AIP Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **55**, 053508 (2014); doi: 10.1063/1.4871554
View online: http://dx.doi.org/10.1063/1.4871554
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/55/5?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS 55, 053508 (2014)

Nonlocal symmetries of the Hirota-Satsuma coupled Korteweg-de Vries system and their applications: Exact interaction solutions and integrable hierarchy

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(Received 27 April 2013; accepted 4 April 2014; published online 12 May 2014)

The nonlocal symmetry is derived from the known Darboux transformation (DT) of the Hirota-Satsuma coupled Korteweg-de Vries (HS-cKdV) system, and infinitely many nonlocal symmetries are given by introducing the internal parameters. By extending the HS-cKdV system to an auxiliary system with five dependent variables, the prolongation is found to localize the so-called seed nonlocal symmetry related to the DT. By applying the general Lie point symmetry method to this enlarged system, we obtain two main results: a new type of finite symmetry transformation is derived, which is different from the initial DT and can generate new solutions from old ones; some novel exact interaction solutions among solitons and other complicated waves including periodic cnoidal waves and Painlevé waves are computed through similarity reductions. In addition, two kinds of new integrable models are proposed from the obtained nonlocal symmetry: the negative HS-cKdV hierarchy by introducing the internal parameters; the integrable models both in lower and higher dimensions by restricting the symmetry constraints. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4871554]

I. INTRODUCTION

Symmetry study is a powerful method in mathematical physics, especially in integrable systems and soliton theory.[1–3] For a given differential system, there are many important applications from one known symmetry, such as obtaining new solutions from old ones via finite transformation,[1–4] reducing dimensions of differential system by similarity reductions,[1–3,5–7] and finding new integrable models by symmetry constraints.[8–11] A lot of studies have been devoted to seeking the general Lie point symmetry by using the classical or nonclassical Lie group method, but not so many work has done in the existence and applications of nonlocal symmetries. The reason is that the construction of nonlocal symmetries itself is a challengeable work, and the finite symmetry transformations and similarity reductions cannot be directly calculated.[13–15] Recently, some effective techniques to find nonlocal symmetries have been proposed and developed in the literature. For a given partial differential equation (PDE) system, Bluman *et al.*[2] presented many methods to find nonlocal symmetries systematically. In particular, they pointed out that any conservation law of a given PDE system yields an equivalent nonlocally related system, and then nonlocal symmetries of the original PDE system can be obtained from Lie symmetries of such a nonlocally related system. In their recent work,[12] they showed that each point symmetry of a PDE system leads to a nonlocally related PDE system whose Lie symmetries could also yield nonlocal symmetries of the original PDE system. In integrable system, one can obtain nonlocal symmetry by recursion operators and their inverses,[15–19] the conformal invariant form (Schwartz form),[10,11] Darboux transformation (DT),[20–22] Bäcklund transformation (BT),[23] pseudopotentials,[4] and potential system.[2,3] Most recently, taking the well-known Korteweg-de Vries (KdV) equation as an example, Hu, Lou, and Chen[22,23] have obtained

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some interesting results from the nonlocal symmetry related to the known DT and BT, respectively. Through introducing potential[3] and pseudopotential-type symmetry,[24–26] the localization of nonlocal symmetry, which possesses the close prolongation, extends the application of nonlocal symmetries. Based on this method, the original differential equation needs to be expanded into close prolonged systems.[4,22,23] These prolonged systems lead to new finite symmetry transformations. For example, the finite symmetry transformations obtained from nonlocal symmetry related to DT and the initial DT are different but possess the same infinitesimal form.[22] Besides, several novel exact interaction solutions among solitons and other complicated waves including periodic cnoidal waves and Painlevé waves[22,23] can be derived, which have potential applications in many fields.[27–29]

On the other hand, it is an important topic in symmetry study to find new integrable models. For the general integrable model, the recursion operator[30–32] is a valid and effective method to produce the integrable hierarchy. Lou[16–19] has extended the negative integrable hierarchy by utilizing the inverse recursion operator for $(1+1)$-dimensional integrable models and introducing the internal parameters for $(2+1)$-dimensional integrable models.[10,11] Cao[33–35] has developed a systematic approach to find finite-dimensional integrable system by making use of the nonlinearization of Lax pair under certain constraints between potentials and eigenfunctions. With the help of this nonlinearization approach, a variety of new confocal involutive systems are constructed in the study of $(1+1)$-dimensional soliton equations. It is also worth mentioning that by restricting a symmetry constraint to the Lax pair of the soliton equation, one can not only obtain the lower dimensional integrable models from higher ones, but can also embed the lower ones into higher dimensional integrable models.[10,11,20,21]

In this paper, we focus on the nonlocal symmetries of the Hirota-Satsuma coupled KdV (HS-cKdV) system and their applications. First, the nonlocal symmetry is derived from the known DT of the HS-cKdV system, and infinitely many nonlocal symmetries are obtained by introducing the internal parameters. The prolongation of the nonlocal symmetry related to the DT is found by extending the HS-cKdV system to an auxiliary system with five dependent variables. Then, using the classical Lie symmetry method on the enlarged system, a new type of finite symmetry transformations, the second DT, is derived directly, which can generate new solutions from old ones. Several novel exact interaction solutions among solitons and other complicated waves including periodic cnoidal waves and Painlevé waves are computed through similarity reductions. Another contribution of this paper is to extend the HS-cKdV system to new integrable models from the nonlocal symmetry related to the DT in two aspects: (1) the negative HS-cKdV hierarchy by introducing the internal parameter; (2) the integrable models both in finite and infinite dimensions by restricting the nonlocal symmetry constraints.

Hirota and Satsuma[36] proposed the HS-cKdV system

$$u_t = \frac{1}{2}u_{xxx} + 3uu_x - 6vv_x, \tag{1}$$

$$v_t = -v_{xxx} - 3uv_x, \tag{2}$$

which describes interactions of two long waves with different dispersion relations. In a subsequent paper,[37] these authors showed that this coupled KdV system is the four-reduction of the celebrated Kadomtsev-Petviashvili (KP) hierarchy and its soliton solutions can be derived from ones of the KP equation. Meanwhile, Wilson[38] observed that the HS-cKdV system is just an example of many integrable systems arising from the Drinfeld-Sokolov theory.[39,40] Many significant properties of the HS-cKdV system have been clarified in the literature: bilinear form,[36,41] Lax pair,[42,47,48] Bäcklund transformations,[43] Darboux transformations,[44–46] Painlevé property,[47,48] infinitely many symmetries and conservation laws,[30] etc.

This paper is organized as follows. In Sec. II, the nonlocal symmetry is derived from the DT of the HS-cKdV system, and more nonlocal symmetries can be produced from one seed symmetry through introducing the inner parameters. In Sec. III, a prolonged system to localize the nonlocal symmetry is performed by extending the HS-cKdV system. The finite symmetry transformations and similar reductions of the prolonged system are presented, and several new exact solutions of the original system are derived. Section IV is devoted to finding the negative HS-cKdV hierarchy and giving the integrable models both in lower and higher dimensions. Conclusions and discussions are given in Sec. V.

II. NONLOCAL SYMMETRIES VIA DARBOUX TRANSFORMATION

It is known that DT is the most direct and yet elementary approach to construct exact solutions of soliton equation. Using this well-known method, one can obtain new solutions from old ones through simple iteration. In this section, we use the known DT to derive the nonlocal symmetries of the HS-cKdV system (1)–(2).

The Lax pair for the HS-cKdV system (1)–(2) reads[42,47,48]

$$\psi_{1xx} = -(u+v)\psi_1 - \lambda\psi_2, \tag{3}$$

$$\psi_{2xx} = -(u-v)\psi_2 + \lambda\psi_1, \tag{4}$$

$$\psi_{1t} = -\frac{1}{2}(u_x - 2v_x)\psi_1 + (u-2v)\psi_{1x} - 2\lambda\psi_{2x}, \tag{5}$$

$$\psi_{2t} = -\frac{1}{2}(u_x + 2v_x)\psi_2 + (u+2v)\psi_{2x} + 2\lambda\psi_{1x}, \tag{6}$$

where $\{u, v\}$ is a solution of Eqs. (1) and (2) and λ is a spectral parameter.

In Refs. 45 and 46, Hu and Liu have constructed the DT for the HS-cKdV system from two different aspects: singularity analysis and reduction of a binary DT.

Proposition 1. The DT of Eqs. (1) and (2) is expressed by[45,46]

$$\bar{u} = u + 2(\ln\theta)_{xx}, \quad \bar{v} = v + \frac{\psi_2\psi_{1x} - \psi_1\psi_{2x}}{\theta}, \tag{7}$$

with

$$\theta_x = \psi_1\psi_2, \quad \theta_t = 2\lambda(\psi_1^2 - \psi_2^2) - 2\psi_{1x}\psi_{2x} - u\psi_1\psi_2. \tag{8}$$

Proposition 2.

$$\sigma_1 = (\sigma_1^u, \sigma_1^v) \equiv \left(\left(\frac{\theta_2}{\theta_1}\right)_{xx}, \frac{W[\phi_2, \tilde{\phi}_1] - W[\phi_1, \tilde{\phi}_2]}{\theta_1} + \frac{\theta_2 W[\phi_1, \phi_2]}{\theta_1^2}\right) \tag{9}$$

is a symmetry of the HS-cKdV system (1)–(2) with $\{u, v\}$ replaced by $\{U, V\}$, where $\phi_1, \phi_2, \tilde{\phi}_1, \tilde{\phi}_2$ and θ_1, θ_2 satisfy the following equations:

$$\phi_{1xx} = -(\bar{U} + \bar{V})\phi_1, \quad \phi_{1t} = -\frac{1}{2}(\bar{U}_x - 2\bar{V}_x)\phi_1 + (\bar{U} - 2\bar{V})\phi_{1x}, \tag{10}$$

$$\phi_{2xx} = -(\bar{U} - \bar{V})\phi_2, \quad \phi_{2t} = -\frac{1}{2}(\bar{U}_x + 2\bar{V}_x)\phi_2 + (\bar{U} + 2\bar{V})\phi_{2x}, \tag{11}$$

$$\tilde{\phi}_{1xx} = -(\bar{U} + \bar{V})\tilde{\phi}_1 - \phi_2, \quad \tilde{\phi}_{1t} = -\frac{1}{2}(\bar{U}_x - 2\bar{V}_x)\tilde{\phi}_1 + (\bar{U} - 2\bar{V})\tilde{\phi}_{1x} - 2\phi_{2x}, \tag{12}$$

$$\tilde{\phi}_{2xx} = -(\bar{U} - \bar{V})\tilde{\phi}_2 + \phi_1, \quad \tilde{\phi}_{2t} = -\frac{1}{2}(\bar{U}_x + 2\bar{V}_x)\tilde{\phi}_2 + (\bar{U} + 2\bar{V})\tilde{\phi}_{2x} + 2\phi_{1x}, \tag{13}$$

and

$$\theta_1 = \int \phi_1\phi_2 dx + \alpha(t), \tag{14}$$

$$\theta_2 = \int \phi_1\tilde{\phi}_2 + \phi_2\tilde{\phi}_1 dx + \beta(t), \tag{15}$$

where $\bar{U} \equiv U - 2(\ln\theta_1)_{xx}$, $\bar{V} \equiv V - \frac{[\phi_2,\phi_1]}{\theta_1}$, $W[a, b] \equiv ab_x - ba_x$, and $\alpha(t)$, $\beta(t)$ are functions of t.

