Pfaffian-Type Soliton Solution to a Multi-Component Coupled Ito Equation *

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Multi-soliton solution to a multi-component coupled Ito system is investigated based on the Hirota bilinear method. By virtue of the perturbation method, we firstly derive one- and two-soliton solutions for the coupled Ito system possessing four components. Then the multi-soliton solution for the multi-component coupled Ito system is summarized into a general form expressed by pfaffians. Finally, this general pfaffian-type soliton solution is proved by pfaffian techniques.

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The so-called Ito equation

$$u_{tt} + u_{xxxt} + 6u_x u_t + 3uu_{xt} + 3u_{xx} \int_{-\infty}^{x} u_t dx' = 0 \tag{1}$$

was introduced by Ito[1] as an extension of the KdV equation. It has the following bilinear form

$$D_t(D_t + D_x^3) f \cdot f = 0, \tag{2}$$

through the dependent variable transformation

$$u = 2(\ln f)_{xx},$$

where Hirota's bilinear differential operators[2] are defined by

$$D_x^n D_t^m (a \cdot b) = \Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big)^n \cdot \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\Big)^m a(x,t) b(x',t')\Big|_{x=x',t=t'}.$$

In Ref. [3], Tam *et al.* introduced two coupled Ito systems

$$\begin{aligned} u_t &= v_x, \\ v_t &= -2v_{xxx} - 6(uv)_x - 12ww_x + 6p_x, \\ w_t &= w_{xxx} + 3uw_x, \\ p_t &= p_{xxx} + 3up_x, \end{aligned} \tag{3}$$

and

$$\begin{aligned} u_t &= v_x, \\ v_t &= -2v_{xxx} - 6(uv)_x - 6(wp)_x, \\ w_t &= w_{xxx} + 3uw_x, \\ p_t &= p_{xxx} + 3up_x, \end{aligned} \tag{4}$$

which are reduced to the Ito Eq. (1) when $w = 0$ and $p = 0$, respectively. It was shown in Ref. [3] that the coupled system Eq. (3) with $p = 0$ is a special case of the $(6, 2)$-reduction of the two component BKP hierarchy. Moreover, 3- and 4-soliton solution for system Eq. (3) with $p = 0$ and 3-soliton solution for system Eq. (4) were derived. In addition, a new type of 3-soliton solution with constant boundary condition at infinity was found in Ref. [4] for system Eq. (3) with $p = 0$. Karasu *et al.*[5] applied Painlevé test to a generalized coupled Ito system and showed that this system passes the Painlevé test for integrability under five distinct cases, of which two are the coupled Ito systems (3) and (4). Recently, Balakhnev[6] proposed another vector coupled Ito system and presented its bilinear Bäcklund transformation and Lax pair.

In this Letter, we consider multi-soliton solution to a multi-component generalization of the coupled Ito system

$$\begin{aligned} &u_t = v_x, \\ &v_t + 2v_{xxx} + 6uv_x + 6vu_x = 6 \sum_{1\le j<k\le N} c_{jk}(\phi_j\phi_k)_x, \\ &\phi_{i,t} - \phi_{i,xxx} - 3u\phi_{i,x} = 0,\ i = 1, 2, \ldots, N, \end{aligned} \tag{5}$$

which is a special case of multi-component generalizations of the coupled Ito systems Eqs. (3) and (4) and was conjectured to be integrable in Ref. [5].

The aim of the present work is to find the multi-soliton solution to the multi-component coupled Ito system Eq. (5) by using the Hirota bilinear method. First, by introducing the dependent variable transformations

$$u = 2(\ln f)_{xx},\ v = 2(\ln f)_{xt},\ \phi_i = \frac{g_i}{f}. \tag{6}$$

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Equation (5) can be transformed to the following bilinear forms

$$(D_t^2 + 2D_tD_x^3)f\cdot f - 6\sum_{1\le j\le k\le N} c_{jk}g_jg_k = \lambda f^2, \quad (7)$$

$$(D_t - D_x^3)g_i\cdot f = 0,\ i = 1, 2, \ldots, N, \quad (8)$$

where λ is an integration constant.

