0}\left(\frac{\partial p_i}{\partial u_j}\frac{\partial q_i}{\partial u_k} - \frac{\partial q_i}{\partial u_j}\frac{\partial p_i}{\partial u_k}\right)\mathrm{d}u_j \wedge \mathrm{d}u_k.$$

we have

$$a_{jk}(u) = \sum_{i=1}^{N_0}\left(\frac{\partial p_i}{\partial u_j}\frac{\partial q_i}{\partial u_k} - \frac{\partial q_i}{\partial u_j}\frac{\partial p_i}{\partial u_k}\right), \quad 1 \le j < k \le 2N_0.$$

By condition (a)

$$a_{kj}(u) = -\sum_{i=1}^{N_0}\left(\frac{\partial p_i}{\partial u_j}\frac{\partial q_i}{\partial u_k} - \frac{\partial q_i}{\partial u_j}\frac{\partial p_i}{\partial u_k}\right), \quad 1 \le j < k \le 2N_0.$$

Meanwhile, as

$$q_i = \sum_{k=1}^{2N_0} \frac{\partial q_i}{\partial u_k}u_k, \quad p_i = \sum_{k=1}^{2N_0} \frac{\partial p_i}{\partial u_k}u_k,$$

so

$$\sum_{k=1}^{2N_0} a_{jk}u_k = \sum_{k=1}^{N_0} \frac{\partial p_i}{\partial u_k}q_i - \frac{\partial q_i}{\partial u_j}p_i, \quad \forall\, 1 \le j \le 2N_0. \tag{3.2}$$

Now according to theorem in [6] and condition (c), we know that operator C is the variation of some functional $\mathscr{H}[u]$, that is

$$C[u] = \delta\mathscr{H}/\delta u, \quad \mathscr{H}[u] = \int_\Omega H[x,u]\,\mathrm{d}x.$$

By the condition (a) and implicit function theorem, u is a smooth function of $(p_1, \ldots, p_{N_0}, q_1, \ldots, q_{N_0})^T$, so

$$c[x,u] = \frac{\delta\mathscr{H}}{\delta u} = \left(-\sum_{i=1}^{N_0}\left(\frac{\delta\widetilde{\mathscr{H}}}{\delta p_i}\frac{\partial p_i}{\partial u_1} + \frac{\delta\widetilde{\mathscr{H}}}{\delta q_i}\frac{\partial q_i}{\partial u_1}\right), \ldots, -\sum_{i=1}^{N_0}\frac{\delta\widetilde{\mathscr{H}}}{\delta p_i}\frac{\partial p_i}{\partial u_{2N_0}} + \frac{\delta\widetilde{\mathscr{H}}}{\delta q_i}\frac{\partial q_i}{\partial u_{2N_0}}\right)\right)^T,$$

where

$$\widetilde{\mathscr{H}}[p_1, \ldots, p_{N_0}, q_1, \ldots, q_{N_0}] = \int_\Omega \widetilde{H}[x, p_1, \ldots, p_{N_0}, q_1, \ldots, q_{N_0}]\,\mathrm{d}x,$$

$$\widetilde{H}[x, p_1, \ldots, p_{N_0}, q_1, \ldots, q_{N_0}] = -H[x, u(p_1, \ldots, p_{N_0}, q_1, \ldots, q_{N_0})].$$

Therefore

$$c_j[x,u] = \frac{\delta\mathscr{H}}{\delta u_j} = -\sum_{i=1}^{N_0}\frac{\delta\widetilde{\mathscr{H}}}{\delta p_i}\frac{\partial p_i}{\partial u_j} + \frac{\delta\widetilde{\mathscr{H}}}{\delta q_i}\frac{\partial q_i}{\partial u_j}, \quad j = 1, \ldots, 2N_0, \tag{3.3}$$

Combined with (3.2) and (3.3), Eq. (2.1) becomes the following equation

$$\sum_{k=1}^{2N_0}\sum_{i=1}^{N_0}\left(\frac{\partial p_i}{\partial u_j}\frac{\partial q_i}{\partial u_k}-\frac{\partial q_i}{\partial u_j}\frac{\partial p_i}{\partial u_k}\right)u_k-\sum_{i=1}^{N_0}\left(\frac{\delta\widetilde{\mathscr{H}}}{\delta p_i}\frac{\partial p_i}{\partial u_j}+\frac{\delta\widetilde{\mathscr{H}}}{\delta q_i}\frac{\partial q_i}{\partial u_j}\right)=0 \qquad (3.4)'$$

or equivalently

$$\sum_{i=1}^{N_0}\left(\frac{\partial p_i}{\partial u_j}q_i-\frac{\partial q_i}{\partial u_j}p_i\right)-\frac{\delta\widetilde{\mathscr{H}}}{\delta u_j}=0. \qquad (3.4)$$

Thus Eq. (2.1) is the Euler-Lagrange equation of the following functional $\mathscr{F}[u]$,

$$\mathscr{F}[u]=\int_\Omega\left(\sum_{i=1}^{N_0}p_i(u)q_i(u)-\tilde{H}[x,p_1(u),\ldots,p_{N_0}(u),q_1(u),\ldots,q_{N_0}(u)]\right)\mathrm{d}x,$$

and the proof is completed.

