Integrability of an extended (2+1)-dimensional shallow water wave equation with Bell polynomials*

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We investigate the extended (2+1)-dimensional shallow water wave equation. The binary Bell polynomials are used to construct bilinear equation, bilinear Bäcklund transformation, Lax pair, and Darboux covariant Lax pair for this equation. Moreover, the infinite conservation laws of this equation are found by using its Lax pair. All conserved densities and fluxes are given with explicit recursion formulas. The N-soliton solutions are also presented by means of the Hirota bilinear method.

Keywords: binary Bell polynomials, Darboux covariant Lax pair, bilinear Bäcklund transformation, infinite conservation laws

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1. Introduction

As is well known, the investigation of integrability for a nonlinear evolution equation (NLEE) can be regarded as a pretest and the first step towards its exact solvability. Among the methods employed to study the integrability of NLEEs is the Hirota bilinear method,[1–7] which is a powerful tool for constructing soliton solutions of NLEEs. The key of the Hirota bilinear method is the construction of a bilinear form for a given NLEE, however, this process is not trivial. It relies on choosing suitable variable transformations, but there is no general rule to find these transformations. Lambert and his co-workers[8,9] established a deep link between the Bell polynomials and the integrability of NLEEs, which provides us a lucid and systematic way to study the bilinear form, bilinear Bäcklund transformation (BT), Lax pair, and Darboux covariant Lax pair for NLEEs. By using the binary Bell polynomial approach, one may obtain, on one hand, results such as the Bell polynomial expressions (in the P or $\mathscr{Y}$ polynomial form), Bell polynomial type BT, and Lax pair, and on the other hand, the connection between the Bell polynomials and the Hirota bilinear method, namely, the Bell polynomial expressions can be cast into the bilinear form, and the Bell polynomial type BT can be mapped into the bilinear BT. Then, both the Bell polynomial type and the bilinear BT can lead to the corresponding Lax pairs. Furthermore, the Darboux covariant Lax pair can also be obtained under a certain gauge transformation, and its form is invariant. More recently, this method is extended to investigate the nonisospectral and variable-coefficient KdV equation, the supersymmetric KdV equation, the supersymmetric two-Boson equation, the extended Lotka–Volterra equation, etc.[10–18]

In fluid dynamics, the shallow water wave equation is utilized as a mathematical description of regular and generalized solitary waves in shallow water. The higher-order dispersive and higher-dimensional generalized nonlinear models are useful in analyzing and obtaining the modulation theory, the existence and stability of solitary waves, bores, and shocks, as well as other integrable properties.[19] The purpose of this paper is to use the binary Bell polynomial approach and the Hirota bilinear method to study the integrability of the following extended (2+1)-dimensional shallow water wave equation:[20]

$$u_{x,t} - 4u_{x,y}u_x - 2u_{2x}u_y + u_{3x,y} + \alpha u_{x,y} = 0, \tag{1}$$

where (x, y) and t are the scaled space and time coordinates, respectively, $u(x, y, t)$ is the amplitude or elevation of the relevant wave, the subscripts denote the corresponding derivatives, and α is an arbitrary constant. Equation (1) can be used to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y axis with a long wave propagating along the x axis. As a reduced equation of Eq. (1), the integrability of the so-called breaking soliton equation has been investigated in many literatures.[21–31] But for Eq. (1), to the best of our knowledge, it has not been studied in detail except for a few previous works.[20] In addition, the study on the conservation laws of (2+1)-dimensional equations is still lacking in contrast with the (1+1)-dimensional case. The existence of infinite local conservation laws can be considered as one of the many remarkable properties that characterize the soliton equations, namely, the more conservation laws one finds, the closer one gets to the complete solution. Generally, the infi-

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nite conservation laws or conserved quantities for both continuous systems and discrete systems can be derived from BT,[32] Lax pair,[32–34] formal solutions of eigenfunctions,[35] scattering problem,[36] trace identity,[37] quasi-diffierential operator based on the Sato theory,[38] etc. Based on the binary Bell polynomials, Fan[10] introduced a method for deriving the infinite conservation laws of NLEEs, the key is to turn the associated Lax pair into a Riccati type equation and a divergence-type equation.