For simplicity, we choose $\lambda = 0$ in Eqs. (7) and (8), and then we can obtain

$$(D_t^2 + 2D_tD_x^3)f\cdot f = 6\sum_{1\le j\le k\le N} c_{jk}g_jg_k, \quad (9)$$

$$(D_t - D_x^3)g_i\cdot f = 0,\ i = 1, 2, \ldots, N. \quad (10)$$

By letting $N = 3$, Eq. (5) are expressed explicitly as

$$\begin{aligned}&u_t = v_x,\\ &v_t + 2v_{xxx} + 6uv_x + 6vu_x\\ &= 6\Big(c_{12}\phi_1\phi_2 + c_{13}\phi_1\phi_3 + c_{23}\phi_2\phi_3\Big)_x,\\ &\phi_{i,t} - \phi_{i,xxx} - 3u\phi_{i,x} = 0,\ i = 1, 2, 3,\end{aligned}$$

and the corresponding bilinear Eqs. (9) and (10) become

$$\begin{aligned}&(D_t^2 + 2D_tD_x^3)f\cdot f\\ &= 6c_{12}g_1g_2 + 6c_{13}g_1g_3 + 6c_{23}g_2g_3,\\ &(D_t - D_x^3)g_i\cdot f = 0,\quad i = 1, 2, 3.\end{aligned}$$

By using the perturbational method,[2] the solution which has one soliton for $\phi_i (i = 1, 2, 3)$ is given by

$$\begin{aligned}g_1 &= \exp(\eta_1) + \alpha_{12}\alpha_{13}\alpha_{23}\beta_{23}\exp(\eta_1+\eta_2+\eta_3),\\ g_2 &= \exp(\eta_2) - \alpha_{12}\alpha_{13}\alpha_{23}\beta_{13}\exp(\eta_1+\eta_2+\eta_3),\\ g_3 &= \exp(\eta_3) + \alpha_{12}\alpha_{13}\alpha_{23}\beta_{12}\exp(\eta_1+\eta_2+\eta_3),\\ f &= 1 + \alpha_{12}\beta_{12}\exp(\eta_1+\eta_2)\\ &\quad + \alpha_{13}\beta_{13}\exp(\eta_1+\eta_3)\\ &\quad + \alpha_{23}\beta_{23}\exp(\eta_2+\eta_3),\end{aligned}$$

with

$$\alpha_{\mu\nu} = \frac{p_\mu - p_\nu}{p_\mu + p_\nu},\quad \beta_{\mu\nu} = \frac{c_{\mu\nu}}{p_\mu^6 - p_\nu^6},$$

where $\eta_\mu = p_\mu x + p_\mu^3 t + \eta_\mu^0$, p_μ and η_μ^0 are arbitrary parameters for $\mu, \nu = 1, 2, 3$.

The solution, which has one soliton for ϕ_1 and ϕ_2, two soliton for ϕ_3, is derived as follows:

$$\begin{aligned}g_1 &= \exp(\eta_1) + \alpha_{12}\alpha_{13}\alpha_{23}\beta_{23}\exp(\eta_1+\eta_2+\eta_3)\\ &\quad + \alpha_{12}\alpha_{14}\alpha_{24}\beta_{24}\exp(\eta_1+\eta_2+\eta_4),\\ g_2 &= \exp(\eta_2) - \alpha_{12}\alpha_{23}\alpha_{13}\beta_{13}\exp(\eta_1+\eta_2+\eta_3)\\ &\quad - \alpha_{12}\alpha_{24}\alpha_{14}\beta_{14}\exp(\eta_1+\eta_2+\eta_4),\\ g_3 &= \exp(\eta_3) + \exp(\eta_4)\\ &\quad + \alpha_{12}\alpha_{13}\alpha_{23}\beta_{12}\exp(\eta_1+\eta_2+\eta_3)\\ &\quad + \alpha_{12}\alpha_{14}\alpha_{24}\beta_{12}\exp(\eta_1+\eta_2+\eta_4)\\ &\quad + \alpha_{13}\alpha_{14}\alpha_{34}\beta_{13}\frac{(p_3^6 - p_4^6)}{(p_1^6 - p_4^6)}\exp(\eta_1+\eta_3+\eta_4)\\ &\quad + \alpha_{23}\alpha_{24}\alpha_{34}\beta_{23}\frac{(p_3^6 - p_4^6)}{(p_2^6 - p_4^6)}\exp(\eta_2+\eta_3+\eta_4),\\ f &= 1 + \alpha_{12}\beta_{12}\exp(\eta_1+\eta_2)\\ &\quad + \alpha_{13}\beta_{13}\exp(\eta_1+\eta_3) + \alpha_{14}\beta_{14}\exp(\eta_1+\eta_4)\\ &\quad + \alpha_{23}\beta_{23}\exp(\eta_2+\eta_3) + \alpha_{24}\beta_{24}\exp(\eta_2+\eta_4)\\ &\quad + \alpha_{13}\alpha_{23}\beta_{13}\beta_{23}\alpha_{12}\alpha_{14}\alpha_{24}\alpha_{34}\frac{(p_1^6 - p_2^6)(p_3^6 - p_4^6)}{(p_1^6 - p_4^6)(p_2^6 - p_4^6)}\\ &\quad \cdot\exp(\eta_1+\eta_2+\eta_3+\eta_4),\end{aligned}$$

with

$$\alpha_{\mu\nu} = \frac{p_\mu - p_\nu}{p_\mu + p_\nu},\quad \beta_{\mu\nu} = \frac{c_{\mu\nu}}{p_\mu^6 - p_\nu^6},\quad c_{\mu 4} \equiv c_{\mu 3},$$

where $\eta_\mu = p_\mu x + p_\mu^3 t + \eta_\mu^0$, p_μ and η_μ^0 are arbitrary parameters for $\mu, \nu = 1, 2, 3, 4$.

Similar to the coupled modified KdV equation,[7] the coupled derivative modified KdV equation,[8] the vector potential KdV equation and the vector Ito equation,[9] the multi-component (derivative) coupled integrable dispersionless equation,[10,11] the multi-component higher-order Ito equation[12] and the multi-component Hirota-Satsuma coupled KdV equation,[13] we find out that multi-soliton solution to the multi-component coupled Ito system Eq. (5) can also be expressed by pfaffians. It is thought that all these coupled systems, together with their pfaffian-type soliton solutions, are reduced from the multi-component BKP/DKP hierarchy. Bearing this in mind, we explicitly present this pfaffian-type soliton solution to the system Eq. (5) and prove it in this section.

Assuming that ϕ_i has M_i solitons and letting $L = M_1 + M_2 + \ldots + M_N$, we define the elements of the pfaffians as follows:

$$\begin{aligned}&\mathrm{pf}(d_n, a_\mu) = p_\mu^n\exp(\eta_\mu),\\ &\mathrm{pf}(a_\mu, a_\nu) = \frac{p_\mu - p_\nu}{p_\mu + p_\nu}\exp(\eta_\mu+\eta_\nu),\\ &\mathrm{pf}(a_\mu, b_\nu) = \delta_{\mu\nu}, \mathrm{pf}(b_\mu, \beta_i) = \begin{cases}1, & \text{if } b_\mu \in B_i,\\ 0, & \text{if } b_\mu \notin B_i,\end{cases}\\ &\mathrm{pf}(b_\mu, b_\nu) = -\frac{c_{jk}}{p_\mu^6 - p_\nu^6}(b_\mu \in B_j,\ b_\nu \in B_k),\\ &\mathrm{pf(otherwise)} \equiv 0,\end{aligned}$$

where $\eta_\mu = p_\mu x + p_\mu^3 t + \eta_\mu^0$, the elements of the sets B_i $(i = 1, 2, \ldots N)$ are chosen out of $\{b_1, b_2, \ldots b_L\}$ satisfying

$$M_i = \text{number of elements in the set} B_i,$$
$$B_i \cap B_j = \varnothing,$$
$$\bigcup_{i=1}^{N} B_i = \{b_1, b_2, \ldots b_L\}.$$

Based on the results in the above section, the multi-soliton solution to the system Eq. (5), which contains M_i solitons in component ϕ_i, is summarized into the following theorem.

