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Dual Hierarchies of a Multi-Component Camassa–Holm System*

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Abstract *In this paper, we derive the bi-Hamiltonian structure of a multi-component Camassa–Holm system, which associates with the multi-component AKNS hierarchy and multi-component KN hierarchy via the tri-Hamiltonian duality method. Furthermore, the spectral problems of the dual hierarchies may be obtained.*

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1 Introduction

In 1993, the Camassa–Holm (CH) equation

$$m_t + um_x + 2u_x m = 0\,, \quad m = u - u_{xx}\,, \tag{1}$$

was derived by Camassa and Holm from an approximation to the incompressible Eluer equations.[1] Like the KdV equation, the CH equation is integrable with Lax pair and bi-Hamiltonian structure,[2] an unusual feature is that it admits peakon solutions.[3–4] It is interesting that the CH equation is associated with the first negative flow of the KdV hierarchy by reciprocal transformation,[5] and the Hamiltonian pair for it can be constructed by rearranging that of the KdV equation. Via this connection, the spectral problem for the CH equation can be obtained from that of the KdV equation. Motivated by the remarkable property of the CH equation, many other CH systems have been constructed[6–10] and studied.[11–15]

Recently, Xia and Qiao presented a multi-component CH system,[16]

$$\begin{aligned}\vec{m}_t &= \frac{1}{(s+1)^2}[\vec{m}(\vec{v}+\vec{v}_x)^{\mathrm{T}}(\vec{u}-\vec{u}_x) \\ &\quad + (\vec{u}-\vec{u}_x)(\vec{v}+\vec{v}_x)^{\mathrm{T}}\vec{m}]\,, \\ \vec{n}_t &= -\frac{1}{(s+1)^2}[\vec{n}(\vec{u}-\vec{u}_x)^{\mathrm{T}}(\vec{v}+\vec{v}_x) \\ &\quad + (\vec{v}+\vec{v}_x)(\vec{u}-\vec{u}_x)^{\mathrm{T}}\vec{n}]\,, \\ \vec{m} &= \vec{u}-\vec{u}_{xx}, \quad \vec{n} = \vec{v}-\vec{v}_{xx}\,,\end{aligned} \tag{2}$$

where $\vec{u} = (u_1, u_2, \ldots, u_s)$, $\vec{v} = (v_1, v_2, \cdots, v_s)$, $\vec{m} = (m_1, m_2, \ldots, m_s)$, $\vec{n} = (n_1, n_2, \ldots, n_s)$ and T is the transpose of a vector. They found that the system (2) possessed a Lax pair and infinitely many conservation laws, and discussed the peakon solutions as $s = 2$. When $s = 1$, the bi-Hamiltonian structure of the system (2) was considered in Ref. [17]. The multi-component CH system (2) is bi-Hamiltonian as a by-product of the results in this paper.

Olver and Rosenau constructed CH systems via the tri-Hamiltonian duality method that rearranging the Hamiltonian operators of the classical soliton equations in an algorithmic manner.[18] They called the CH systems the dual hierarchies of the associated soliton equations. Indeed, the method of rearranging the Hamiltonian operators appeared in the earlier work of Fuchssteiner and Fokas.[19] And vice versa, a proper recombination of Hamiltonian operators of the CH systems can also generate the classical soliton hierarchies. The aim of this paper is to construct the dual hierarchies of the CH system (2).

The paper is arranged as follows: In Sec. 2, we derive the bi-Hamiltonian structure of the $2s$-component CH system (2), and construct the dual hierarchies of it using the tri-Hamiltonian duality method.[18] In Sec. 3, we study the dual versions of a two-component ($s = 1$) CH system and a four-component ($s = 2$) CH system by recombining their Hamiltonian operators. In Appendix, we present the detail proof of the Jacobi identity for the operator $\mathcal{J}$ (13) as well as the compatibility with the Hamiltonian operator $\mathcal{K}$ (12) by the trivector technique of Olver.[20]

2 Dual Hierarchies of the Multi-Component Camassa–Holm System

In this section, we derive the bi-Hamiltonian structure of the multi-component CH system (2) and consider its dual hierarchies using the tri-Hamiltonian duality approach.[18] Moreover, via this connection, we recover the spectral problems of the dual hierarchies.