This paper is arranged as follows. With the help of the binary Bell polynomials and the symbolic computation, we systematically construct the bilinear form, N-soliton solutions, bilinear BT, Lax pair, Darboux covariant Lax pair, and the infinite conservation laws of Eq. (1) in Sections 2–5, respectively. The conclusion is given in Section 6.

2. Bilinear representation and N-soliton solutions

In this section, we construct the bilinear form and N-soliton solutions of Eq. (1) by using the binary Bell polynomials and the Hirota bilinear method, respectively. The main tool used in this paper is a class of generalized multi-dimensional binary Bell polynomials, the details can be found in the work of Bell *et al.*[8,9,39]

Theorem 1 Under the following transformation:

$$u=-2(\ln F)_x, \tag{2}$$

equation (1) can be bilinearized into

$$(D_x^4+D_xD_z)F\cdot F=0, \tag{3a}$$
$$\left(D_xD_t+\frac{2}{3}D_x^3D_y-\frac{1}{3}D_yD_z+\alpha D_xD_y\right)F\cdot F=0, \tag{3b}$$

where $F=F(x,y,z,t)$, α is an arbitrary constant, and D denotes the Hirota operator.

Proof Invariance of Eq. (1) under the scale transformations

$$x\to\lambda x,\ \ y\to\lambda^{r_1}y,\ \ t\to\lambda^{r_2}t,\ \ u\to\lambda^{r_3}u \tag{4}$$

with $r_3=-1$ shows that a dimensionless field $q=q(x,y,z,t)$ can be related to field u by setting

$$u=cq_x, \tag{5}$$

where c is a free constant to be chosen so that equation (1) connects with the P-polynomials.[8] Substituting Eq. (5) into Eq. (1) and integrating it with respect to x yield

$$\begin{aligned}E(q)\ =\ &q_{x,t}+\frac{2}{3}(q_{3x,y}+3q_{2x}q_{x,y})\\&+\frac{1}{3}\partial_x^{-1}\partial_y(q_{4x}+3q_{2x}^2)+\alpha q_{x,y}=0,\end{aligned} \tag{6}$$

if we set $c=-1$ according to the P-polynomials.[8] Then, we introduce an auxiliary variable z and impose a subsidiary constraint condition

$$q_{4x}+3q_{2x}^2+q_{x,z}=0, \tag{7}$$

which implies that Eq. (6) can be rewritten in the local form by eliminating the effect of the integration ∂_x^{-1}.

In terms of the subsidiary constraint condition (7), equation (6) becomes

$$E(q)=q_{x,t}+\frac{2}{3}(q_{3x,y}+3q_{2x}q_{x,y})-\frac{1}{3}q_{y,z}+\alpha q_{x,y}=0. \tag{8}$$

Now according to the P-polynomials,[8] equations (7) and (8) are cast into a pair of equations in the form of P-polynomials

$$P_{4x}(q)+P_{x,z}(q)=0, \tag{9a}$$
$$P_{x,t}(q)+\frac{2}{3}P_{3x,y}(q)-\frac{1}{3}P_{y,z}(q)+\alpha P_{x,y}=0. \tag{9b}$$

Finally, by using the the identity which links the $\mathscr{Y}$-polynomials and Hirota D operator,[8] and the following change of dependent variable

$$q=2\ln F\Leftrightarrow u=-q_x=-2(\ln F)_x, \tag{10}$$

P-polynomials type equation (9) immediately produces the bilinear system (3) of Eq. (1).