Theorem 1: the bilinear Eqs (9) and (10) have the following pfaffian solutions

$$f = \mathrm{pf}(a_1, a_2, \ldots, a_L, b_1, b_2, \ldots, b_L) \triangleq \mathrm{pf}(\cdot), \tag{11}$$

$$g_i = \mathrm{pf}(d_0, a_1, a_2, \ldots, a_L, b_1, b_2, \ldots, b_L, \beta_i) \triangleq \mathrm{pf}(d_0, \cdot, \beta_i). \tag{12}$$

Proof: following the procedure as in Refs. [7–9], we can derive the following pfaffians' rules for g_i and f

$$\frac{\partial f}{\partial x} = \mathrm{pf}(d_0, d_1, \cdot), \tag{13}$$
$$\frac{\partial^2 f}{\partial x^2} = \mathrm{pf}(d_0, d_2, \cdot), \tag{14}$$
$$\frac{\partial^3 f}{\partial x^3} = \mathrm{pf}(d_0, d_3, \cdot) + \mathrm{pf}(d_1, d_2, \cdot), \tag{15}$$
$$\frac{\partial f}{\partial t} = \mathrm{pf}(d_0, d_3, \cdot) - 2\mathrm{pf}(d_1, d_2, \cdot), \tag{16}$$
$$\frac{\partial^2 f}{\partial x \partial t} = \mathrm{pf}(d_0, d_4, \cdot) - \mathrm{pf}(d_1, d_3, \cdot), \tag{17}$$
$$\frac{\partial^3 f}{\partial x^2 \partial t} = \mathrm{pf}(d_0, d_5, \cdot) - \mathrm{pf}(d_2, d_3, \cdot), \tag{18}$$
$$\frac{\partial^4 f}{\partial x^3 \partial t} = \mathrm{pf}(d_0, d_6, \cdot) + \mathrm{pf}(d_1, d_5, \cdot) - \mathrm{pf}(d_2, d_4, \cdot), \tag{19}$$
$$\frac{\partial^2 f}{\partial t^2} = \mathrm{pf}(d_0, d_6, \cdot) - 2\mathrm{pf}(d_1, d_5, \cdot) + 2\mathrm{pf}(d_2, d_4, \cdot), \tag{20}$$

and

$$\frac{\partial g_i}{\partial x} = \mathrm{pf}(d_1, \cdot, \beta_i), \tag{21}$$
$$\frac{\partial^2 g_i}{\partial x^2} = \mathrm{pf}(d_2, \cdot, \beta_i), \tag{22}$$
$$\frac{\partial^3 g_i}{\partial x^3} = \mathrm{pf}(d_3, \cdot, \beta_i) + \mathrm{pf}(d_0, d_1, d_2, \cdot, \beta_i), \tag{23}$$
$$\frac{\partial g_i}{\partial t} = \mathrm{pf}(d_3, \cdot, \beta_i) - 2\mathrm{pf}(d_0, d_1, d_2, \cdot, \beta_i). \tag{24}$$