Proof. Let $\phi_1 \equiv \psi_1(x, t, 0)$, $\phi_2 \equiv \psi_2(x, t, 0)$, $\tilde{\phi}_1 \equiv \psi_{1\lambda}(x, t, 0)$, $\tilde{\phi}_2 \equiv \psi_{2\lambda}(x, t, 0)$, $\theta_1 \equiv \theta(x, t, 0)$, and $\theta_2 \equiv \theta_\lambda(x, t, 0)$. Furthermore, defining $U = u + 2(\ln\theta_1)_{xx}$ and $V = v + \frac{[\phi_2,\phi_1]}{\theta_1}$, we know that $\{U, V\}$ is a solution of the HS-cKdV system (1) and (2). By formally expanding $\bar{u}$ and $\bar{v}$ in powers of λ, we obtain

$$\bar{u} = U + \lambda\left[\left(2\frac{\partial^2}{\partial x^2}\ln\psi\right)_\lambda\Big|_{\lambda=0}\right] + O(\lambda^2) = U + \lambda\sigma_1^u + O(\lambda^2), \tag{16}$$

$$\bar{v} = V + \lambda\left[\left(\frac{W[\psi_2,\psi_1]}{\theta}\right)_\lambda\Big|_{\lambda=0}\right] + O(\lambda^2) = V + \lambda\sigma_1^v + O(\lambda^2). \tag{17}$$

Thus (σ_1, σ_2) is a symmetry of HS-cKdV system (1) and (2) with respect to U and V. Substituting $u = U - 2(\ln\theta_1)_{xx}$ and $v = V - \frac{[\phi_2,\phi_1]}{\theta_1}$ in (1) and (2), one can get (10)–(15). □

Furthermore, a direct calculation shows that if $\{\phi_1,\phi_2\}$ satisfies (10) and (11), then

$$\tilde{\phi}_1 = F_1\phi_1\int\frac{1}{\phi_1^2}dx - \phi_1\int\Big[\frac{1}{\phi_1^2}\int\phi_1\phi_2 dx\Big]dx + F_2\phi_1, \tag{18}$$

$$\tilde{\phi}_2 = G_1\phi_2\int\frac{1}{\phi_2^2}dx - \phi_2\int\Big[\frac{1}{\phi_2^2}\int\phi_2\phi_2 dx\Big]dx + G_2\phi_2, \tag{19}$$

are solutions of (12) and (13), where F_1, F_2, G_1, and G_2 are arbitrary constants.

On the other hand, from Eqs. (10) and (11), one can find that if $\{\phi_1, \phi_2\}$ is a solution of (10) and (11), then $\{\phi_1, \phi_2\}$ has the following relations to Lax pair (3)–(5) with $\lambda = 0$:

$$\phi_1 = -A_1\int\frac{1}{\psi_1^2}dx + A_2,\quad \phi_2 = \frac{\psi_2}{\psi_1}(A_3\int\frac{1}{\psi_2^2}dx + A_4), \tag{20}$$

with the constraint

$$A_1A_4\int\frac{1}{\psi_1^2}dx - A_2A_3\int\frac{1}{\psi_2^2}dx + A_1A_3\int\frac{1}{\psi_1^2}dx\int\frac{1}{\psi_2^2}dx - A_2A_4 = \frac{\psi_{1x}}{\psi_1\psi_2}, \tag{21}$$

where A_1, A_2, A_4, and A_4 are arbitrary constants.

Substituting (14)–(21) in the symmetry (9), we obtain a different nonlocal symmetry of the HS-cKdV system. In other words, the final expression of this nonlocal symmetry is exhibited by ψ_1, ψ_2 in Lax pair (3)–(6).

Proposition 3. $\sigma_2 = (\sigma_2^u, \sigma_2^v)$ is a nonlocal symmetry of HS-cKdV system (1)–(2), where

$$\sigma_2^u = 2A_1A_3\Lambda_1\int\frac{1}{\psi_1^2}dx\int\frac{1}{\psi_2^2}dx + 2A_1(A_4\Lambda_1 + A_3\frac{\psi_1}{\psi_2})\int\frac{1}{\psi_1^2}dx - 2A_3(A_2\Lambda_1 - A_1\frac{\psi_2}{\psi_1})\int\frac{1}{\psi_2^2}dx$$
$$-2A_2A_4\Lambda_1 + 2A_1A_4\frac{\psi_2}{\psi_1} - 2A_2A_3\frac{\psi_1}{\psi_2}, \tag{22}$$

$$\sigma_2^v = -A_1A_3\Lambda_2\int\frac{1}{\psi_1^2}dx\int\frac{1}{\psi_2^2}dx - A_1(A_4\Lambda_2 + A_3\frac{\psi_1}{\psi_2})\int\frac{1}{\psi_1^2}dx + A_3(A_2\Lambda_2 + A_1\frac{\psi_2}{\psi_1})\int\frac{1}{\psi_2^2}dx$$
$$+A_2A_4\Lambda_2 + A_1A_4\frac{\psi_2}{\psi_1} + A_2A_3\frac{\psi_1}{\psi_2}, \tag{23}$$

and $\Lambda_1 = (\psi_1\psi_2)_x$, $\Lambda_2 = W[\psi_1, \psi_2]$, and ψ_1, ψ_2 satisfy Eqs. (3)–(6) with $\lambda = 0$.

Moreover, it is easily verified that Lax pair (3)–(6) is invariant under the transformations:

$$\psi_1 \to \bar{\psi}_1 = \psi_1 \int \frac{1}{\psi_1^2} dx, \quad \psi_2 \to \bar{\psi}_2 = \psi_2 \int \frac{1}{\psi_2^2} dx. \tag{24}$$

With the aid of above transformations (24), the nonlocal symmetry σ_2 in Proposition 3 can be rewritten by the form: $-A_1A_3\Lambda(\bar{\psi}_1, \bar{\psi}_2) - A_1A_4\Lambda(\bar{\psi}_1, \psi_2) + A_2A_3\Lambda(\psi_1, \bar{\psi}_2) + A_2A_4\Lambda(\psi_1, \psi_2)$ with $\Lambda = (2\Lambda_1, \Lambda_2)$. Considering the inverse transformations of (24), we can obtain the following proposition.

Proposition 4. If ψ_1, ψ_2 satisfy Lax pair (3)–(6) with $\lambda = 0$, then

$$\sigma_3 = (\sigma_3^u, \sigma_3^v) \equiv \Big(-2(\psi_1\psi_2)_x, \psi_1\psi_{2x} - \psi_2\psi_{1x}\Big) \tag{25}$$

is a nonlocal symmetry of the HS-cKdV system (1)–(2).

Remark 1. Actually, if ψ_1, ψ_2 satisfy Lax pair (3)–(6) with the arbitrary spectral parameter λ, σ_3 given by (25) is still a symmetry of the HS-cKdV system. This fact is easily verified by direct calculation.

In Refs. 20 and 21, more symmetries were constructed via differentiating a known one with respect to inner parameters. Following this method, one has the following proposition.

Proposition 5. If a λ-dependent function $\sigma_0(\lambda)$ is a symmetry of the HS-cKdV system (1)–(2) with $\lambda \equiv \{\lambda_1, \lambda_2, ...\lambda_r\}$, then

$$\sigma_n \equiv \frac{d^{\{n\}}}{d\lambda^{\{n\}}}\sigma_0(\lambda) \equiv \frac{d^{\{n_1\}}}{d\lambda_1^{\{n_1\}}}\frac{d^{\{n_2\}}}{d\lambda_2^{\{n_2\}}}\cdots\frac{d^{\{n_r\}}}{d\lambda_r^{\{n_r\}}}\sigma_0(\lambda) \tag{26}$$

is also a symmetry of the same HS-cKdV system (1)–(2) for $\{n\} \equiv \{n_1, n_2, \ldots, n_r\}$.

Using Propositions 4 and 5, we can obtain a set of infinitely many new nonlocal symmetries. For instance, if we suppose $\{\psi_1, \psi_2\}$ and $\{\hat{\psi}_1, \hat{\psi}_2\}$ are two solutions of Lax pair (3)–(5), then

$$\sigma(\lambda_1, \lambda_2) \equiv \Big(\sigma^u(\lambda_1, \lambda_2), \sigma^v(\lambda_1, \lambda_2)\Big)$$

with

$$\sigma^u(\lambda_1, \lambda_2) = -2[(\lambda_1\psi_1 + \lambda_2\hat{\psi}_1)(\lambda_1\psi_2 + \lambda_2\hat{\psi}_2)]_x,$$
$$\sigma^v(\lambda_1, \lambda_2) = (\lambda_1\psi_1 + \lambda_2\hat{\psi}_1)(\lambda_1\psi_2 + \lambda_2\hat{\psi}_2)_x - (\lambda_1\psi_2 + \lambda_2\hat{\psi}_2)(\lambda_1\psi_1 + \lambda_2\hat{\psi}_1)_x,$$

and $\frac{\partial^{n_1+n_2}}{\partial\lambda_1^{n_1}\partial\lambda_2^{n_2}}\sigma(\lambda_1, \lambda_2)$ are also symmetries of the HS-cKdV system (1)–(2).

Remark 2. Similar to the KdV equation, the first derivative of the squared eigenfunction in Lax pair is well-known nonlocal symmetry.[11,14,20,49–50] This kind of nonlocal symmetry was also found in other integrable models such as the Kadomtsev-Petviashvili equation,[11,20] the Kaup-Kupershmidt equation,[51] the Boussinesq equation,[52] the Ablowitz-Kaup-Newell-Segur system,[52] and the Toda equation.[52] In the integrable HS-cKdV system, we have constructed similar nonlocal symmetry from its DT. Usually, by using the general DT method, one can obtain new solutions from old solutions. This implies that the DTs of integrable equations reflect some invariant properties of the equations. Thus, based on the invariant properties of differential equations exhibited by DTs, nonlocal symmetries related to Lax pair have been deduced.