In order to better understand and display, we denote $\vec{m}^{\mathrm{T}}$, $\vec{n}^{\mathrm{T}}$, $\vec{u}^{\mathrm{T}}$, $\vec{v}^{\mathrm{T}}$ by M, N, U, V respectively, i.e.,

$$M = (m_1, m_2, \ldots, m_s)^{\mathrm{T}}\,, \quad N = (n_1, n_2, \ldots, n_s)^{\mathrm{T}}\,,$$

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$$U=(u_1,u_2,\ldots,u_s)^{\mathrm{T}}\,,\quad V=(v_1,v_2,\ldots,v_s)^{\mathrm{T}}\,.$$

And then the system (2) can be rewritten as follows

$$\begin{aligned}M_t&=\frac{1}{(s+1)^2}[\langle M,V+V_x\rangle(U-U_x)\\&\quad+\langle U-U_x,V+V_x\rangle M]\,,\\N_t&=-\frac{1}{(s+1)^2}[\langle N,U-U_x\rangle(V+V_x)\\&\quad+\langle V+V_x,U-U_x\rangle N]\,,\end{aligned}$$

$$M=U-U_{xx}\,,\quad N=V-V_{xx}\,,\tag{3}$$

where $\langle,\rangle$ denotes the standard inner product.

As pointed in Ref. [16], the multi-component CH system (3) arises from a zero curvature equation

$$F_t-G_x+[F,G]=0\,,\tag{4}$$

this being the compatibility for the linear system

$$\varphi_x=F\varphi\,,\quad \varphi_t=G\varphi\,,\tag{5}$$

with

$$F=\frac{1}{s+1}\begin{pmatrix}-s & \lambda M^{\mathrm{T}}\\ \lambda N & I_s\end{pmatrix},\quad G=\frac{1}{s+1}\begin{pmatrix}-\lambda^{-2}s+\dfrac{1}{s+1}\langle U-U_x,V+V_x\rangle & \lambda^{-1}(U-U_x)^{\mathrm{T}}\\ \lambda^{-1}(V+V_x) & \lambda^{-2}I_s-\dfrac{1}{s+1}(V+V_x)(U-U_x)^{\mathrm{T}}\end{pmatrix},$$

where λ is the spectral parameter and I_s is the $s\times s$ identity matrix.

In order to obtain the bi-Hamiltonian structure of the system (2), we rewrite the time part of the system (5) as follows

$$V=\frac{1}{s+1}\begin{pmatrix}v_{11} & A\\ B^{\mathrm{T}} & C\end{pmatrix},\tag{6}$$

where v_{11} is a function variable and A,B are both s dimension row vectors depending on vector potentials M, N and λ, C is an $s\times s$ matrix depending on vector potentials M, N and λ.

The compatible condition yields

$$M_t^{\mathrm{T}}=\frac{1}{\lambda}(A+A_x)+\frac{1}{s+1}M^{\mathrm{T}}v_{11}-\frac{1}{s+1}M^{\mathrm{T}}C\,,\tag{7}$$

$$N_t=\frac{1}{\lambda}(B_x^{\mathrm{T}}-B^{\mathrm{T}})-\frac{1}{s+1}Nv_{11}+\frac{1}{s+1}CN\,,\tag{8}$$

$$v_{11}=\frac{\lambda}{s+1}\partial^{-1}(\langle M,B^{\mathrm{T}}\rangle-\langle N,A^{\mathrm{T}}\rangle)\,,\tag{9}$$

$$C=\frac{\lambda}{s+1}\partial^{-1}(NA-B^{\mathrm{T}}M^{\mathrm{T}})\,,\tag{10}$$

where

$$M_t^{\mathrm{T}}=\frac{\partial M^{\mathrm{T}}}{\partial t}\,,\quad B_x^{\mathrm{T}}=\frac{\partial B^{\mathrm{T}}}{\partial x}\,.$$