Then, we start with the r.h.s. of Eq. (9),

$$\begin{aligned}
6 \sum_{1 \le j < k \le N} c_{jk} g_j g_k &= 3 \sum_{j,k=1}^{N} c_{jk} g_j g_k \\
&= 3 \sum_{j,k=1}^{N} c_{jk} \mathrm{pf}(d_0, \cdot, \beta_j) \mathrm{pf}(d_0, \cdot, \beta_k) \\
&= 3 \sum_{j,k=1}^{N} c_{jk} \sum_{\mu,\nu=1}^{L} (-1)^{\mu+\nu} \mathrm{pf}(b_\mu, \beta_j) \mathrm{pf}(b_\nu, \beta_k) \\
&\quad \cdot \mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots) \mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) \\
&= 3 \sum_{\mu,\nu=1}^{L} (-1)^{\mu+\nu} \sum_{j,k=1}^{N} c_{jk} \mathrm{pf}(b_\mu, \beta_j) \mathrm{pf}(b_\nu, \beta_k) \\
&\quad \cdot \mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots) \mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) \\
&= -3 \sum_{\mu,\nu=1}^{L} (-1)^{\mu+\nu} (p_\mu^6 - p_\nu^6) \mathrm{pf}(b_\mu, b_\nu) \\
&\quad \cdot \mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots) \mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) \\
&= -3 \sum_{\mu,\nu=1}^{L} (-1)^{\mu+\nu} p_\mu^6 \mathrm{pf}(b_\mu, b_\nu) \\
&\quad \cdot \mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots) \mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) \\
&\quad - 3 \sum_{\nu,\mu=1}^{L} (-1)^{\mu+\nu} p_\nu^6 \mathrm{pf}(b_\nu, b_\mu) \\
&\quad \cdot \mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots) \mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) \\
&= -6 \sum_{\mu,\nu=1}^{L} (-1)^{\mu+\nu} p_\mu^6 \mathrm{pf}(b_\mu, b_\nu) \mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots) \\
&\quad \cdot \mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots),
\end{aligned} \tag{25}$$

where $\mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots) \equiv \mathrm{pf}(d_0, a_1, a_2, \ldots, a_L, b_1, b_2, \ldots, \hat{b}_\mu, \ldots, b_L)$ and $\hat{\alpha}$ indicate that the letter α is missing.

Next, a substitution of the following pfaffian identity into Eq. (25),

$$\sum_{\nu=1}^{L} (-1)^{\mu+\nu} \mathrm{pf}(b_\mu, b_\nu) \mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) = (-1)^L \mathrm{pf}(d_0, \ldots, \hat{a}_\mu, \ldots), \tag{26}$$

which can be deduced by the expansion of a vanishing pfaffian $\mathrm{pf}(d_0, \cdot, b_\mu)$ on b_μ, yields

$$6 \sum_{1 \le j < k \le N} c_{jk} g_j g_k = 6(-1)^{L+1} \sum_{\mu=1}^{L} p_\mu^6 \mathrm{pf}(d_0, \ldots, \hat{a}_\mu, \ldots) \mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots). \tag{27}$$