III. LOCALIZATION OF NONLOCAL SYMMETRY AND EXACT SOLUTIONS

A. Localization of nonlocal symmetry from DT

First, we rewrite Eq. (25) as

$$\sigma^u = -2\psi_1\psi_{2x} - 2\psi_2\psi_{1x}, \quad \sigma^v = \psi_1\psi_{2x} - \psi_2\psi_{1x}, \tag{27}$$

which contains derivative terms ψ_{1x} and ψ_{2x}. By introducing new dependent variables $\varphi_1 \equiv \varphi_1(x, t)$ and $\varphi_2 \equiv \varphi_2(x, t)$ with

$$\varphi_1 = \psi_{1x}, \quad \varphi_2 = \psi_{2x}, \tag{28}$$

the above symmetry (27) is converted into

$$\sigma^u = -2\psi_1\varphi_2 - 2\psi_2\varphi_1, \quad \sigma^v = \psi_1\varphi_2 - \psi_2\varphi_1. \tag{29}$$

Then, we list the linearized equations of Eqs. (3) and (4) and (28) with $\lambda = 0$ as follows:

$$\begin{aligned} &\sigma^{\psi_1}_{xx} + (u+v)\sigma^{\psi_1} + (\sigma^u + \sigma^v)\psi_1 = 0, \\ &\sigma^{\psi_2}_{xx} + (u-v)\sigma^{\psi_2} + (\sigma^u - \sigma^v)\psi_2 = 0, \\ &\sigma^{\psi_1}_{x} - \sigma^{\varphi_1} = 0, \quad \sigma^{\psi_2}_{x} - \sigma^{\varphi_2} = 0, \end{aligned} \tag{30}$$

where σ^u and σ^v are given by (29), and σ^{ψ_1}, σ^{ψ_2}, σ^{φ_1}, and σ^{φ_2} denote the symmetries of ψ_1, ψ_2, φ_1, and φ_2, respectively.

Referring to Eqs. (3) and (4), one can easily deduce that solution of (30) has the form

$$\sigma^{\psi_1} = p\psi_1, \quad \sigma^{\psi_2} = p\psi_2, \tag{31}$$

$$\sigma^{\varphi_1} = \psi_1^2\psi_2 + p\varphi_1, \quad \sigma^{\varphi_2} = \psi_1\psi_2^2 + p\varphi_2, \tag{32}$$

where $p \equiv p(x, t)$ is a new potential variable. This potential variable p makes the prolonged system close completely, and it satisfies identically the compatibility conditions:

$$p_x = \psi_1\psi_2, \quad p_t = -u\psi_1\psi_2 - 2\varphi_1\varphi_2. \tag{33}$$

Furthermore, the linearized equation of its symmetry σ^p reads

$$\sigma^p_x = \psi_1\sigma^{\psi_2} + \psi_2\sigma^{\psi_1}, \tag{34}$$

and a straightforward calculation shows that σ^p has the simple form

$$\sigma_p = p^2. \tag{35}$$

Finally, the prolongation for nonlocal symmetry (27) is successfully localized by introducing variables ψ_1, ψ_2, φ_1, φ_2, and p with the vector form

$$\begin{aligned} V = &-2(\psi_1\varphi_2 + \psi_2\varphi_1)\frac{\partial}{\partial u} + (\psi_1\varphi_2 - \psi_2\varphi_1)\frac{\partial}{\partial v} + p\psi_1\frac{\partial}{\partial \psi_1} + p\psi_2\frac{\partial}{\partial \psi_2} + (\psi_1^2\psi_2 + p\varphi_1)\frac{\partial}{\partial \varphi_1} \\ &+(\psi_1\psi_2^2 + p\varphi_2)\frac{\partial}{\partial \varphi_2} + p^2\frac{\partial}{\partial p}. \end{aligned} \tag{36}$$

Here, it is worth mentioning that the differential equation that the introduced variable p needs to be satisfied is nothing but the Schwartz form of the HS-cKdV system[47,48]

$$3\mathcal{H}_x^3 - 6\mathcal{H}\mathcal{H}_x\mathcal{H}_{xx} + 16\mathcal{H}^3\mathcal{S}_x + 4\mathcal{H}^2\mathcal{H}_{xxx} + 4\mathcal{H}^2\Big(2\mathcal{C}_t - (\mathcal{C}^2)_x - 4\mathcal{C}\mathcal{S}_x - \mathcal{C}_{xxx}\Big) = 0, \tag{37}$$

where $\mathcal{H} = \mathcal{S} + 2\mathcal{C}$, $\mathcal{S} = \frac{p_{xxx}}{p_x} - \frac{3p_{xx}^2}{p_x^2}$, and $\mathcal{C} = -\frac{p_t}{p_x}$ are all invariant under Möbius (conformal) transformation. The reason is that the finite Möbius transformation

$$p \to \frac{a+bp}{c+dp} \ \ (ad \neq bc)$$

possess its infinitesimal transformation $p \to p + \epsilon p^2$ in special case $a = 0$, $b = c = 1$, and $d = -\epsilon$. Usually, the corresponding Schwartz form of a given differential equation is derived by utilizing singularities analysis method. The above result suggests that it can also be obtained through localization of the nonlocal symmetry from DT. This method may provide a different way to find Schwartz form of some integrable models, especially discrete integrable models which have some known DTs.

B. Finite symmetry transformation

For the prolonged system from symmetry (27), it is natural to seek the corresponding finite transformation of the local symmetries. According to Lie's first theorem, we need to solve the following initial value problem:

$$\begin{aligned}
&\frac{d\hat{u}(\epsilon)}{d\epsilon} = -2\hat{\psi}_1\hat{\varphi}_2 - 2\hat{\psi}_2\hat{\varphi}_1, \quad \frac{d\hat{v}(\epsilon)}{d\epsilon} = \hat{\psi}_1\hat{\varphi}_2 - \hat{\psi}_2\hat{\varphi}_1, \quad \frac{d\hat{\psi}_1(\epsilon)}{d\epsilon} = \hat{p}\hat{\psi}_1,\\
&\frac{d\hat{\psi}_2(\epsilon)}{d\epsilon} = \hat{p}\hat{\psi}_2, \quad \frac{d\hat{\varphi}_1(\epsilon)}{d\epsilon} = \hat{\psi}_1^2\hat{\psi}_2 + \hat{p}\hat{\varphi}_1, \quad \frac{d\hat{\varphi}_2(\epsilon)}{d\epsilon} = \hat{\psi}_1\hat{\psi}_2^2 + \hat{p}\hat{\varphi}_2, \quad \frac{d\hat{p}(\epsilon)}{d\epsilon} = \hat{p}^2,\\
&\hat{u}(0) = u, \quad \hat{v}(0) = v, \quad \hat{\psi}_1(0) = \psi_1, \quad \hat{\psi}_2(0) = \psi_2, \quad \hat{\varphi}_1(0) = \varphi_1, \quad \hat{\varphi}_2(0) = \varphi_2, \quad \hat{p}(0) = p,
\end{aligned} \tag{38}$$

where ϵ is the group parameter.

By solving the initial value problem (38), we arrive at the symmetry group theorem as follows:

Theorem 1. If $\{u, v, \psi_1, \psi_2, \varphi_1, \varphi_2, p\}$ is the solution of the extended system consisting of (1)–(6), (28), and (33) with $\lambda = 0$, so is $\{\hat{u}, \hat{v}, \hat{\psi}_1, \hat{\psi}_2, \hat{\varphi}_1, \hat{\varphi}_2, \hat{p}\}$

$$\begin{aligned}
&\hat{u} = u - \frac{2\epsilon(\psi_1\varphi_2 + \psi_2\varphi_1)}{1-\epsilon p} - \frac{2\epsilon^2\psi_1^2\psi_2^2}{(1-\epsilon p)^2}, \quad \hat{v} = v + \frac{\epsilon(\psi_1\varphi_2 + \psi_2\varphi_1)}{1-\epsilon p} + \frac{\epsilon^2\psi_1^2\psi_2^2}{(1-\epsilon p)^2},\\
&\hat{\psi}_1 = \frac{\psi_1}{1-\epsilon p}, \quad \hat{\varphi}_1 = \frac{\varphi_1}{1-\epsilon p} + \frac{\epsilon\psi_2\psi_1^2}{(1-\epsilon p)^2}, \quad \hat{\psi}_2 = \frac{\psi_2}{1-\epsilon p}, \quad \hat{\varphi}_2 = \frac{\varphi_2}{1-\epsilon p} + \frac{\epsilon\psi_1\psi_2^2}{(1-\epsilon p)^2},\\
&\hat{p} = \frac{p}{1-\epsilon p},
\end{aligned} \tag{39}$$

with ϵ is an arbitrary group parameter.

For example, starting from the simple solution $\{u = \frac{\lambda_1^2}{2} - \frac{\lambda_2^2}{2}, v = \frac{\lambda_2^2}{2} - \frac{\lambda_1^2}{2}\}$ of (1) and (2), we can derive the special solutions for the introduced dependent variables from (3)–(6), (28), and (33) under the condition $\lambda = 0$,

$$\begin{aligned}
\psi_1 &= \cosh\left(\lambda_1 x + \frac{\lambda_1(\lambda_1^2 - 3\lambda_2^2)}{2}t + \xi_{10}\right),\\
\psi_2 &= \cosh\left(\lambda_2 x + \frac{\lambda_2(\lambda_2^2 - 3\lambda_1^2)}{2}t + \xi_{20}\right),\\
\phi_1 &= \lambda_1 \sinh\left(\lambda_1 x + \frac{\lambda_1(\lambda_1^2 - 3\lambda_2^2)}{2}t + \xi_{10}\right),\\
\phi_2 &= \lambda_2 \sinh\left(\lambda_2 x + \frac{\lambda_2(\lambda_2^2 - 3\lambda_1^2)}{2}t + \xi_{20}\right),\\
p &= \frac{\sinh\xi_1}{2\lambda_1 - 2\lambda_2} + \frac{\sinh\xi_2}{2\lambda_1 + 2\lambda_2} + \xi_0.
\end{aligned} \tag{40}$$