By using the Hirota bilinear method and the symbolic computation, the N-soliton solutions of Eq. (1) are obtained as

$$\begin{aligned}&u=-2\left[\ln\left(\sum_{\mu=0,1}\exp\left(\sum_{j=1}^{n}\mu_j\xi_j+\sum_{1\le j<l}^{n}\mu_j\mu_lA_{jl}\right)\right)\right]_x,\\&\xi_j=\kappa_jx-\frac{\omega_j}{\alpha+\kappa_j^2}y+\omega_jt,\ \ \alpha+\kappa_j^2\neq0,\\&\mathrm{e}^{A_{jl}}=\frac{(\kappa_j-\kappa_l)^2}{(\kappa_j+\kappa_l)^2},\ \ j<l,\ \ j,l=1,2,3,\ldots,\end{aligned} \tag{11}$$

where $\sum_{\mu=0,1}$ indicates the summation over all possible combinations of $\mu_j=0,1$ $(j=1,2,\ldots)$.

3. Bilinear BT and associated Lax pair

Theorem 2 If F is a known solution of bilinear equation (3), then $G=G(x,y,z,t)$ satisfying

$$(D_x^2-\lambda)F\cdot G=0, \tag{12a}$$
$$(D_t+D_x^2D_y+3\lambda D_y+\alpha D_y-\mu)F\cdot G=0 \tag{12b}$$

is a novel solution of bilinear equation (3) (μ is an arbitrary constant). Thus system (12) is a bilinear BT for Eq. (1).

Proof In order to obtain the bilinear BT of Eq. (1), suppose that

$$q=2\ln G,\ \ \bar{q}=2\ln F \tag{13}$$

are two different solutions of Eq. (6). Then, we consider the two-field condition

$$\begin{aligned} & E(\bar{q})-E(q) \\ = & (\bar{q}-q)_{x,t}+\frac{2}{3}[(\bar{q}-q)_{3x,y}+3(\bar{q}-q)_{2x}(\bar{q}-q)_{x,y}] \\ & +\frac{1}{3}\partial_x^{-1}\partial_y[(\bar{q}-q)_{4x}+3(\bar{q}-q)_{2x}^2]+\alpha(\bar{q}-q)_{x,y} \\ = & 0. \end{aligned} \tag{14}$$

The two-field condition (14) can be regarded as an ansatz for a bilinear BT and may produce the required transformation under appropriate additional constraints. To this end, we introduce two new variables

$$w=\frac{(\bar{q}+q)}{2}=\ln(FG),\quad v=\frac{(\bar{q}-q)}{2}=\ln\left(\frac{F}{G}\right), \tag{15}$$

with which condition (14) can be rewritten as

$$\begin{aligned} E(\bar{q})-E(q) &= E(w+v)-E(w-v) \\ &= 2v_{x,t}+2\alpha v_{x,y}+2v_{3x,y}+4w_{2x}v_{x,y}+4w_{x,y}v_{2x} \\ &\quad +4\partial_x^{-1}(w_{2x}v_{2x,y}+w_{2x,y}v_{2x}) \\ &= 2\partial_x[\mathscr{Y}_t(v)+\alpha\mathscr{Y}_y(v)+\mathscr{Y}_{2x,y}(v,w)]+R(v,w) \\ &= 0, \end{aligned} \tag{16}$$

with

$$\begin{aligned} R(v,w) &= -2\partial_x[(w_{2x}+v_x^2)v_y]+4w_{2x}v_{x,y}-4w_{2x,y}v_x \\ &\quad +4\partial_x^{-1}[w_{2x}v_{2x,y}+w_{2x,y}v_{2x}]. \end{aligned} \tag{17}$$