On the other hand, by using Eqs. (13)–(20), we can derive the following formula

$$\begin{aligned}
&\Big(2\frac{\partial^2 f}{\partial t^2}+4\frac{\partial^3 f}{\partial x^3\partial_t}\Big)\cdot 0-4\frac{\partial^3 f}{\partial x^3}\frac{\partial f}{\partial t}\\
&\quad-12\frac{\partial^3 f}{\partial x^2\partial t}\frac{\partial f}{\partial x}+12\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial x\partial t}-2\Big(\frac{\partial f}{\partial t}\Big)^2\\
&=6\mathrm{pf}(d_0,d_6,\cdot)\mathrm{pf}(d_0,d_0,\cdot)-4[\mathrm{pf}(d_0,d_3,\cdot)\\
&\quad+\mathrm{pf}(d_1,d_2,\cdot)][\mathrm{pf}(d_0,d_3,\cdot)-2\mathrm{pf}(d_1,d_2,\cdot)]\\
&\quad-12[\mathrm{pf}(d_0,d_5,\cdot)-\mathrm{pf}(d_2,d_3,\cdot)]\mathrm{pf}(d_0,d_1,\cdot)\\
&\quad+12\mathrm{pf}(d_0,d_2,\cdot)[\mathrm{pf}(d_0,d_4,\cdot)-\mathrm{pf}(d_1,d_3,\cdot)]\\
&\quad-2[\mathrm{pf}(d_0,d_3,\cdot)-2\mathrm{pf}(d_1,d_2,\cdot)]\\
&\quad\cdot[\mathrm{pf}(d_0,d_3,\cdot)-2\mathrm{pf}(d_1,d_2,\cdot)]\\
&=6\mathrm{pf}(d_0,d_6,\cdot)\mathrm{pf}(d_0,d_0,\cdot)\\
&\quad-6\mathrm{pf}(d_0,d_3,\cdot)\mathrm{pf}(d_0,d_3,\cdot)\\
&\quad+12\mathrm{pf}(d_0,d_2,\cdot)\mathrm{pf}(d_0,d_4,\cdot)\\
&\quad-12\mathrm{pf}(d_0,d_1,\cdot)\mathrm{pf}(d_0,d_5,\cdot)\\
&=6\Big\{\sum_{\mu=1}^{L}(-1)^{\mu}\mathrm{pf}(d_6,a_\mu)\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\Big\}\\
&\quad\times\Big\{\sum_{\nu=1}^{L}(-1)^{\nu}\mathrm{pf}(d_0,a_\nu)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\Big\}\\
&\quad-6\Big\{\sum_{\mu=1}^{L}(-1)^{\mu}\mathrm{pf}(d_3,a_\mu)\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\Big\}\\
&\quad\times\Big\{\sum_{\nu=1}^{L}(-1)^{\nu}\mathrm{pf}(d_3,a_\nu)\mathrm{pf}(d_0\ldots,\hat{a}_\nu,\ldots)\Big\}\\
&\quad+12\Big\{\sum_{\mu=1}^{L}(-1)^{\mu}\mathrm{pf}(d_2,a_\mu)\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\Big\}\\
&\quad\times\Big\{\sum_{\nu=1}^{L}(-1)^{\nu}\mathrm{pf}(d_4,a_\nu)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\Big\}\\
&\quad-12\Big\{\sum_{\mu=1}^{L}(-1)^{\mu}\mathrm{pf}(d_1,a_\mu)\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\Big\}\\
&\quad\times\Big\{\sum_{\nu=1}^{L}(-1)^{\nu}\mathrm{pf}(d_5,a_\nu)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\Big\}\\
&\quad=6\sum_{\mu,\nu=1}^{L}(-1)^{\mu+\nu}[\mathrm{pf}(d_6,a_\mu)\mathrm{pf}(d_0,a_\nu)\\
&\quad-\mathrm{pf}(d_3,a_\mu)\mathrm{pf}(d_3,a_\nu)+2\mathrm{pf}(d_2,a_\mu)\mathrm{pf}(d_4,a_\nu)\\
&\quad-2\mathrm{pf}(d_1,a_\mu)\mathrm{pf}(d_5,a_\nu)]\\
&\quad\times\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\\
&=6\sum_{\mu,\nu=1}^{L}(-1)^{\mu+\nu}\Big\{[p_\mu^6+p_\mu^3p_\nu^3+2p_\mu^2p_\nu^2(p_\mu^2+p_\nu^2)\\
&\quad+2p_\mu p_\nu(p_\mu^4+p_\nu^4)]\mathrm{pf}(a_\mu,a_\nu)\Big\}\\
&\quad\times\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots).
\end{aligned}\tag{28}$$

Due to the fact that

$$\begin{aligned}
&\sum_{\mu,\nu=1}^{L}(-1)^{\mu+\nu}p_\mu^3p_\nu^3\mathrm{pf}(a_\mu,a_\nu)\\
&\quad\cdot\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\\
&=\sum_{\nu,\mu=1}^{L}(-1)^{\nu+\mu}p_\nu^3p_\mu^3[-\mathrm{pf}(a_\mu,a_\nu)]\\
&\quad\cdot\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\\
&=-\sum_{\mu,\nu=1}^{L}(-1)^{\mu+\nu}p_\mu^3p_\nu^3\mathrm{pf}(a_\mu,a_\nu)\\
&\quad\cdot\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots),
\end{aligned}$$