Substituting the equalities (9) and (10) to Eqs. (7) and (8), we have

$$\begin{pmatrix}M\\N\end{pmatrix}_t=\Bigl(\frac{1}{\lambda}\mathcal{K}+\frac{\lambda}{(s+1)^2}\mathcal{J}\Bigr)\begin{pmatrix}B^{\mathrm{T}}\\A^{\mathrm{T}}\end{pmatrix},\tag{11}$$

with

$$\mathcal{K}=\begin{pmatrix}\mathbf{0}_{s\times s} & (\partial+1)I_s\\ (\partial-1)I_s & \mathbf{0}_{s\times s}\end{pmatrix},\tag{12}$$

$$\mathcal{J}=\begin{pmatrix}M\partial^{-1}M^{\mathrm{T}}+(M\partial^{-1}M^{\mathrm{T}})^{\mathrm{T}} & -M\partial^{-1}N^{\mathrm{T}}-M^{\mathrm{T}}\partial^{-1}NI_s\\ -N\partial^{-1}M^{\mathrm{T}}-N^{\mathrm{T}}\partial^{-1}MI_s & N\partial^{-1}N^{\mathrm{T}}+(N\partial^{-1}N^{\mathrm{T}})^{\mathrm{T}}\end{pmatrix},\tag{13}$$

where $\mathbf{0}_{s\times s}$ is the $s\times s$ zero matrix.

In the following, we show that the operators $\mathcal{K}$, $\mathcal{J}$ are Hamiltonian operators and form a bi-Hamiltonian pair. The Jacobi identity and compatibility conditions for the operators $\mathcal{K}$, $\mathcal{J}$ may be checked using the multivector approach to Hamiltonian systems in infinite dimensions, as described in the work of Olver.[20]

Theorem 1 The multi-component CH system (3) can be written in the bi-Hamiltonian form

$$\begin{pmatrix}M\\N\end{pmatrix}_t=\mathcal{K}\begin{pmatrix}\dfrac{\delta H_1}{\delta M}\\[2mm] \dfrac{\delta H_1}{\delta N}\end{pmatrix}=\mathcal{J}\begin{pmatrix}\dfrac{\delta H_0}{\delta M}\\[2mm] \dfrac{\delta H_0}{\delta N}\end{pmatrix},\tag{14}$$

using the operators $\mathcal{K}$ and $\mathcal{J}$ (12)–(13) and

$$\begin{aligned}H_0&=\frac{1}{(s+1)^2}\int\langle U_{xx}-U_x,N\rangle\,\mathrm{d}x\,,\\H_1&=\frac{1}{(s+1)^2}\int\langle U-U_x,V+V_x\rangle\langle U-U_x,N\rangle\,\mathrm{d}x\,.\end{aligned}$$

Proof The equalities (12) and (13) imply that the operators $\mathcal{K}$, $\mathcal{J}$ are skew-symmetric. Furthermore $\mathcal{K}$ is a Hamiltonian operator. Hence, we need to prove that the Jacobi identity for $\mathcal{J}$ and compatibility of $\mathcal{J}$ with $\mathcal{K}$. Taking $\theta_1=(\theta_{11},\ldots,\theta_{1s})^{\mathrm{T}}$, $\theta_2=(\theta_{21},\ldots,\theta_{2s})^{\mathrm{T}}$ as the basic uni-vectors corresponding to M, N respectively, we know that the operator $\mathcal{J}$ is a Hamiltonian operator if

$$\mathrm{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}})=0\,,\tag{15}$$

where

$$\theta=\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$$

and $\Theta_{\mathcal{J}}$ is the associated bi-vector of $\mathcal{J}$.