Substituting (40) into (39) yields the non-trivial solution of HS-cKdV system (1)–(2):

$$u = -\frac{\lambda_1^2}{2} - \frac{\lambda_2^2}{2} + \frac{\epsilon[(\lambda_1-\lambda_2)\sinh\xi_1 + (\lambda_1+\lambda_2)\sinh\xi_2]}{\epsilon(\frac{\sinh\xi_1}{2\lambda_1-2\lambda_2} + \frac{\sinh\xi_2}{2\lambda_1+2\lambda_2} + \xi_0) - 1} - \frac{\epsilon^2(\cosh\xi_1 + \cosh\xi_2)^2}{16[\epsilon(\frac{\sinh\xi_1}{2\lambda_1-2\lambda_2} + \frac{\sinh\xi_2}{2\lambda_1+2\lambda_2} + \xi_0) - 1]^2}, \tag{41}$$

$$v = -\frac{\lambda_1^2}{2} + \frac{\lambda_2^2}{2} - \frac{\epsilon[(\lambda_1-\lambda_2)\sinh\xi_1 + (\lambda_1+\lambda_2)\sinh\xi_2]}{2\epsilon(\frac{\sinh\xi_1}{2\lambda_1-2\lambda_2} + \frac{\sinh\xi_2}{2\lambda_1+2\lambda_2} + \xi_0) - 2} + \frac{\epsilon^2(\cosh\xi_1 + \cosh\xi_2)^2}{32[\epsilon(\frac{\sinh\xi_1}{2\lambda_1-2\lambda_2} + \frac{\sinh\xi_2}{2\lambda_1+2\lambda_2} + \xi_0) - 1]^2}, \tag{42}$$

with

$$\xi_1 = (\lambda_1 - \lambda_2)[x + \frac{\lambda_1^2 + 4\lambda_1\lambda_2 + \lambda_2^2}{2}t] + \xi_{10} - \xi_{20},$$
$$\xi_2 = (\lambda_1 + \lambda_2)[x + \frac{\lambda_1^2 - 4\lambda_1\lambda_2 + \lambda_2^2}{2}t] + \xi_{10} + \xi_{20},$$

where ξ_0, ξ_{10}, and ξ_{20} are arbitrary constants.

It is necessary to point out that the finite transformation exhibited in **Theorem 1** is distinct from the original DT. Indeed, it is equivalent to the Levi transformation, or the second type of Darboux transformation. At the algebraic level, two different types of finite transformation possess the same infinitesimal expression (27). Besides, the last equation of (39) is nothing but the corresponding Möbius transformation in the previous analysis about the Schwartz form of the original system (37).

C. Similarity reductions of the prolonged system

In this section, our main goal is to find novel solutions of the original HS-cKdV system through constructing exact solutions of the prolonged system. Alternatively, we employ the classical Lie symmetry method to search for similarity reductions of the whole prolonged system.

Accordingly, we consider the one-parameter group of infinitesimal transformations in $\{x, t, u, v, \psi_1, \psi_2, \varphi_1, \varphi_2, p\}$ given by

$$\{x, t, u, v, \psi_1, \psi_2, \varphi_1, \varphi_2, p\} \to \{x+\epsilon X, t+\epsilon T, u+\epsilon U, v+\epsilon V, \psi_1+\epsilon\psi_1, \psi_2+\epsilon\Psi_2, \varphi_1+\epsilon\Phi_1, \varphi_2+\epsilon\Phi_2, p+\epsilon P\}$$

with

$$\begin{aligned}
&\sigma^u = Xu_x + Tu_t - U, \quad \sigma^v = Xv_x + Tv_t - V,\\
&\sigma^{\psi_1} = X\psi_{1x} + T\psi_{1t} - \Psi_1, \quad \sigma^{\psi_2} = X\psi_{2x} + T\psi_{2t} - \Psi_2,\\
&\sigma^{\varphi_1} = X\varphi_{1x} + T\varphi_{1t} - \Phi_1, \quad \sigma^{\varphi_2} = X\varphi_{2x} + T\varphi_{2t} - \Phi_2,\\
&\sigma^p = Xp_x + Tp_t - P,
\end{aligned} \tag{43}$$

where $X, T, U, V, \Psi_1, \Psi_2, \Phi_1, \Phi_2$, and P are functions with respect to $\{x, t, u, v, \psi_1, \psi_2, \varphi_1, \varphi_2, p\}$, and ϵ is a small parameter. Substituting (43) into the symmetry equations, i.e., the linearized equations of the prolonged system

$$\begin{aligned}
&\sigma_t^u - \frac{1}{2}\sigma_{xxx}^u - 3u\sigma_x^u - 3\sigma^u u_x + 6v\sigma_x^v + 6\sigma^v v_x = 0, \quad \sigma_t^v + \sigma_{xxx}^v + 3\sigma^u v_x + 3u\sigma_x^v = 0,\\
&\sigma_{xx}^{\psi_1} + (u+v)\sigma^{\psi_1} + (\sigma^u + \sigma^v)\psi_1 = 0, \quad \sigma_{xx}^{\psi_2} + (u-v)\sigma^{\psi_2} + (\sigma^u - \sigma^v)\psi_2 = 0,\\
&\sigma_t^{\psi_1} + \frac{1}{2}\sigma^{\psi_1}(u_x - 2v_x) + \frac{1}{2}\psi_1(\sigma_x^u - 2\sigma_x^v) - (u-2v)\sigma_x^{\psi_1} - (\sigma^u - 2\sigma^v)\psi_{1x} = 0,\\
&\sigma_t^{\psi_2} + \frac{1}{2}\sigma^{\psi_2}(u_x + 2v_x) + \frac{1}{2}\psi_2(\sigma_x^u + 2\sigma_x^v) - (u+2v)\sigma_x^{\psi_2} - (\sigma^u + 2\sigma^v)\psi_{2x} = 0,\\
&\sigma_x^{\psi_1} - \sigma^{\varphi_1} = 0, \quad \sigma_x^{\psi_2} - \sigma^{\varphi_2} = 0, \quad \sigma_x^p - \psi_1\sigma^{\psi_2} - \psi_2\sigma^{\psi_1} = 0,\\
&\sigma_t^p + u(\psi_1\sigma^{\psi_2} + \psi_2\sigma^{\psi_1}) + \sigma^u\psi_1\psi_2 + 2\phi_1\sigma^{\varphi_2} + 2\phi_2\sigma^{\varphi_1} = 0,
\end{aligned} \tag{44}$$

then collecting the coefficients of the dependent variables and their partial derivatives, and setting all of them to zero, we arrive at a system of overdetermined linear equations for the infinitesimals $\{X, T, U, V, \Psi_1, \Psi_2, \Phi_1, \Phi_2, P\}$. By solving these equations, one can get

$$\begin{aligned}
&X = c_1 x + c_4, \quad T = 3c_1 t + c_2,\\
&U = -2c_1 u - 2c_3(\psi_1\varphi_2 + \psi_2\varphi_1), \quad V = -2c_1 v + c_3(\psi_1\phi_2 - \psi_2\phi_1),\\
&\Psi_1 = c_3 p\psi_1 + c_5\psi_1, \quad \Phi_1 = c_3(\psi_1^2\psi_2 + p\varphi_1) - (c_1 - c_5)\varphi_1,\\
&\Psi_2 = c_3 p\psi_2 + c_6\psi_2, \quad \Phi_2 = c_3(\psi_1\psi_2^2 + p\varphi_2) - (c_1 - c_6)\varphi_2,\\
&P = c_3 p^2 + (c_5 + c_6)p + c_1 p + c_7,
\end{aligned} \tag{45}$$

where c_i $(i = 1...7)$ are arbitrary constants. When $c_1 = c_2 = c_4 = c_5 = c_6 = c_7 = 0$, the degenerated symmetry is just the one expressed by (36), and when $c_3 = c_5 = c_6 = c_7 = 0$, the related symmetry is only the general Lie point symmetry of the HS-cKdV system (1)–(2).

To give some corresponding group invariant solutions, one need to solve the following characteristic equations:

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U} = \frac{dv}{V} = \frac{d\psi_1}{\Psi_1} = \frac{d\psi_2}{\Psi_2} = \frac{d\varphi_1}{\Phi_1} = \frac{d\varphi_2}{\Phi_2} = \frac{dp}{P}. \tag{46}$$

Next, we consider several different similarity reductions arising from (46) under the condition $c_3 \neq 0$ in detail.

Reduction 1. $c_1 \neq 0$

Without loss of generality, we assume $c_2 = c_4 = c_5 = c_6 = 0$ and redefine $k^2 = \frac{c_1^2 - 4c_3c_7}{36c_1^2}$. Then, two different situations, $k \neq 0$ and $k = 0$, need to be further considered, respectively.

Case 1 $k \neq 0$. We obtain similarity solutions

$$\begin{aligned}
u &= \frac{U(z)}{t^{\frac{2}{3}}} + \frac{2c_3}{9c_1^2k^2t^{\frac{2}{3}}}\exp(-\frac{2}{3}P(z))\big\{3c_1k[\Psi_1(z)\Phi_2(z) + \Psi_2(z)\Phi_1(z)]\tanh z_1\\
&\quad - c_3\Psi_1^2(z)\Psi_2^2(z)\mathrm{sech}^2 z_1\big\},\\
v &= \frac{V(z)}{t^{\frac{2}{3}}} + \frac{c_3}{3c_1kt^{\frac{2}{3}}}\exp(-\frac{2}{3}P(z))[\Psi_1(z)\Phi_2(z) - \Psi_2(z)\Phi_1(z)]\tanh z_1,\\
\psi_1 &= \frac{\Psi_1(z)}{t^{\frac{1}{6}}}\exp(-\frac{1}{6}P(z))\mathrm{sech} z_1, \quad \psi_2 = \frac{\Psi_2(z)}{t^{\frac{1}{6}}}\exp(-\frac{1}{6}P(z))\mathrm{sech} z_1,\\
\varphi_1 &= \frac{1}{3c_1kt^{\frac{1}{2}}}\exp(-\frac{1}{2}P(z))[3c_1k\Phi_1(x)\mathrm{sech} z_1 + c_3\Psi_1^2(z)\Psi_2(z)\tanh z_1],\\
\varphi_2 &= \frac{1}{3c_1kt^{\frac{1}{2}}}\exp(-\frac{1}{2}P(z))[3c_1k\Phi_2(x)\mathrm{sech} z_1 + c_3\Psi_1(z)\Psi_2^2(z)\tanh z_1],\\
p &= -\frac{c_1}{2c_3}(1 + 6k\tanh z_1),
\end{aligned} \tag{47}$$

with $z_1 = k(\ln t + P(z))$, and the similarity variable $z = x/\sqrt[3]{t}$.