In order to decouple the two-field condition (16) into a pair of constraints, we impose such a constraint which enables us to express $R(v,w)$ in the form of x-derivative of $\mathscr{Y}$-polynomials.[8] The simplest possible choice of such a constraint is

$$\mathscr{Y}_{2x}(v,w)=w_{2x}+v_x^2=\lambda, \tag{18}$$

where λ is an arbitrary constant. Thus, $R(v,w)$ can be expressed as

$$R(v,w)=6\lambda v_{x,y}=6\lambda\partial_x\mathscr{Y}_y(v) \tag{19}$$

under the following relations:

$$w_{2x}=\lambda-v_x^2,\quad w_{3x}=-2v_xv_{2x},\quad w_{2x,y}=-2v_xv_{x,y}. \tag{20}$$

Then, combining relations (16)–(19), we deduce a coupled system of $\mathscr{Y}$-polynomials

$$\mathscr{Y}_{2x}(v,w)-\lambda=0, \tag{21a}$$
$$\partial_x[\mathscr{Y}_t(v)+\mathscr{Y}_{2x,y}(v,w)+\alpha\mathscr{Y}_y(v)+3\lambda\mathscr{Y}_y(v)]=0, \tag{21b}$$

where the second equation is useful for constructing conservation laws later. Based on the identity linking $\mathscr{Y}$-polynomials and Hirota D operator,[8] system (21) immediately leads to the bilinear BT (12).

Theorem 3 Under condition (21), Eq. (1) admits a Lax pair

$$\psi_{2x}+q_{2x}\psi-\lambda\psi=0, \tag{22a}$$
$$\psi_t-2u_y\psi_x-u_x\psi_y+\psi_{2x,y}+(\alpha+3\lambda)\psi_y-\mu\psi=0, \tag{22b}$$

where u is a solution of Eq. (1).

Proof In order to linearize the Bell polynomial system (21) into a Lax pair, we use the Hopf–Cole transformation $v=\ln\psi$, then we have

$$\begin{aligned} &\mathscr{Y}_t(v)=\frac{\psi_t}{\psi},\quad \mathscr{Y}_{2x}(v,w)=q_{2x}+\frac{\psi_{2x}}{\psi}, \\ &\mathscr{Y}_{2x,y}(v,w)=\frac{2q_{x,y}\psi_x}{\psi}+\frac{q_{2x}\psi_y}{\psi}+\frac{\psi_{2x,y}}{\psi}, \end{aligned} \tag{23}$$

which make system (21) linearized into a Lax pair with double parameters λ and μ

$$L_1\psi\equiv(\partial_x^2+q_{2x})\psi=\lambda\psi, \tag{24a}$$
$$\begin{aligned} \psi_t+L_2\psi &\equiv [\partial_t+\partial_y\partial_x^2+2q_{x,y}\partial_x \\ &+ (q_{2x}+\alpha+3\lambda)\partial_y-\mu]\psi=0, \end{aligned} \tag{24b}$$

this is equivalent to

$$\psi_{2x}-u_x\psi-\lambda\psi=0, \tag{25a}$$
$$\psi_t-2u_y\psi_x-u_x\psi_y+\psi_{2x,y}+(\alpha+3\lambda)\psi_y-\mu\psi=0. \tag{25b}$$

It is easy to check that the integrability condition of system (25) is

$$\begin{aligned} & [L_1-\lambda,\partial_t+L_2] \\ = & u_{x,t}-4u_{x,y}u_x-2u_{2x}u_y+u_{3x,y}+\alpha u_{x,y}=0, \end{aligned} \tag{26}$$

which is just Eq. (1). Thus we call system (22) the Lax pair of Eq. (1).