To check whether $\mathcal{K}$ and $\mathcal{J}$ form a bi-Hamiltonian pair, we only need to prove

$$\mathrm{PrV}_{\mathcal{K}\theta}(\Theta_{\mathcal{J}})=0\,.\tag{16}$$

The proof of the theorem is rather technical and lengthy, so are given in Appendix.

In the following, we will study the dual hierarchies of the multi-component CH system (3) by recombining the

Hamiltonian operators $\mathcal{K}$ and $\mathcal{J}$ in Eqs. (12) and (13) accordingly.

The dual Hamiltonian operators of the operators $\mathcal{K}$ and $\mathcal{J}$ are obtained by the following procedure. Transferring the terms ∂I_s from the operator $\mathcal{K}$ to the corresponding elements of the operator $\mathcal{J}$ and replacing M, N by Q, R respectively in the operator $\mathcal{J}$, we get two Hamiltonian operators

$$\widehat{\mathcal{K}} = \begin{pmatrix} \mathbf{0}_{s\times s} & I_s \\ -I_s & \mathbf{0}_{s\times s} \end{pmatrix}, \quad \widehat{\mathcal{J}} = \begin{pmatrix} Q\partial^{-1}Q^{\mathrm{T}} + (Q\partial^{-1}Q^{\mathrm{T}})^{\mathrm{T}} & (\partial - Q^{\mathrm{T}}\partial^{-1}R)I_s - Q\partial^{-1}R^{\mathrm{T}} \\ (\partial - R^{\mathrm{T}}\partial^{-1}Q)I_s - R\partial^{-1}Q^{\mathrm{T}} & R\partial^{-1}R^{\mathrm{T}} + (R\partial^{-1}R^{\mathrm{T}})^{\mathrm{T}} \end{pmatrix},$$

where $Q = (q_1, q_2, \ldots, q_s)^{\mathrm{T}}$, $R = (r_1, r_2, \ldots, r_s)^{\mathrm{T}}$.

The above Hamiltonian pair $\widehat{\mathcal{K}}$, $\widehat{\mathcal{J}}$ is nothing but the bi-Hamiltonian pair for the multi-component AKNS hierarchy.[21]

In the equality (11), for the Hamiltonian operators $\mathcal{K}$ and $\mathcal{J}$, we make the following transformation

$$x \to \frac{1}{s+1}\lambda x, \quad M \to Q, \quad N \to R. \tag{17}$$

After the above transformation, the equality (11) becomes

$$\begin{pmatrix} Q \\ R \end{pmatrix}_t = \Bigl(\frac{1}{\lambda}\widehat{\mathcal{K}} + \frac{1}{s+1}\widehat{\mathcal{J}}\Bigr)\begin{pmatrix} B^{\mathrm{T}} \\ A^{\mathrm{T}} \end{pmatrix}, \tag{18}$$

which leads to the Hamiltonian operators $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{J}}$. Therewith, the spatial part of the linear problem (5) is transformed into

$$\varphi_x = F\varphi, \quad F = \begin{pmatrix} \dfrac{-s}{\lambda} & Q^{\mathrm{T}} \\ R & \dfrac{1}{\lambda}I_s \end{pmatrix}, \tag{19}$$

which, by the transformation $\lambda \to -1/\lambda$, may be reformulated as

$$\varphi_x = F\varphi, \quad F = \begin{pmatrix} \lambda s & Q^{\mathrm{T}} \\ R & -\lambda I_s \end{pmatrix}. \tag{20}$$

The above spectral problem is just the one for the multi-component AKNS hierarchy (see (1.2) in Ref. [22] for details).