one has

$$\begin{aligned}
&\sum_{\mu,\nu=1}^{L}(-1)^{\mu+\nu}p_\mu^3p_\nu^3\mathrm{pf}(a_\mu,a_\nu)\\
&\quad\cdot\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)=0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{\mu,\nu=1}^{L}(-1)^{\mu+\nu}p_\mu^2p_\nu^2(p_\mu^2+p_\nu^2)\mathrm{pf}(a_\mu,a_\nu)\\
&\quad\cdot\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)=0,\\
&\sum_{\mu,\nu=1}^{L}(-1)^{\mu+\nu}p_\mu p_\nu(p_\mu^4+p_\nu^4)\mathrm{pf}(a_\mu,a_\nu)\\
&\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)=0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&-4\frac{\partial^3 f}{\partial x^3}\frac{\partial f}{\partial t}-12\frac{\partial^3 f}{\partial x^2\partial t}\frac{\partial f}{\partial x}\\
&+12\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial x\partial t}-2\Big(\frac{\partial f}{\partial t}\Big)^2\\
&=6\sum_{\mu,\nu=1}^{L}(-1)^{\mu+\nu}p_\mu^6\mathrm{pf}(a_\mu,a_\nu)\\
&\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\\
&=6\sum_{\mu=1}^{L}(-1)^{\mu}p_\mu^6\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\\
&\times\Big[\sum_{\nu=1}^{L}(-1)^{\nu}\mathrm{pf}(a_\mu,a_\nu)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\Big].
\end{aligned}\tag{29}$$

Furthermore, we note that the following identity can be substituted into the term within bracket

$$\begin{aligned}
&\sum_{\nu=1}^{L}(-1)^{\nu}\mathrm{pf}(a_\mu,a_\nu)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\\
&=\mathrm{pf}(d_0,a_\mu)\mathrm{pf}(\cdot)+(-1)^{L+\mu+1}\mathrm{pf}(d_0,\ldots,\hat{b}_\mu,\ldots),
\end{aligned}\tag{30}$$

which is obtained from the expansion of the vanishing pfaffian $(a_\mu,d_0,\cdot)$ on a_μ.

Consequently, we have

$$
\begin{aligned}
&-4\frac{\partial^3 f}{\partial x^3}\frac{\partial f}{\partial t}-12\frac{\partial^3 f}{\partial x^2\partial t}\frac{\partial f}{\partial x}\\
&+12\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial x\partial t}-2\Big(\frac{\partial f}{\partial t}\Big)^2\\
=&6\sum_{\mu=1}^{L}(-1)^{\mu}p_{\mu}^{6}\mathrm{pf}(d_0,\ldots,\hat{a}_{\mu},\ldots)\\
&[\mathrm{pf}(d_0,a_{\mu})\mathrm{pf}(\cdot)+(-1)^{L+\mu+1}\mathrm{pf}(d_0,\ldots,\hat{b}_{\mu},\ldots)]\\
&=-6\mathrm{pf}(d_0,d_6,\cdot)\mathrm{pf}(\cdot)+6(-1)^{L+1}\\
&\cdot\sum_{\mu=1}^{L}p_{\mu}^{6}\mathrm{pf}(d_0,\ldots,\hat{a}_{\mu},\ldots)\mathrm{pf}(d_0,\ldots,\hat{b}_{\mu},\ldots)\\
&=-\Big(2\frac{\partial^2 f}{\partial t^2}+4\frac{\partial^3 f}{\partial x^3\partial_t}\Big)\cdot f+6\sum_{1\le j<k\le N}c_{jk}g_jg_k. \qquad (31)
\end{aligned}
$$

Thus the proof of the first bilinear equation is finished.