Substituting (47) into the prolonged equations leads to

$$
\begin{aligned}
U(z) &= -\frac{\Psi_{1zz}(z)}{2\Psi_1(z)} - \frac{\Psi_{2zz}(z)}{2\Psi_2(z)} - \frac{c_3}{9c_1k^2}\exp(-\frac{1}{3}P(z))[\Psi_1(z)\Psi_{2z}(z) + \Psi_2(z)\Psi_{1z}(z)] \\
&\quad -\frac{c_3^2(1+12k^2)}{108c_1^2k^4}\exp(-\frac{2}{3}P(z))\Psi_1^2(z)\Psi_2^2(z), \\
V(z) &= -\frac{\Psi_{1zz}(z)}{2\Psi_1(z)} + \frac{\Psi_{2zz}(z)}{2\Psi_2(z)} + \frac{c_3}{18c_1k^2}\exp(-\frac{1}{3}P(z))[\Psi_1(z)\Psi_{2z}(z) - \Psi_2(z)\Psi_{1z}(z)], \\
\Phi_1 &= \exp(\frac{1}{3}P(z))\Psi_1(z) + \frac{c_3}{18c_1k^2}\Psi_1^2(z)\Psi_2(z), \\
\Phi_2 &= \exp(\frac{1}{3}P(z))\Psi_2(z) + \frac{c_3}{18c_1k^2}\Psi_1(x)\Psi_2^2(z), \\
\Psi_1(z) &= \sqrt{-\frac{3c_1k^2}{c_3}P_z(z)}\exp(\frac{1}{6}P(z) + Q(z)), \\
\Psi_2(z) &= \sqrt{-\frac{3c_1k^2}{c_3}P_z(z)}\exp(\frac{1}{6}P(z) - Q(z)),
\end{aligned}
\tag{48}
$$

where $P_z(z) \equiv P_1(z)$ and $Q_z(z) \equiv Q_1(z)$ satisfy the ordinary differential equations

$$
\begin{aligned}
&6P_{1zz}(z)P_1(z) - 9P_{1z}^2(z) - 12P_1(z) + 4P_1^2(z)z + 36P_1^2(z)Q_1^2(z) - 12k^2P_1^4(z) = 0, \\
&3Q_{1zz}P_1(z) - 4P_1(z)Q_1(z)z + 9Q_1(z) - 24P_1(z)Q_1^3(z) = 0.
\end{aligned}
\tag{49}
$$

Eliminating $P_1(z)$ and its derivative terms in first equation through the second equation in (49), we obtain a fourth-order ordinary differential equation about the variable $Q_1(z)$. By using the Ablowitz-Ramani-Segur (ARS) algorithm, there are two possible branches: $Q_1(z) = Q_{10}/(z - z_0)$ with Q_{10}=$\{\pm\frac{1}{2}\}$, and the resonant points occur at $\{-1, 1, 4, 5\}$. Then the detailed calculation shows equations (49) pass the Painlevé test.

Case 2 $k = 0$. We obtain similarity solutions

$$
\begin{aligned}
u &= \frac{U(z)}{t^{\frac{2}{3}}} + \frac{2c_3}{3c_1t^{\frac{2}{3}}z_1}[\Psi_1(z)\Phi_2(z) + \Psi_2(z)\Phi_1(z)] - \frac{2c_3^2}{9c_1^2t^{\frac{2}{3}}z_1^2}\Psi_1^2(z)\Psi_2^2(z), \\
v &= \frac{V(z)}{t^{\frac{2}{3}}} - \frac{c_3}{3c_1t^{\frac{2}{3}}z_1}[\Psi_1(z)\Phi_2(z) - \Psi_2(z)\Phi_1(z)], \\
\phi_1 &= \frac{\Phi_1(z)}{t^{\frac{1}{2}}z_1} - \frac{c_3\Psi_1^2(z)\Psi_2(z)}{3c_1t^{\frac{1}{2}}z_1^2}, \quad \psi_1 = \frac{\Psi_1(z)}{t^{\frac{1}{6}}z_1}, \\
\phi_2 &= \frac{\Phi_2(z)}{t^{\frac{1}{2}}z_1} - \frac{c_3\Psi_1(z)\Psi_2^2(z)}{3c_1t^{\frac{1}{2}}z_1^2}, \quad \psi_2 = \frac{\Psi_2(z)}{t^{\frac{1}{6}}z_1}, \\
p &= -\frac{c_1}{2c_3} - \frac{3c_1}{c_3z_1},
\end{aligned}
\tag{50}
$$

with $z_1 = \ln t + P(z)$, and the similarity variable $z = x/\sqrt[3]{t}$.

Substituting (50) into the prolonged equations leads to

$$
\begin{aligned}
&U(z) = -\frac{\Psi_{1zz}(z)}{2\Psi_1(z)} - \frac{\Psi_{2zz}(z)}{2\Psi_2(z)}, \quad V(z) = -\frac{\Psi_{1zz}(z)}{2\Psi_1(z)} + \frac{\Psi_{2zz}(z)}{2\Psi_2(z)},\\
&\Phi_1 = \Psi_{1z}(z), \quad \Psi_1(z) = \sqrt{\frac{3c_1}{c_3}P_z(z)}\exp(Q(z)),\\
&\Phi_2 = \Psi_{2z}(z), \quad \Psi_2(z) = \sqrt{\frac{3c_1}{c_3}P_z(z)}\exp(-Q(z)),
\end{aligned} \tag{51}
$$

where $P_z(z) \equiv P_1(z)$ and $Q_z(z) \equiv Q_1(z)$ satisfy the ordinary differential equations

$$
\begin{aligned}
&6P_{1zz}(z)P_1(z) - 9P_{1z}^2(z) - 12P_1(z) + 4P_1^2(z)z + 36P_1^2(z)Q_1^2(z) = 0,\\
&3Q_{1zz}P_1(z) - 4P_1(z)Q_1(z)z + 9Q_1(z) - 24P_1(z)Q_1^3(z) = 0.
\end{aligned} \tag{52}
$$

For Eqs. (52), eliminating $P_1(z)$ and its derivative terms in first equation through the second equation, we obtain a fourth-order ordinary differential equation about the variable $Q_1(z)$. Similar to the above subcase, we get two possible branches: $Q_1(z) = Q_{10}/(z - z_0)$ with $Q_{10}=\{\pm\frac{1}{2}\}$, and the resonant points appear at $\{-1, 1, 4, 5\}$. Then the detailed calculation shows that Eq. (52) also possess Painlevé property.

From the results in **Reduction 1**, one can observe that the final exact solutions include hyperbolic function, Painlevé solution, and rational function. These solutions represent the wave interactions among solitary wave, Painlevé wave, and rational wave for the HS-cKdV system.

Reduction 2. $c_1 = 0$.

For simplicity, let $c_2 = 1$ and $l^2=\frac{(c_5+c_6)^2}{4}-c_3c_7$. Then, two subcases $l \neq 0$ and $l = 0$ are taken into account in this subsection.

Case 1 $l \neq 0$. We derive similarity solutions

$$
\begin{aligned}
&u = U(z) - \frac{2c_3}{l}[\Psi_1(z)\Phi_2(z) + \Psi_2(z)\Phi_1(z)]\tanh z_1 + \frac{2c_3}{l^2}\Psi_1^2(z)\Psi_2^2(z)\mathrm{sech}^2 z_1,\\
&v = V(z) + \frac{c_3}{l}[\Psi_1(z)\Phi_2(z) - \Psi_2(z)\Phi_1(z)]\tanh z_1,\\
&\psi_1 = \exp(\frac{c_5 - c_6}{2}t)\Psi_1(z)\mathrm{sech} z_1, \quad \psi_2 = \exp(\frac{c_6 - c_5}{2}t)\Psi_2(z)\mathrm{sech} z_1,\\
&\varphi_1 = \exp(\frac{c_5 - c_6}{2}t)\mathrm{sech} z_1[\Phi_1(z) + \frac{c_3}{l}\Psi_1^2(z)\Psi_2(z)\tanh z_1],\\
&\varphi_2 = \exp(\frac{c_6 - c_5}{2}t)\mathrm{sech} z_1[\Phi_2(z) + \frac{c_3}{l}\Psi_1(z)\Psi_2^2(z)\tanh z_1],\\
&p = -\frac{1}{2c_3}(c_5 + c_6 + 2l\tanh z_1),
\end{aligned} \tag{53}
$$

with $z_1 = l(t + P(z))$, and the similarity variable $z = x - c_4t$.

Substituting (53) into the prolonged equations leads to

$$U(z) = -\frac{\Psi_{1zz}(z)}{2\Psi_1(z)} - \frac{\Psi_{2zz}(z)}{2\Psi_2(z)} - \frac{c_3^2}{l^2}\Psi_1^2(z)\Psi_2^2(z), \quad V(z) = -\frac{\Psi_{1zz}(z)}{2\Psi_1(z)} + \frac{\Psi_{2zz}(z)}{2\Psi_2(z)},$$

$$\Phi_1 = \Psi_{1z}(z), \quad \Psi_1(z) = \sqrt{\frac{-l^2}{c_3 P_1(z)}}\exp(\int Q_1(z)dz), \tag{54}$$

$$\Phi_2 = \Psi_{2z}(z), \quad \Psi_2(z) = \sqrt{\frac{-l^2}{c_3 P_1(z)}}\exp(-\int Q_1(z)dz),$$

$$P_1(z) = \frac{4c_4}{3} + \frac{8}{3}Q_1^2(z) - \frac{2Q_{1zz}(z) + c_5 - c_6}{6Q_1(z)},$$

where $Q_1(z)$ satisfies the ordinary differential equation

$$Q_{1z}^2(z) = a_0 + a_1 Q_1(z) + a_2 Q_1^2(z) + a_3 Q_1^3(z) + 4Q_1^4(z), \tag{55}$$

with $a_0 = \frac{16l^2}{a_3^2}$, $a_1 = c_6 - c_5$, and $a_2 = 4c_4$.

To demonstrate this kind of solution more clearly, we give one special case by solving Eq. (55). For instance, a simple solution of (55) takes the form

$$Q_1(z) = b_0 + b_1\mathrm{sn}(hz, m). \tag{56}$$

It leads to the solution of the HS-cKdV system (1) and (2) as follows:

$$\begin{aligned} u &= -6b_0^2 + \frac{1}{4}h^2(1+m^2) + \frac{(16b_0^4 - h^4m^2) + 2b_0mh[8b_0^2 - (1+m^2)h^2]\mathrm{sn}(h\xi, m)}{[2b_0 + mh\mathrm{sn}(h\xi, m)]^2} \\ &\quad - \frac{\mu mh^2\mathrm{cn}(h\xi, m)\mathrm{dn}(h\xi, m)\tanh[2\mu b_0(t+z_1)]}{[2b_0 + mh\mathrm{sn}(h\xi, m)]^2} + \frac{\mu^2\mathrm{sech}^2[2\mu b_0(t+z_1)]}{2[2b_0 + mh\mathrm{sn}(h\xi, m)]^2}, \\ v &= \frac{1}{2}\mu\tanh[2\mu b_0(t+z_1)], \end{aligned} \tag{57}$$

with $z_1 = \frac{1}{4b_0}\int_0^{\xi_0}[2b_0 + mh\mathrm{sn}(h\xi, m)]^{-1}dz$, $\xi_0 = x - [6b_0^2 - \frac{1}{4}h^2(1+m^2)]t$, and $\mu = [(4b_0^2 - h^2)(4b_0^2 - m^2h^2)]^{\frac{1}{2}}$. Here, sn, cn, and dn are usual Jacobian elliptic functions with modulus m.