On the other hand, if we transfer the terms $-I_s$ and I_s instead of the terms ∂I_s from $\mathcal{K}$ to the corresponding elements of the $\mathcal{J}$, and replace M, N by Q, R respectively as well, we have

$$\widetilde{\mathcal{K}} = \begin{pmatrix} \mathbf{0}_{s\times s} & \partial I_s \\ \partial I_s & \mathbf{0}_{s\times s} \end{pmatrix}, \quad \widetilde{\mathcal{J}} = \begin{pmatrix} Q\partial^{-1}Q^{\mathrm{T}} + (Q\partial^{-1}Q^{\mathrm{T}})^{\mathrm{T}} & (1 - Q^{\mathrm{T}}\partial^{-1}R)I_s - Q\partial^{-1}R^{\mathrm{T}} \\ -(1 + R^{\mathrm{T}}\partial^{-1}Q)I_s - R\partial^{-1}Q^{\mathrm{T}} & R\partial^{-1}R^{\mathrm{T}} + (R\partial^{-1}R^{\mathrm{T}})^{\mathrm{T}} \end{pmatrix},$$

which are just the compatible Hamiltonian operators of multi-component KN hierarchy. Furthermore, after the transformation

$$x \to \lambda x, \quad M \to (s+1)Q, \quad N \to \frac{s+1}{\lambda}R, \tag{21}$$

Eq. (11) yields

$$\begin{pmatrix} Q \\ R \end{pmatrix}_t = \Bigl(\frac{\lambda}{s+1}\widetilde{\mathcal{K}} + \frac{1}{s+1}\widetilde{\mathcal{J}}\Bigr)\begin{pmatrix} B^{\mathrm{T}} \\ A^{\mathrm{T}}/\lambda \end{pmatrix}. \tag{22}$$

The spectral problem of the CH system (3) becomes

$$\varphi_x = F\varphi, \quad F = \frac{1}{s+1}\begin{pmatrix} -\dfrac{s}{\lambda} & (s+1)Q^{\mathrm{T}} \\ \dfrac{s+1}{\lambda}R & \dfrac{1}{\lambda}I_s \end{pmatrix}, \tag{23}$$

which, by the transformation

$$\lambda \to -\frac{1}{\lambda(s+1)}, \quad R \to -\frac{1}{(s+1)}R,$$

leads to

$$\varphi_x = F\varphi, \quad F = \begin{pmatrix} \lambda s & Q^{\mathrm{T}} \\ \lambda R & -\lambda I_s \end{pmatrix}. \tag{24}$$

The spectral problem (24) is nothing but the one of the multi-component KN hierarchy.

3 Dual Hierarchies of the Two-Component CH System and Four-Component Camassa–Holm System

In this section, we consider the dual hierarchies of the two-component CH system and four-component CH system.

3.1 *Dual Hierarchies of the Two-Component Camassa–Holm System*

As $s = 1$, the multi-component CH system (3) is

$$\begin{aligned} m_t &= \frac{1}{2}m(uv - u_x v_x + uv_x - u_x v), \\ n_t &= -\frac{1}{2}n(uv - u_x v_x + uv_x - u_x v), \\ m &= u - u_{xx}, \ n = v - v_{xx}, \end{aligned} \tag{25}$$

which appears in the bi-Hamiltonian form (14) with

$$\begin{aligned} \mathcal{K} &= \begin{pmatrix} 0 & \partial + 1 \\ \partial - 1 & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} m\partial^{-1}m & -m\partial^{-1}n \\ -n\partial^{-1}m & n\partial^{-1}n \end{pmatrix}, \\ H_0 &= \frac{1}{2}\int (u_{xx} - u_x)n\,\mathrm{d}x, \\ H_1 &= \frac{1}{4}\int (u - u_x)^2(v + v_x)n\,\mathrm{d}x. \end{aligned} \tag{26}$$