According to the above result, we can now consider when will Eq. (2.1) be ordered and analytically represented by Hamiltonian canonical system.

Corollary 3.1. When conditions (a)–(c) are satisfied, Eq. (2.1) can be locally ordered and analytically represented by Hamiltonian canonical system of (p_i, q_i) $(i=1,\ldots,N_0)$, where A is the inverese matrix of the Jacobi transposed matrix of $(p_1,\ldots,p_{N_0},q_1,\ldots,q_{N_0})$ with respect to u, B is the transformation of $u\to(p_1,\ldots,p_{N_0},q_1,\ldots,q_{N_0})$.

Proof. According to Theorem 3.1, Eq. (2.1) is the Euler-Lagrange equation of $\mathscr{F}[u]$. By the sufficient proof of this Theorem, we see that Eq. (2.1) is just the Eq. (3.1) or (3.1)′, or equivalently

$$\sum_{i=1}^{N_0}\left(q_i-\frac{\delta\mathscr{H}}{\delta p_i}\right)\frac{\partial p_i}{\partial u_j}-\left(p_i+\frac{\delta\mathscr{H}}{\delta q_i}\right)\frac{\partial q_i}{\partial u_j}=0.$$

Multiplying this equation with the inverse matrix of the Jacobi transposed matrix of $(p_1,\ldots,p_{N_0},q_1,\ldots,q_{N_0})^T$ with respect to u, this equation becomes

$$q_i-\frac{\delta\mathscr{H}}{\delta p_i}=0,\qquad -p_i-\frac{\delta\mathscr{H}}{\delta q_i}=0,$$

where $i=1,2,\ldots,N_0$. Therefore, after new transformation B form u to $(p_1,\ldots,p_{N_0},q_1,\ldots,q_{N_0})$, this equation is just the Hamiltonian canonical system (*) for (p_i,q_i), which completes the proof.

Meanwhile we can also have

Theorem 3.2. The necessary and sufficient condition for Eq. (2.1) to be locally ordered and analytically represented by Hamitonian canonical system about x_{k_0} is that there locally exist diffeomorphisms $A, B: \Omega\subset\mathbf{R}^{2N_0}\to\mathbf{R}^{2N_0}$ such that operator $A\circ(a_{ij})B^{-1}$ is linear, anti-symmetric and nondegenrate, and $c[\tilde{u}]=A(c_1[x,B^{-1}\tilde{u}],\ldots,c_{2N_0}[x,B^{-1}\tilde{u}])^T$ is the potential operator (or symmetric equivalently) with respect to $\tilde{u}$. Specially, if the Eq. (2.1) is linear with all the constant coefficients, A and B are linear operator: $\mathbf{R}^{2N_0}\to\mathbf{R}^{2N_0}$, then the representation is defined globally.

Proof. (1) Sufficiently, if there exist transformations A and B as above, it's easy to see that equation

$$A\circ(a_{ij})B^{-1}\tilde{u}+c[\tilde{u}]=0 \qquad (3.5)$$

satisfies the conditions (a)~(c) in Theorem 3.1, so according to Corollary 3.1, Eq. (3.5) can finally be locally ordered and analytically represented by Hamiltonian canonical system. Specially, as Eq. (2.1) is linear, by the proof of the Darboux theorem in [5] the result is easy to be obtained globally.

(ii) Conversely, by Definition 2.2, there are diffeomorphisms $\widetilde{A}$ and B such that Eq. (2.1) can be represented by Eq. (2.2) which is just one Hamiltonian system (*) for $\widetilde{u}=(p^1,\ldots,p^{N_0},q^1,\ldots,q^{N_0})$. Multiply this equations with the Jacobi transposed matrix of $(p^1,\ldots,p^{N_0},q^1,\ldots,q^{N_0})^T$ for u. According to the sufficient proof of Theorem 3.1 with the comparison of Eq. (3.1), Eq. (3.1)′, Eq. (3.4) and Eq. (3.4)′, it is easy to see that this new equation is just the Euler-Lagrange equation of functional $\mathscr{F}[u]$ as in Theorem 3.1. Therefore, the sufficient condition is satisfied for this new equation of u which means that the necessary condition is obviously proved when operator A is taken as the multiplication of $\widetilde{A}$ by the transformation corresponding to the Jacobi transposed matrix of $(p^1,\ldots,p^{N_0},q^1,\ldots,q^{N_0})^T$.

Finally, when transformations A and B are permutations, the criteria of the corresponding represention can be obtained which has been discussed in [7].

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1 Dept. of Math., East China Normal Univ., Shanghai 200062, China.

2 Dept. of Math., Dalian Univ. of Technology, Dalian, 116024, China.