From the expression of the exact solution (57), we know that it potentially reflects the wave interaction between the soliton and the cnoidal periodic wave. As Shin has mentioned,[27–29] these soliton + cnoidal wave solutions can be easily applicable to the analysis of physical processes.

The dynamic behaviors of the soliton + cnoidal wave solution (57) for two different choices of the parameters b_0, h, and m are illustrated in Figs. 1 and 2. In Fig. 1, when $m \neq 1$, the component u exhibits a bell-shaped bright soliton propagating on a cnoidal wave background, whereas, periodic wave in the component v occurs at the corner of a kink-shaped soliton. When $m = 1$, the Jacobian elliptic periodic functions in (57) are reduced to the general hyperbolic functions, the two-soliton is displayed distinctly in Fig. 2.

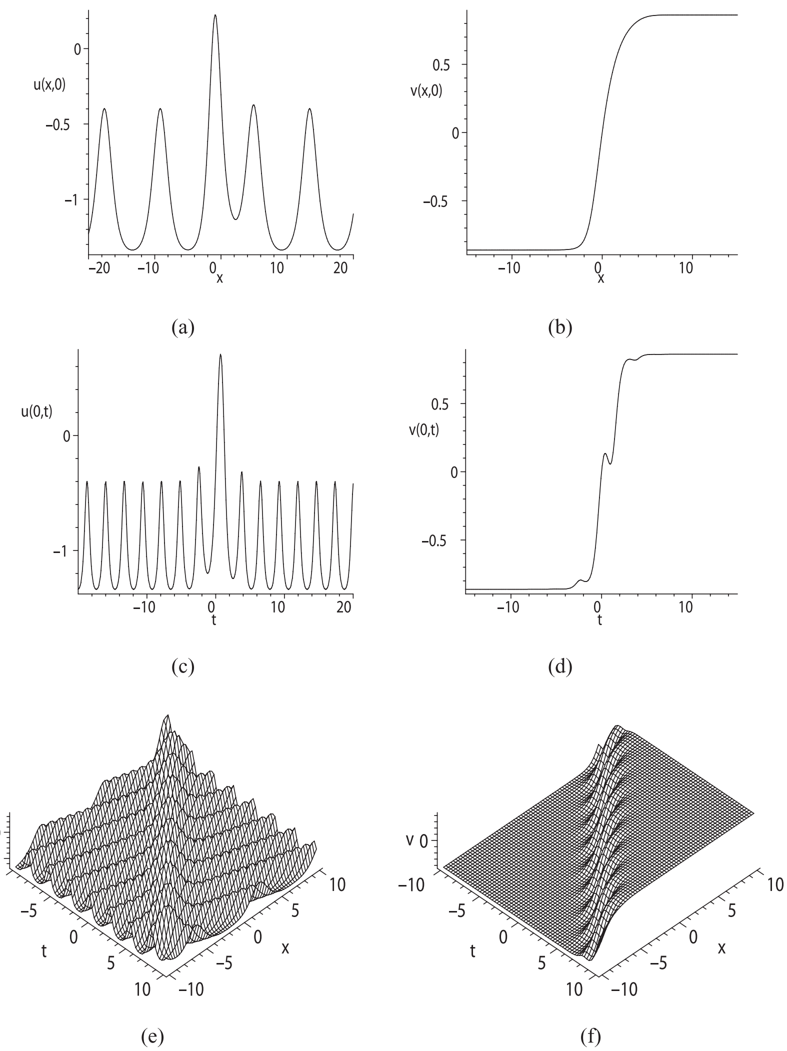

FIG. 1. The wave propagation plots of the HS-cKdV system given by (57), with the parameters $b_0 = 0.75$, $h = 0.85$ and $m = 0.65$. (a) and (b) The wave propagation pattern of the wave along x axis at $t = 0$; (c) and (d) The wave propagation pattern of the wave along t axis at $x = 0$; (e) and (f) The two-dimensional perspective view of the corresponding solution.

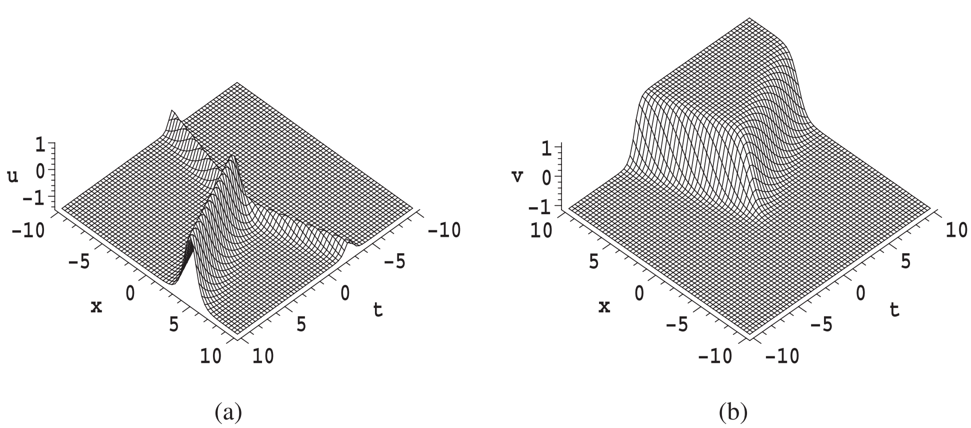

FIG. 2. The wave propagation plots of the HS-cKdV system given by (57), with the parameters $b_0 = 0.8$, $h = 0.6$, and $m = 1$. (a) and (b) The two-dimensional perspective view of the corresponding solution.

Case 2 $l = 0$. We obtain similarity solutions

$$\begin{aligned}
u &= U(z) + \frac{2c_3}{z_1}[\Psi_1(z)\Phi_2(z) + \Psi_2(z)\Phi_1(z)] - \frac{2c_3^2}{z_1^2}\Psi_1^2(z)\Psi_2^2(z)\\
v &= V(z) + \frac{c_3}{z_1}[\Psi_2(z)\Phi_1(z) - \Psi_1(z)\Phi_2(z)],\\
\psi_1 &= \exp(\frac{c_5 - c_6}{2}t)\frac{\Psi_1(z)}{z_1}, \quad \varphi_1 = \exp(\frac{c_5 - c_6}{2}t)[\frac{\Phi_1(z)}{z_1} - \frac{c_3\Psi_1^2(z)\Psi_2(z)}{z_1^2}],\\
\psi_2 &= \exp(\frac{c_6 - c_5}{2}t)\frac{\Psi_2(z)}{z_1}, \quad \varphi_2 = \exp(\frac{c_6 - c_5}{2}t)[\frac{\Phi_2(z)}{z_1} - \frac{c_3\Psi_1(z)\Psi_2^2(z)}{z_1^2}],\\
p &= -\frac{c_5 + c_6}{2c_3} - \frac{1}{c_3 z_1},
\end{aligned} \tag{58}$$

with $z_1 = t + P(z)$, and the similarity variable $z = x - c_4 t$.

Substituting (58) into the prolonged equations leads to

$$\begin{aligned}
U(z) &= -\frac{\Psi_{1zz}(z)}{2\Psi_1(z)} - \frac{\Psi_{2zz}(z)}{2\Psi_2(z)}, \quad V(z) = -\frac{\Psi_{1zz}(z)}{2\Psi_1(z)} + \frac{\Psi_{2zz}(z)}{2\Psi_2(z)},\\
\Phi_1 &= \Psi_{1z}(z), \quad \Psi_1(z) = \sqrt{\frac{1}{c_3 P_1(z)}}\exp(\int Q_1(z)dz),\\
\Phi_2 &= \Psi_{2z}(z), \quad \Psi_2(z) = \sqrt{\frac{1}{c_3 P_1(z)}}\exp(-\int Q_1(z)dz),\\
P_1(z) &= \frac{4c_4}{3} + \frac{8}{3}Q_1^2(z) - \frac{2Q_{1zz}(z) + c_5 - c_6}{6Q_1(z)},
\end{aligned} \tag{59}$$

where $Q_1(z)$ satisfies the ordinary differential equation

$$Q_{1z}^2(z) = a_1 Q_1(z) + a_2 Q_1^2(z) + a_3 Q_1^3(z) + 4Q_1^4(z), \tag{60}$$

with $a_1 = c_6 - c_5$ and $a_2 = 4c_4$.

In this subcase, the final exact solutions consisting of Jacobian elliptic periodic function and rational function denote the interactions between elliptic periodic wave and rational wave for the HS-cKdV system.

Remark 3. In Ref. 2, it has been shown that any conservation law of a given PDE system can yield an equivalent nonlocally related augmented PDE system, which is called as a potential system. Then, considering the general Lie symmetries of such a nonlocally related system, nonlocal symmetries of the original PDE system can be obtained directly. Here, we focus on the integrable HS-cKdV system which satisfies identically the compatibility of its Lax pair. If we view the spectral functions in Lax pair as potential (pseudopotential) variables, the extended PDE system including the given HS-cKdV system (1)–(2), Lax pair (3)–(6), and introduced potential equations (28) and (33) can be viewed as a potential system. Moreover, by computing the general Lie symmetry of this augmented system, nonlocal symmetries of the HS-cKdV system are also derived (see (45), which contains nonlocal symmetry and the general Lie symmetry of the HS-cKdV system). From this perspective, nonlocal symmetry for the HS-cKdV system can be obtained by using the same method in Ref. 2.