From the results in Sec. 2, we know the dual Hamiltonian pairs of the operator (26) are respectively

$$\widehat{\mathcal{K}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \widehat{\mathcal{J}} = \begin{pmatrix} q\partial^{-1}q & \partial - q\partial^{-1}r \\ \partial - r\partial^{-1}q & r\partial^{-1}r \end{pmatrix}, \tag{27}$$

$$\widetilde{\mathcal{K}} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad \widetilde{\mathcal{J}} = \begin{pmatrix} q\partial^{-1}q & 1 - q\partial^{-1}r \\ -1 - r\partial^{-1}q & r\partial^{-1}r \end{pmatrix}, \tag{28}$$

which are the Hamiltonian pairs for the AKNS hierarchy and KN hierarchy respectively. The associated spectral problems of the dual hierarchies are respectively

$$\varphi_x = U\varphi\,,\quad U = \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix}, \tag{29}$$

$$\varphi_x = U\varphi\,,\quad U = \begin{pmatrix} \lambda & q \\ \lambda r & -\lambda \end{pmatrix}, \tag{30}$$

which can be deduced via the connection.

Remark 1 In Ref. [23], Ma and Zhou studied the Hamiltonian operator

$$M = \begin{pmatrix} \alpha_1 q\partial^{-1}q & \alpha_2 + \alpha_3\partial - \alpha_1 q\partial^{-1}r \\ -\alpha_2 + \alpha_3\partial - \alpha_1 r\partial^{-1}q & \alpha_1 r\partial^{-1}r \end{pmatrix}, \tag{31}$$

where α_1, α_2, α_3 are arbitrary constants. They gave the Hamiltonian pairs (27) and (28) for the AKNS hierarchy and KN hierarchy starting form Hamiltonian operator (31). Indeed the Hamiltonian operator (31) can lead to another Hamiltonian pair (26) for the two-component CH system (25).

3.2 *Dual Hierarchies of the Four-Component Camassa–Holm System*

When $s = 2$, the multi-component CH system (3) becomes

$$\begin{aligned}
m_{1t} &= \frac{1}{9}\{m_1[2(u_1 - u_{1x})(v_1 + v_{1x}) + (u_2 - u_{2x})(v_2 + v_{2x})] + m_2(u_1 - u_{1x})(v_2 + v_{2x})\}\,,\\
m_{2t} &= \frac{1}{9}\{m_2[(u_1 - u_{1x})(v_1 + v_{1x}) + 2(u_2 - u_{2x})(v_2 + v_{2x})] + m_1(u_2 - u_{2x})(v_1 + v_{1x})\}\,,\\
n_{1t} &= -\frac{1}{9}\{n_1[2(u_1 - u_{1x})(v_1 + v_{1x}) + (u_2 - u_{2x})(v_2 + v_{2x})] + n_2(u_2 - u_{2x})(v_1 + v_{1x})\}\,,\\
n_{2t} &= -\frac{1}{9}\{n_2[(u_1 - u_{1x})(v_1 + v_{1x}) + 2(u_2 - u_{2x})(v_2 + v_{2x})] + n_1(u_1 - u_{1x})(v_2 + v_{2x})\}\,,\\
m_1 &= u_1 - u_{1xx}\,,\quad m_2 = u_2 - u_{2xx}\,,\quad n_1 = v_1 - v_{1xx}\,,\quad n_2 = v_2 - v_{2xx}\,,
\end{aligned} \tag{32}$$

which can be written as the bi-Hamiltonian form (14), using the

$$\mathcal{K} = \begin{pmatrix} 0 & 0 & \partial+1 & 0 \\ 0 & 0 & 0 & \partial+1 \\ \partial-1 & 0 & 0 & 0 \\ 0 & \partial-1 & 0 & 0 \end{pmatrix},\quad \mathcal{J} = \begin{pmatrix} 2m_1\partial^{-1}m_1 & \mathcal{J}_{12} & \mathcal{J}_{13} & -m_1\partial^{-1}n_2 \\ -\mathcal{J}_{12}^* & 2m_2\partial^{-1}m_2 & -m_2\partial^{-1}n_1 & \mathcal{J}_{24} \\ -\mathcal{J}_{13}^* & -n_1\partial^{-1}m_2 & 2n_1\partial^{-1}n_1 & \mathcal{J}_{34} \\ -n_2\partial^{-1}m_1 & -\mathcal{J}_{24}^* & -\mathcal{J}_{34}^* & 2n_2\partial^{-1}n_2 \end{pmatrix},$$