The proof of the second bilinear equation is much easier than the first one. Substituting the relations (21)–(24) into the l.h.s. of Eq. (10), we arrive at

$$
\begin{aligned}
&(D_t-D_x^3)g_i\cdot f=\Big(\frac{\partial}{\partial t}g_i-\frac{\partial^3}{\partial x^3}g_i\Big)f\\
&+3\frac{\partial^2}{\partial x^2}g_i\frac{\partial}{\partial x}f-3\frac{\partial}{\partial x}g_i\frac{\partial^2}{\partial x^2}f\\
&-g_i\Big(\frac{\partial}{\partial t}f-\frac{\partial^3}{\partial x^3}f\Big)\\
&=-3\mathrm{pf}(d_0,d_1,d_2,\cdot,\beta_i)\mathrm{pf}(\cdot)\\
&+3\mathrm{pf}(d_2,\cdot,\beta_i)\mathrm{pf}(d_0,d_1,\cdot)\\
&-3\mathrm{pf}(d_1,\cdot,\beta_i)\mathrm{pf}(d_0,d_2,\cdot,b_L)\\
&+3\mathrm{pf}(d_0,\cdot,\beta_i)\mathrm{pf}(d_1,d_2,\cdot), \qquad (32)
\end{aligned}
$$

which vanishes by the pfaffian identity. Thus we prove the second bilinear equation.

In summary, we have investigated the multi-soliton solution for the multi-component coupled Ito system based on the Hirota bilinear method. Using the perturbation method, we firstly derive a few soliton solutions of the coupled Ito system for $N=3$. Furthermore, the multi-soliton solution for the multi-component coupled Ito system is given by pfaffians and is proved by pfaffian techniques.

Finally, it is known that, based on their bilinear forms, some single component soliton equations can be generalized into multi-component ones. Examples include the coupled KdV equation,[14] the coupled sine-Gordon equation,[15] the coupled modified KdV equation,[7,15] the coupled derivative modified KdV equation,[8] the vector potential KdV equation and the vector Ito equation,[9] the multi-component (derivative) coupled integrable dispersionless equation,[10,11] the multi-component higher-order Ito equation,[12] the vector asymmetrical NNV equation,[16] the multi-component Hirota-Satsuma KdV equation[13] and the multi-component Ito system studied in the present work. By using this bilinear procedure, we can deduce some other coupled systems such as the coupled derivative KdV and Ito equations. Furthermore, all these coupled systems are expected to be reduced from the multi-component BKP/DKP hierarchies. We will explore these issues and report our results in the future.

References

[1] Ito M 1980 *J. Phys. Soc. Jpn.* **49** 771
[2] Hirota R 2004 *The Direct Method in Soliton Theory* (Cambridge: Cambridge University Press)
[3] Tam H W, Hu X B and Wang D L 1999 *J. Phys. Soc. Jpn.* **68** 369
[4] Tam H W, Ma W X, Hu X B and Wang D L 2000 *J. Phys. Soc. Jpn.* **69** 45
[5] Karasu-Kalkanli A, Karasu A and Sakovich S Y 2001 *J. Phys. Soc. Jpn.* **70** 1165
[6] Balakhnev M J 2011 *J. Phys. Soc. Jpn.* **80** 045002
[7] Iwao M and Hirota R 1997 *J. Phys. Soc. Jpn.* **66** 577
[8] Iwao M and Hirota R 2000 *J. Phys. Soc. Jpn.* **69** 59
[9] Hirota R, Hu X B and Tang X Y 2003 *J. Math. Anal. Appl.* **288** 326
[10] Vinet L and Yu G F 2013 *J. Phys. A: Math. Theor.* **46** 175205
[11] Yu G F 2014 *J. Phys. Soc. Jpn.* **83** 074003
[12] Zhao H Q 2013 *Appl. Math. Lett.* **26** 681
[13] Chen J C, Chen Y, Feng B F and Zhu H M 2014 *Appl. Math. Lett.* **37** 15
[14] Yoneyama T 1984 *Prog. Theor. Phys.* **72** 1081
[15] Hirota R and Ohta Y 1991 *J. Phys. Soc. Jpn.* **60** 798
[16] Yu G F and Tam H W 2008 *J. Math. Anal. Appl.* **344** 593