4. Darboux covariant Lax pair

Theorem 4 By using the associated Lax pair (22) and assuming that parameter λ is an arbitrary constant, equation (1) admits a kind of Darboux covariant Lax pair as follows:

$$\tilde{L}_1\psi=\lambda\psi,\quad \tilde{L}_1=\partial_x^2+\tilde{q}_{2x}, \tag{27a}$$
$$\begin{aligned} &(\partial_t+\tilde{L}_{2,\mathrm{cov}})\psi=0, \\ &\tilde{L}_{2,\mathrm{cov}}=4\partial_y\partial_x^2+2\tilde{q}_{x,y}\partial_x+(4\tilde{q}_{2x}+\alpha)\partial_y+3\tilde{q}_{2x,y}-\mu, \end{aligned} \tag{27b}$$

whose form is Darboux covariant, namely,

$$T(L_1-\lambda)(q)T^{-1}=(\tilde{L}_1-\lambda)(\tilde{q}), \tag{28a}$$
$$T(\partial_t+L_{2,\mathrm{cov}})(q)T^{-1}=(\partial_t+\tilde{L}_{2,\mathrm{cov}})(\tilde{q}), \tag{28b}$$

with $\tilde{q}=q+2\ln\psi$, under a certain gauge transformation

$$T=\psi\partial_x\psi^{-1}=\partial_x-\sigma,\quad \sigma=\partial_x\ln\psi. \tag{29}$$

The integrability condition of Darboux covariant Lax pair (27) precisely gives rise to Eq. (1)

$$\begin{aligned}&[L_1-\lambda,\partial_t+L_{2,\mathrm{cov}}]\\ =&\ u_{x,t}-4u_{x,y}u_x-2u_{2x}u_y+u_{3x,y}+\alpha u_{x,y}\\ =&\ 0,\end{aligned}\tag{30}$$

which implies that Lax equation (27) also gives a Lax pair for Eq. (1).

Proof Suppose that ψ is a solution of eigenvalue equation (24a). It is well-known that the gauge transformation

$$T=\psi\partial_x\psi^{-1}=\partial_x-\sigma,\quad \sigma=\partial_x\ln\psi \tag{31}$$

maps operator $L_1(q)-\lambda$ into a similar operator

$$T(L_1(q)-\lambda)T^{-1}=\tilde{L}_1(\tilde{q})-\lambda, \tag{32}$$

which satisfies the covariance condition

$$\tilde{L}_1(\tilde{q})=L_1(\tilde{q}=q+\Delta q), \tag{33}$$

with $\Delta q=2\ln\psi$. But it can be verified that a similar property does not hold for evolution equation (24b). Howerer, we can find another third-order operator $L_{2,\mathrm{cov}}(q)$ with appropriate coefficients, such that $\partial_t+L_{2,\mathrm{cov}}(q)$ can be mapped by gauge transformation (31) into a similar operator $\tilde{L}_{2,\mathrm{cov}}(\tilde{q})$ which satisfies the covariance condition

$$\tilde{L}_{2,\mathrm{cov}}(\tilde{q})=L_{2,\mathrm{cov}}(\tilde{q}=q+\Delta q). \tag{34}$$

Suppose that ψ is a solution of the following Lax pair:

$$L_1\psi=\lambda\psi,\quad \psi_t+L_{2,\mathrm{cov}}\psi=0, \tag{35}$$

where

$$L_{2,\mathrm{cov}}=4\partial_y\partial_x^2+b_1\partial_x+b_2\partial_y+b_3, \tag{36}$$

and b_1,b_2, and b_3 are functions to be determined. It suffices that the transformation T maps operator $\partial_t+L_{2,\mathrm{cov}}$ into a similar one

$$T(\partial_t+L_{2,\mathrm{cov}})T^{-1}=\partial_t+\tilde{L}_{2,\mathrm{cov}}, \tag{37}$$

where

$$\tilde{L}_{2,\mathrm{cov}}=4\partial_y\partial_x^2+\tilde{b}_1\partial_x+\tilde{b}_2\partial_y+\tilde{b}_3, \tag{38}$$

and $\tilde{b}_1,\tilde{b}_2$, and $\tilde{b}_3$ satisfy the following covariant condition:

$$\tilde{b}_j=b_j(q)+\Delta b_j=b_j(q+\Delta q),\quad j=1,2,3. \tag{39}$$

By virtue of Eqs. (35) and (37), we find

$$\Delta b_1=\tilde{b}_1-b_1=4\sigma_y, \tag{40a}$$

$$\Delta b_2=\tilde{b}_2-b_2=8\sigma_x, \tag{40b}$$

$$\Delta b_3=\tilde{b}_3-b_3=8\sigma_{x,y}+\sigma\Delta b_1+b_{1,x}, \tag{40c}$$

and

$$b_{2,x}+4\sigma_{2x}+8\sigma\sigma_x=0, \tag{41a}$$

$$\begin{aligned}&\sigma_t+4\sigma_{2x,y}+b_1\sigma_x+12\sigma_x\sigma_y+b_2\sigma_y+b_{3,x}\\ &+\ 8\sigma\sigma_{x,y}+\sigma b_{1,x}+4\sigma^2\sigma_y=0.\end{aligned}\tag{41b}$$

According to relation (39), we need to determine b_1,b_2, and b_3 in the form of polynomials in terms of derivatives of q

$$b_j=F_j(q,q_x,q_y,q_{x,y},q_{2x},q_{2y},q_{2x,y},\ldots),\quad j=1,2,3, \tag{42}$$

such that

$$\Delta F_j=F_j(q+\Delta q,q_x+\Delta q_x,\ldots)-F_j(q,q_x,q_y,\ldots)=\Delta b_j, \tag{43}$$

with $\Delta q_{kx,ly}=2(\ln\psi)_{kx,ly}$, $(k,l=1,2,\ldots)$, and Δb_j being determined by relations (40) and (41). In order to satisfy the first condition

$$\begin{aligned}\Delta b_1\ &=\ \Delta F_1=F_{1,q}\Delta q+F_{1,q_x}\Delta q_x+F_{1,q_y}\Delta q_y+\cdots\\ &=\ 4\sigma_y=2\Delta q_{x,y},\end{aligned}\tag{44}$$

we choose

$$b_1=F_1(q_{x,y})=2q_{x,y}+c_1, \tag{45}$$

with c_1 being an arbitrary constant. Proceeding in the same way, function b_2 can be determined as

$$b_2=F_1(q_{2x})=4q_{2x}+c_2. \tag{46}$$

with c_2 being an arbitrary constant.

We can find that relation (40c) contains the term

$$b_{1,x}=2q_{2x,y}, \tag{47}$$

which should be eliminated so that Δb_3 admits the form (43). Following the eigenvalue equation (35), we can find the relation

$$q_{2x,y}=-\sigma_{x,y}-2\sigma\sigma_y. \tag{48}$$

Substituting Eqs. (45) and (48) into Eq. (40c) yields

$$\Delta b_3=6\sigma_{x,y}=3\Delta q_{2x,y}. \tag{49}$$

It can be verified that the third condition

$$\Delta F_3=F_{3,q}\Delta q+F_{3,q_x}\Delta q_x+F_{3,q_y}\Delta q_y+\cdots=\Delta b_3 \tag{50}$$

can be satisfied if we choose

$$b_3=F_3(q_{2x,y})=3q_{2x,y}+c_3, \tag{51}$$

where c_3 is an arbitrary constant.

Setting $c_1=0, c_2=\alpha, c_3=-\mu$ in Eqs. (45), (46), and (51), we find from Eq. (35) the following Darboux covariant evolution equation:

$$\psi_t + L_{2,\mathrm{cov}}\psi = 0,$$
$$L_{2,\mathrm{cov}} = 4\partial_y\partial_x^2 + 2q_{x,y}\partial_x + (4q_{2x}+\alpha)\partial_y + 3q_{2x,y} - \mu, \quad (52)$$

which is in agreement with Eq. (41). Moreover, the relation between operators $L_{2,\mathrm{cov}}$ and L_2 is given by

$$L_{2,\mathrm{cov}} = L_2 + 3\partial_y(L_1-\lambda). \quad (53)$$

The integrability condition (30) of Darboux covariant Lax pair (35) precisely gives rise to Eq. (1)

In contrast to the original Lax procedure, the above technique of generating Eq. (1) is through the construction of Darboux covariant linear systems. The results suggest that we can obtain higher operators in a similar way, which are Darboux covariant with respect to L_1, so we can obtain higher-order members of Eq. (1).