$$\begin{aligned}
H_0 &= \frac{1}{9}\int (u_{1xx} - u_{1x})n_1 + (u_{2xx} - u_{2x})n_2\,\mathrm{d}x\,,\\
H_1 &= \frac{1}{9}\int [(u_1 - u_{1x})(v_1 + v_{1x}) + (u_2 - u_{2x})(v_2 + v_{2x})][(u_1 - u_{1x})n_1 + (u_2 - u_{2x})n_2]\,\mathrm{d}x\,,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{J}_{12} &= m_1\partial^{-1}m_2 + m_2\partial^{-1}m_1\,,\quad \mathcal{J}_{13} = -(2m_1\partial^{-1}n_1 + m_2\partial^{-1}n_2)\,,\\
\mathcal{J}_{24} &= -(2m_2\partial^{-1}n_2 + m_1\partial^{-1}n_1)\,,\quad \mathcal{J}_{34} = n_1\partial^{-1}n_2 + n_2\partial^{-1}n_1\,.
\end{aligned}$$

After applying the tri-Hamiltonian duality method to the Hamiltonian operators $\mathcal{K}$ and $\mathcal{J}$, we get the duality Hamiltonian operators

$$\widehat{\mathcal{K}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\quad \widehat{\mathcal{J}} = \begin{pmatrix} 2q_1\partial^{-1}q_1 & \widehat{\mathcal{J}_{12}} & \widehat{\mathcal{J}_{13}} & -q_1\partial^{-1}r_2 \\ -\widehat{\mathcal{J}_{12}}^* & 2q_2\partial^{-1}q_2 & -q_2\partial^{-1}r_1 & \widehat{\mathcal{J}_{24}} \\ -\widehat{\mathcal{J}_{31}}^* & -r_1\partial^{-1}q_2 & 2r_1\partial^{-1}r_1 & \widehat{\mathcal{J}_{34}} \\ -r_2\partial^{-1}q_1 & -\widehat{\mathcal{J}_{24}}^* & -\widehat{\mathcal{J}_{34}}^* & 2r_2\partial^{-1}r_2 \end{pmatrix}, \tag{33}$$

$$\widetilde{\mathcal{K}} = \begin{pmatrix} 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & \partial \\ \partial & 0 & 0 & 0 \\ 0 & \partial & 0 & 0 \end{pmatrix},\quad \widetilde{\mathcal{J}} = \begin{pmatrix} 2q_1\partial^{-1}q_1 & \widetilde{\mathcal{J}_{12}} & \widetilde{\mathcal{J}_{13}} & -q_1\partial^{-1}r_2 \\ -\widetilde{\mathcal{J}_{12}}^* & 2q_2\partial^{-1}q_2 & -q_2\partial^{-1}r_1 & \widetilde{\mathcal{J}_{24}} \\ -\widetilde{\mathcal{J}_{31}}^* & -r_1\partial^{-1}q_2 & 2r_1\partial^{-1}r_1 & \widetilde{\mathcal{J}_{34}} \\ -r_2\partial^{-1}q_1 & -\widetilde{\mathcal{J}_{24}}^* & -\widetilde{\mathcal{J}_{34}}^* & 2r_2\partial^{-1}r_2 \end{pmatrix}, \tag{34}$$