IV. INTEGRABLE MODELS FROM NONLOCAL SYMMETRY

A. Negative HS-cKdV hierarchy

As is known, the existence of infinitely many symmetries naturally leads to the existence of integrable hierarchies. For the general hierarchies, one can obtain them from a trivial symmetry of the original integrable system through the recursion operator. For example, using the recursion operator,[30–32] the HS-cKdV system has the following high order symmetry:

$$u_\tau = (\frac{1}{2}u_{4x} + 5uu_x^2 - 5vv_{xx} + \frac{5}{2}u_x^2 - 8uv^2 + 5u^3)_x - v^2u_x, \tag{61}$$

$$v_\tau = -(v_{4x} + 2u_xv_x + 6uv_{xx} + \frac{5}{3}v^3)_x - 3u_{xx}v_x - 5u^2v_x. \tag{62}$$

It is remarkable that the reduction $v = 0$ in this symmetry gives the Lax equation.[53]

Recently, Lou[16–19] has extended some (1+1)-dimensional integrable models to negative hierarchies through a set of infinitely many nonlocal symmetries which can be obtained from the kernels of a reversible recursion operator. However, it is difficult to derive the inverse of the known recursion operator, especially for the high dimensional system and the high order operator. Here, we introduce the internal parameter to obtain the negative HS-cKdV hierarchy.[10,11]

Starting from the nonlocal symmetry (25), we let

$$K_0^u(\lambda) \equiv -2(\psi_1\psi_2)_x, \quad K_0^v(\lambda) \equiv \psi_1\psi_{2x} - \psi_2\psi_{1x}, \tag{63}$$

where ψ_1 and ψ_2 are determined by Lax pair (3)–(6) with $\lambda \neq 0$. Because the parameter λ is an arbitrary constant, we can treat it as a small parameter and expand $K_0^u(\lambda)$ and $K_0^v(\lambda)$ as a series in λ:

$$K_0^u(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}K_0^u(\lambda)\bigg|_{\lambda=0}\lambda^n, \quad K_0^v(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}K_0^v(\lambda)\bigg|_{\lambda=0}\lambda^n. \tag{64}$$

Substituting (64) into the corresponding symmetry equation of the HS-cKdV system (1)–(2), one can conclude that

$$K_n^u(\lambda) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}K_0^u(\lambda)\bigg|_{\lambda=0}, \quad K_n^v(\lambda) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}K_0^v(\lambda)\bigg|_{\lambda=0} \tag{65}$$

must be a symmetry of the HS-cKdV system for all n ($n = 0, 1, 2...$). Meanwhile, we request $\psi_1 = \psi_1(\lambda)$ and $\psi_2 = \psi_2(\lambda)$ have the formal series

$$\psi_1 = \sum_{k=0}^{\infty}\psi_1[k]\lambda^k, \quad \psi_2 = \sum_{k=0}^{\infty}\psi_2[k]\lambda^k, \tag{66}$$

where $\psi_1[k]$ and $\psi_2[k]$ are λ independent and should be determined later.

Substituting (65) and (66) into (3) and (4), and collecting the coefficients of λ yields

$$(\partial^2 + u + v)\psi_1[0] = 0, \quad (\partial^2 + u + v)\psi_1[k] = -\psi_2[k-1], \tag{67}$$

$$(\partial^2 + u - v)\psi_2[0] = 0, \quad (\partial^2 + u - v)\psi_2[k] = \psi_1[k-1]. \tag{68}$$

Then, $\psi_1[k]$ and $\psi_2[k]$ can be solved recursively as

$$\psi_1[k] = L_1^{-1}\psi_2[k-1] = L_1^{-1}L_2^{-1}\psi_1[k-2]\cdots = \begin{cases}(L_2L_1)^{-\frac{k}{2}}\psi_1[0], & k \text{ is even}\\ (L_2L_1)^{-\frac{k-1}{2}}L_1^{-1}\psi_2[0], & k \text{ is odd}\end{cases}, \tag{69}$$

$$\psi_2[k] = L_2^{-1}\psi_1[k-1] = L_2^{-1}L_1^{-1}\psi_2[k-2]\cdots = \begin{cases}(L_1L_2)^{-\frac{k}{2}}\psi_2[0], & k \text{ is even}\\ (L_1L_2)^{-\frac{k-1}{2}}L_2^{-1}\psi_1[0], & k \text{ is odd}\end{cases}, \tag{70}$$

$$L_1 = -\partial^2 - u - v, \quad L_2 = \partial^2 + u - v,$$

which transform (66) into

$$\psi_1 = \sum_{k_1=0,2,4\cdots}^{\infty}(L_2L_1)^{-\frac{k_1}{2}}\psi_1[0]\lambda^{k_1} + \sum_{k_2=1,3,5\cdots}^{\infty}(L_2L_1)^{-\frac{k_2-1}{2}}L_1^{-1}\psi_2[0]\lambda^{k_2}, \tag{71}$$

$$\psi_2 = \sum_{k_3=0,2,4\cdots}^{\infty}(L_1L_2)^{-\frac{k_3}{2}}\psi_2[0]\lambda^{k_3} + \sum_{k_4=1,3,5\cdots}^{\infty}(L_1L_2)^{-\frac{k_4-1}{2}}L_2^{-1}\psi_1[0]\lambda^{k_4}. \tag{72}$$

Finally, the following nonlocal symmetries can be obtained:

$$\begin{cases} K_n^u = -2\sum_{k_1=0,2,4\ldots}^{n}(\Xi_1\Xi_3)_x - 2\sum_{k_2=1,3,5\ldots}^{n-1}(\Xi_2\Xi_4)_x, \\ K_n^v = \sum_{k_1=0,2,4\ldots}^{n}(\Xi_1\Xi_{3x} - \Xi_3\Xi_{1x}) + \sum_{k_2=1,3,5\ldots}^{n-1}(\Xi_2\Xi_{4x} - \Xi_4\Xi_{2x}), \end{cases} \quad n \text{ is even}, \tag{73}$$

$$\begin{cases} K_n^u = -2\sum_{k_1=0,2,4\ldots}^{n-1}(\Xi_1\Xi_6)_x - 2\sum_{k_2=1,3,5\ldots}^{n}(\Xi_2\Xi_5)_x, \\ K_n^v = \sum_{k_1=0,2,4\ldots}^{n-1}(\Xi_1\Xi_{6x} - \Xi_6\Xi_{1x}) + \sum_{k_2=1,3,5\ldots}^{n}(\Xi_2\Xi_{5x} - \Xi_5\Xi_{2x}), \end{cases} \quad n \text{ is odd}, \tag{74}$$

where

$$\Xi_1=(L_2L_1)^{-\frac{k_1}{2}}\psi_1[0], \quad \Xi_2=(L_2L_1)^{-\frac{k_2-1}{2}}L_1^{-1}\psi_2[0],$$
$$\Xi_3=(L_1L_2)^{-\frac{n-k_1}{2}}\psi_2[0], \quad \Xi_4=(L_1L_2)^{-\frac{n-k_2-1}{2}}L_2^{-1}\psi_1[0],$$
$$\Xi_5=(L_1L_2)^{-\frac{n-k_2}{2}}\psi_2[0], \quad \Xi_6=(L_1L_2)^{-\frac{n-k_1-1}{2}}L_2^{-1}\psi_1[0].$$

From the set of the nonlocal symmetries, the negative HS-cKdV hierarchy (the flow equations of the HS-cKdV equation corresponding to the nonlocal symmetries) follows immediately

$$\begin{cases} u_t = K_n^u, \quad v_t = K_n^v, \\ L_1\psi_1[0] = 0, \quad L_2\psi_2[0] = 0, \end{cases} \tag{75}$$

where K_n^u and K_n^v are expressed by (73) and (74).

Using the Miura transformation[47,48]

$$u = -\frac{1}{2}F_{xx} - \frac{1}{4}F_x^2 - \frac{1}{4}G_x^2, \quad v = -\frac{1}{2}G_{xx} - \frac{1}{2}F_xG_x, \tag{76}$$

the first member of the negative HS-cKdV hierarchy (75) is transformed to a coupled sinh-Gordon system:

$$\begin{aligned} &F_{xt} = \sinh(F) + f, \quad G_{xt} = \sinh(F) + g, \\ &[(4s-1)e^F - f]F_x + [\frac{1}{2}\sinh(F) - g]G_x - f_x = 0, \\ &[f - g - 2se^F](F_x - G_x) + f_x - g_x - 2se^F F_x = 0, \end{aligned} \tag{77}$$

where s is an arbitrary constant. When $s = \frac{1}{4}$ and $G = 0$, (77) is just the usual sinh-Gordon equation.

Further, the complicated integrodifferential hierarchy (75) can be written as a simple equivalent differential equation system ($(L_2L_1)^{m+1}P_m = \psi_1[0]$, $(L_1L_2)^mL_1Q_m = \psi_2[0]$)

$$\begin{cases} u_t = -2\sum_{j_1=0}^{m}(\Xi_1\Xi_3)_x - 2\sum_{j_2=0}^{m-1}(\Xi_2\Xi_4)_x, \\ v_t = \sum_{j_1=0}^{m}(\Xi_1\Xi_{3x} - \Xi_3\Xi_{1x}) + \sum_{j_2=0}^{m-1}(\Xi_2\Xi_{4x} - \Xi_4\Xi_{2x}), \end{cases} \quad n = 2m, \tag{78}$$

$$\begin{cases} u_t = -2\sum_{j_1=0}^{m}(\Xi_1\Xi_6)_x - 2\sum_{j_2=0}^{m}(\Xi_2\Xi_5)_x, \\ v_t = \sum_{j_1=0}^{m}(\Xi_1\Xi_{6x} - \Xi_6\Xi_{1x}) + \sum_{j_2=0}^{m}(\Xi_2\Xi_{5x} - \Xi_5\Xi_{2x}), \end{cases} \quad n = 2m+1, \tag{79}$$

$$L_1(L_2L_1)^{m+1}P_m = 0, \quad L_2(L_1L_2)^mL_1Q_m = 0,$$

with

$$\Xi_1 = (L_2 L_1)^{m-j_1+1} P_m, \quad \Xi_2 = (L_2 L_1)^{m-j_2} Q_m,$$
$$\Xi_3 = (L_1 L_2)^{j_1} L_1 Q_m, \quad \Xi_4 = (L_1 L_2)^{j_2+1} L_1 P_m,$$
$$\Xi_5 = (L_1 L_2)^{j_2} L_1 Q_m, \quad \Xi_6 = (L_1 L_2)^{j_1} L_1 P_m.$$

In Refs. 20 and 21, the first member of the negative HS-cKdV system is given. Now we provide the whole negative HS-cKdV hierarchy by the simple differential form.