5. Infinite conservation laws

Theorem 5 Equation (1) admits infinite conservation laws

$$\mathscr{I}_{n,t} + \mathscr{F}_{n,x} + \mathscr{G}_{n,y} = 0, \quad n = 1,2,\ldots. \quad (54)$$

The conversed densities $\mathscr{I}_n'$s are given by the recursion formulas

$$\mathscr{I}_1 = -\frac{1}{2}q_{2x} = \frac{1}{2}u_x, \quad (55a)$$
$$\mathscr{I}_2 = -\frac{1}{2}\mathscr{I}_{1,x} = -\frac{1}{4}u_{2x}, \quad (55b)$$
$$\mathscr{I}_3 = -\frac{1}{2}(\mathscr{I}_1^2 + \mathscr{I}_{2,x}) = \frac{1}{8}(u_{3x}+u_x^2), \quad (55c)$$
$$\mathscr{I}_4 = -\frac{1}{2}(2I_1I_2 + \mathscr{I}_{3,x}) = \frac{1}{4}u_xu_{2x} - \frac{1}{16}u_{4x}, \quad (55d)$$
$$\cdots, \quad (55e)$$
$$\mathscr{I}_n = -\frac{1}{2}\mathscr{I}_{n,x} - \frac{1}{2}\sum_{k=1}^{n} I_kI_{n-k}, \quad n=2,3,\ldots, \quad (55f)$$

the first fluxes $\mathscr{F}_n'$s are given by

$$\mathscr{F}_1 = -\frac{1}{2}u_{2x,y} - u_xu_y, \quad (56a)$$
$$\mathscr{F}_2 = \frac{1}{2}u_{2x}u_y - \frac{1}{2}u_xu_{x,y} + \frac{1}{4}u_{3x,y}, \quad (56b)$$
$$\cdots, \quad (56c)$$
$$\mathscr{F}_n = -2u_y\mathscr{I}_n + 2I_{n+1,y}, \quad n=2,3,\ldots, \quad (56d)$$

and the second fluxes $\mathscr{G}_n'$s are given by

$$\mathscr{G}_1 = -\frac{1}{2}u_x^2 + \frac{\alpha}{2}u_x + u_{3x}, \quad (57a)$$
$$\mathscr{G}_2 = -\frac{1}{2}u_{4x} + \frac{3}{2}u_xu_{2x} - \frac{\alpha}{4}u_{2x}, \quad (57b)$$
$$\cdots, \quad (57c)$$
$$\mathscr{G}_n = I_{n,2x} + 4I_{n+2} + \alpha I_n + 2\sum_{k=1}^{n} I_kI_{n-k}, \quad n=2,3,\ldots. \quad (57d)$$

Proof The conservation laws have actually been hinted in two-field constraint system (21), which can be rewritten in the conserved form

$$w_{2x} + v_x^2 - \lambda = 0, \quad (58a)$$
$$\partial_t(v_x) + \partial_x(v_{2x,y} + 2v_xw_{x,y}) + \partial_y(\alpha v_x + 4\lambda v_x) = 0, \quad (58b)$$

by applying the relation

$$v_{x,t} = \partial_t\mathscr{Y}_x(v) = \partial_x\mathscr{Y}_t(v), \quad (59a)$$
$$v_{x,y} = \partial_x\mathscr{Y}_y(v) = \partial_y\mathscr{Y}_x(v). \quad (59b)$$