B. Lower-dimensional and higher-dimensional integrable systems

From the nonlocal symmetry and Proposition 5 in Sec. II, the HS-cKdV system has the nontrivial nonlocal symmetry as follows:

$$\sigma_N = (\sigma_N^u, \sigma_N^v) \equiv \Big(-\sum_{i=1}^{N} 2a_i(\psi_{1i}\psi_{2i})_x, \sum_{i=1}^{N} a_i(\psi_{1i}\psi_{2ix} - \psi_{2i}\psi_{1ix}) \Big), \tag{80}$$

where a_i, $i = 1, 2, ..., N$, are constants and ψ_{1i}, ψ_{2i} are independent solutions of the Lax pair (3)–(6) with $\lambda = 0$.

1. Lower-dimensional integrable systems

Usually, every symmetry of a higher dimensional model can lead the original one to its lower form. Now, considering

$$u_x = -\sum_{i=1}^{N} 2a_i(\psi_{1i}\psi_{2i})_x, \quad v_x = \sum_{i=1}^{N} a_i(\psi_{1i}\psi_{2ix} - \psi_{2i}\psi_{1ix}) \tag{81}$$

as a symmetry constraint condition and acting it on the x-part of Lax pair (3)–(4) for $\psi_1 = \psi_{1i}$ and $\psi_2 = \psi_{2i}$, we have the lower dimensional 2N-component differential system

$$\psi_{1i}\psi_{1ixxx} - \psi_{1ix}\psi_{1ixx} - \sum_{n=1}^{N} a_n(\psi_{1n}\psi_{2nx} + 3\psi_{2n}\psi_{1nx})\psi_{1i}^2 = 0, \tag{82}$$

$$\psi_{2i}\psi_{2ixxx} - \psi_{2ix}\psi_{2ixx} - \sum_{n=1}^{N} a_n(\psi_{2n}\psi_{1nx} + 3\psi_{1n}\psi_{2nx})\psi_{2i}^2 = 0. \tag{83}$$

When $N = 1$, $\psi_{10} = \psi_1$, and $\psi_{20} = \psi_2$, (82) and (83) become the following differential equations:

$$\psi_1\psi_{1xxx} - \psi_{1x}\psi_{1xx} - 3a_1\psi_1^2\psi_2\psi_{1x} - a_1\psi_1^3\psi_{2x} = 0, \tag{84}$$

$$\psi_2\psi_{2xxx} - \psi_{2x}\psi_{2xx} - 3a_1\psi_2^2\psi_1\psi_{2x} - a_1\psi_2^3\psi_{1x} = 0, \tag{85}$$

More specifically, when $\psi_1 = \psi_2$, (84) and (85) are reduced to the elliptic equation

$$\psi_{1x}^2 = b_0 + b_1\psi_1^2 + a_1\psi_1^4,$$

where b_0 and b_1 are arbitrary constants. So we call system (84)–(85) a coupled elliptic equation.

Substituting (81) into (5) and (6) and using (82)–(83), the t-part of the Lax pair becomes the generalized 2N-component coupled modified KdV (mKdV) system

$$\psi_{1it} = -\psi_{1ixxx} + 3\sum_{n=1}^{N} a_n(\psi_{1n}\psi_{2n})_x\psi_{1i} - 3\sum_{n=1}^{N} a_n\partial_x^{-1}(\psi_{1n}\psi_{2nx} - \psi_{2n}\psi_{1nx})\psi_{1ix}, \tag{86}$$

$$\psi_{2it} = -\psi_{2ixxx} + 3\sum_{n=1}^{N} a_n(\psi_{1n}\psi_{2n})_x\psi_{2i} + 3\sum_{n=1}^{N} a_n\partial_x^{-1}(\psi_{1n}\psi_{2nx} - \psi_{2n}\psi_{1nx})\psi_{2ix}. \tag{87}$$

When $N = 1$, $a_1 = 1$, $\psi_{10} = \psi_1$, and $\psi_{20} = \psi_2$, (86) and (87) are the generalized coupled mKdV equations,

$$\begin{aligned} \psi_{1t} &= -\psi_{1xxx} + 3\psi_1(\psi_1\psi_2)_x - 3v\psi_{1x}, \\ \psi_{2t} &= -\psi_{2xxx} + 3\psi_2(\psi_1\psi_2)_x + 3v\psi_{2x}, \\ v_x &= \psi_1\psi_{2x} - \psi_2\psi_{1x}. \end{aligned} \tag{88}$$

2. Higher-dimensional integrable systems

It is obvious that system (1)–(2) are invariant under the internal parameter translation, say y translation. That is to say (u_y, v_y) is also a symmetry of the HS-cKdV system. So we take

$$u_y = -\sum_{i=1}^{N} 2a_i(\psi_{1i}\psi_{2i})_x, \quad v_y = \sum_{i=1}^{N} a_i(\psi_{1i}\psi_{2ix} - \psi_{2i}\psi_{1ix}), \tag{89}$$

as a generalized symmetry constraint condition.

Substituting (89) into the x-part of the Lax pair (3)–(4) for $\psi_1 = \psi_{1i}$ and $\psi_2 = \psi_{2i}$ yields a higher dimensional 2N-component differential system

$$\psi_{1ixx} - \psi_{1i}\sum_{n=1}^{N} a_n\partial_y^{-1}(\psi_{1n}\psi_{2nx} + 3\psi_{2n}\psi_{1nx}) = 0, \tag{90}$$

$$\psi_{2ixx} - \psi_{2i}\sum_{n=1}^{N} a_n\partial_y^{-1}(\psi_{2n}\psi_{1nx} + 3\psi_{1n}\psi_{2nx}) = 0. \tag{91}$$

When we take $N = 1$, $a_1 = 1$, $\psi_{10} = \psi_1$, $\psi_{20} = \psi_2$, and use the same Miura transformation (76), the system (90)–(91) is reduced to the same coupled sinh-Gordon equation (77).

Considering (89) and the x-part of the Lax pair (3)–(4) for $\psi_1 = \psi_{1i}$ and $\psi_2 = \psi_{2i}$ produces a higher dimensional 2N-component differential system

$$\psi_{1it} = -\psi_{1ixxx} + 3\sum_{n=1}^{N} a_n\partial_y^{-1}(\psi_{1n}\psi_{2n})_{xx}\psi_{1i} - 3\sum_{n=1}^{N} a_n\partial_y^{-1}(\psi_{1n}\psi_{2nx} - \psi_{2n}\psi_{1nx})\psi_{1ix}, \tag{92}$$

$$\psi_{2it} = -\psi_{2ixxx} + 3\sum_{n=1}^{N} a_n\partial_y^{-1}(\psi_{1n}\psi_{2n})_{xx}\psi_{2i} + 3\sum_{n=1}^{N} a_n\partial_y^{-1}(\psi_{1n}\psi_{2nx} - \psi_{2n}\psi_{1nx})\psi_{2ix}. \tag{93}$$

Taking $N = 1$, $a_1 = 1$, $\psi_{10} = \psi_1$, and $\psi_{20} = \psi_2$, (92) and (93) become

$$\psi_{1t} = -\psi_{1xxx} + 3\psi_1\partial_y^{-1}(\psi_1\psi_2)_{xx} - 3\psi_{1x}\partial_y^{-1}(\psi_1\psi_{2x} - \psi_2\psi_{1x}), \tag{94}$$

$$\psi_{2t} = -\psi_{2xxx} + 3\psi_2\partial_y^{-1}(\psi_1\psi_2)_{xx} + 3\psi_{2x}\partial_y^{-1}(\psi_1\psi_{2x} - \psi_2\psi_{1x}). \tag{95}$$

For $\psi_1 = \psi_2$, $y = x$, system (94) is reduced to the mKdV equation. So we call system (94)–(95) a coupled modified ANNV equation.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we have studied nonlocal symmetries of the HS-cKdV system from DT and obtained the following results.

Starting from the known DT of the HS-cKdV system, several different types of nonlocal symmetries are derived directly. Infinitely many nonlocal symmetries can be obtained by introducing the internal parameters.

Then we focused on how to localize the nonlocal symmetry related to the DT. Through introducing five potential variables, the nonlocal symmetry obtained from the DT is successfully localized to some local ones. This procedure leads the original HS-cKdV system to be extended into the prolonged system. Applying the Lie's first theorem to these local symmetries, the corresponding finite symmetry transformations are derived. These transformations and the initial DT have different group parameters but possess the same infinitesimal forms. Meanwhile, we observe that the DT of the HS-cKdV system is closely related to the Möbius transformation of the Schwartz form. Moreover, it is necessary to point out that according to Ref. 2, if the prolonged system is viewed as a potential system, nonlocal symmetry can be also obtained by computing the general Lie symmetry.

For the prolonged system, the general Lie symmetry transformations and similarity reductions are considered by using the Lie point symmetry method. Several classes of exact interaction solutions among solitons and other complicated waves including periodic cnoidal waves, Painlevé waves, and rational waves are presented. In fact, it is difficult to obtain these types of solutions from the original DT by solving the spectral problem, especially with the seed solution which is neither a constant nor a soliton solution. This may provide us with an alternative way to construct some new solutions for the integrable models with the known DT.

The final work of this paper is to extend the HS-cKdV system to new integrable models from the nonlocal symmetry related to the DT in two aspects. Usually, the existence of infinitely many symmetries suggests the existence of integrable hierarchies. By introducing the internal parameter, the negative HS-cKdV hierarchy is obtained without using the inverse of the known recursion operator. With the help of one kind of special Miura transformation, this hierarchy is transformed to a coupled sinh-Gordon hierarchy. In addition, symmetry constraint approach is one of the most powerful tools to give new integrable models from known ones. Using this method, one can obtain not only the lower dimensional integrable models from higher ones, but also the higher dimensional integrable models from lower ones. Here, both lower and higher dimensional integrable models related to the HS-cKdV system are presented by means of the nonlocal symmetry constraint method.

Using DT to search for nonlocal symmetries of integrable models and then applying them to construct exact solutions and integrable models, are both of considerable interest. However, the concrete integrability for the given lower and higher dimensional models is unknown. Moreover, in Ref. 21, Lou *et al.* utilized the multi-DT to obtain some interesting nonlocal symmetries and constructed various lower and higher dimensional integrable models. A natural problem is how to use those nonlocal symmetries related to the multi-DT to obtain more novel solutions. These matters deserve further study.

ACKNOWLEDGMENTS

We would like to express our sincere thanks to Professor S. Y. Lou and Dr. Y. Q. Li and other members of our discussion group for their valuable comments. This work is supported by the National Natural Science Foundation of China (No. 11275072), Research Fund for the Doctoral Program of Higher Education of China (No. 20120076110024), Innovative Research Team Program of the National Natural Science Foundation of China (No. 61321064), Shanghai Knowledge Service Platform Project (No. ZF1213), and Talent Fund and K. C. Wong Magna Fund in Ningbo University.

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