By introducing a new potential function

$$\eta = (\bar{q}_x - q_x)/2, \quad (60)$$

it follows from relation (15) that

$$v_x = \eta, \quad w_x = q_x + \eta. \quad (61)$$

Substituting Eq. (61) into Eq. (58), we get a Riccati-type equation

$$\eta_x + \eta^2 + q_{2x} = \lambda, \quad (62)$$

and a divergence-type equation

$$\eta_t + \partial_x(\eta_{x,y} + 2\eta q_{x,y} + 2\eta\eta_y) + \partial_y(4\lambda\eta + \alpha\eta) = 0. \quad (63)$$

Suppose that $\lambda = \varepsilon^2$, under the transformation $\eta = \bar{\eta} + \varepsilon$, we have

$$\bar{\eta}_x + \bar{\eta}^2 + 2\varepsilon\bar{\eta} + q_{2x} = 0, \quad (64)$$
$$\bar{\eta}_t + \partial_x(2\bar{\eta}q_{x,y} + 2q_{x,y}\varepsilon + 2\bar{\eta}_y\varepsilon) + \partial_y(\bar{\eta}_{2x} + 2\bar{\eta}\bar{\eta}_x + 4\bar{\eta}\varepsilon^2 + \alpha\bar{\eta}) = 0. \quad (65)$$

Inserting the expansion

$$\bar{\eta} = \sum_{n=1}^{\infty}\mathscr{I}_n(q, q_x, \ldots)\varepsilon^{-n} \quad (66)$$

into Eq. (64) and equating the coefficients for powers of ε, we explicitly obtain the recursion relation (55) for the conserved densities $\mathscr{I}_n'$s.

In addition, substituting expansion (66) into the divergence-type equation (65) leads to

$$\sum_{n=1}^{\infty} I_{n,t}\varepsilon^{-n} + \partial_x\Big(2q_{x,y}\sum_{n=1}^{\infty} I_n\varepsilon^{-n} + 2q_{x,y}\varepsilon + 2\varepsilon\sum_{n=1}^{\infty} I_{n,y}\varepsilon^{-n}\Big) + \partial_y\Big(\sum_{n=1}^{\infty} I_{n,2x}\varepsilon^{-n} + 2\sum_{n=1}^{\infty} I_n\varepsilon^{-n}\sum_{n=1}^{\infty} I_{n,x}\varepsilon^{-n} + 4\varepsilon^2\sum_{n=1}^{\infty} I_n\varepsilon^{-n} + \alpha\sum_{n=1}^{\infty} I_n\varepsilon^{-n}\Big) = 0, \quad (67)$$

which provides us the infinite conservation laws (54)

$$\mathscr{I}_{n,t}+\mathscr{F}_{n,x}+\mathscr{G}_{n,y}=0,\ n=1,2,\ldots. \tag{68}$$

In Eq. (54), the conversed densities $\mathscr{I}_n'$s are given by recursion formulas (55), and the first fluxes $\mathscr{F}_n'$s and the second fluxes $\mathscr{G}_n'$s are obtained by Eqs. (56) and (57), respectively. The first equation of conservation law (54) is exactly Eq. (1).

6. Conclusion

The shallow water wave equations are of current interest in nonlinear mathematical physics, while Eq. (1) describes the (2+1)-dimensional interaction of a Riemann wave propagating along the y axis with a long wave propagating along the x axis in fluids. Employing the binary Bell polynomial approach and the Hirota bilinear method, we construct the bilinear form, N-soliton solutions, Lax pair, bilinear BT, Darboux covariant Lax pair, and infinite conservation laws of Eq. (1). In conclusion, this equation is completely integrable in the sense that it admits the above properties. It should be specially mentioned that once the suitable constraint condition can be found to construct the corresponding bilinear BT for a given NLEE, its other integrable properties, such as Lax pair, Darboux covariant Lax pair, and infinite conservation laws can be obtained more direct and systematic than using the Hirota bilinear method.

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