where

$$\begin{aligned}
\widehat{\mathcal{J}_{12}} &= q_1\partial^{-1}q_2 + q_2\partial^{-1}q_1\,,\\
\widehat{\mathcal{J}_{13}} &= \partial - (2q_1\partial^{-1}r_1 + q_2\partial^{-1}r_2)\,,\\
\widehat{\mathcal{J}_{24}} &= \partial - (2q_2\partial^{-1}r_2 + q_1\partial^{-1}r_1)\,,\\
\widehat{\mathcal{J}_{34}} &= r_1\partial^{-1}r_2 + r_2\partial^{-1}r_1\,,\\
\widetilde{\mathcal{J}_{12}} &= q_1\partial^{-1}q_2 + q_2\partial^{-1}q_1\,,\\
\widetilde{\mathcal{J}_{13}} &= 1 - (2q_1\partial^{-1}r_1 + q_2\partial^{-1}r_2)\,,\\
\widetilde{\mathcal{J}_{24}} &= 1 - (2q_2\partial^{-1}r_2 + q_1\partial^{-1}r_1)\,,\\
\widetilde{\mathcal{J}_{34}} &= r_1\partial^{-1}r_2 + r_2\partial^{-1}r_1\,.
\end{aligned}$$

The $\widehat{\mathcal{K}}, \widehat{\mathcal{J}}$ (33) and $\widetilde{\mathcal{K}}, \widetilde{\mathcal{J}}$ (34) are just the Hamiltonian pairs of the coupled AKNS hierarchy,[24] and coupled KN hierarchy, the spectral problems of them are respectively

$$\varphi_x = U\varphi, \quad U = \begin{pmatrix} 2\lambda & q_1 & q_2 \\ r_1 & -\lambda & 0 \\ r_2 & 0 & -\lambda \end{pmatrix}, \tag{35}$$

$$\varphi_x = U\varphi, \quad U = \begin{pmatrix} 2\lambda & q_1 & q_2 \\ \lambda r_1 & -\lambda & 0 \\ \lambda r_2 & 0 & -\lambda \end{pmatrix}. \tag{36}$$

Remark 2 The coupled nonlinear Schrödinger equation[25] is a reduction of the coupled AKNS hierarchy. In fact, it can be reduced to the coupled MKdV equation[26] and the Sasa–Satsuma equation[27] under the constraints $r_1 = q_1$, $r_2 = q_2$ and $r_1 = q_2$, $r_2 = q_1$ respectively.[28]

Furthermore, through the Dirac reductions of the Hamiltonian operators $\mathcal{J}$ and $\mathcal{K}\mathcal{J}^{-1}\mathcal{K}$ under the corresponding constraints, one can get a Hamiltonian structure and a symplectic structure for the coupled MKdV equation and the Sasa–Satsuma equation respectively.

A natural question is what are the reduced systems of the four-component CH system (32) and their corresponding Hamiltonian structures. Besides, it is worthwhile to investigate the reciprocal transformations between the CH systems (3), (25), (32) and their dual hierarchies.

Appendix

First, we prove that the operator $\mathcal{J}$ is Hamiltonian, namely to verify (15). To simplify the presentation and calculations, we introduce $M_i, N_i (i = 1, 2, \ldots, s)$ as

$$\mathcal{J}\theta = \begin{pmatrix} M_1 & \cdots & M_s & N_1 & \cdots & N_s \end{pmatrix}^{\mathrm{T}} = \sum_{i=1}^{s} \begin{pmatrix} m_1\partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) + m_i\partial^{-1}(m_1\theta_{1i} - n_i\theta_{21}) \\ \vdots \\ m_s\partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) + m_i\partial^{-1}(m_s\theta_{1i} - n_i\theta_{2s}) \\ -n_1\partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) - n_i\partial^{-1}(m_i\theta_{11} - n_1\theta_{2i}) \\ \vdots \\ -n_s\partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) - n_i\partial^{-1}(m_i\theta_{1s} - n_s\theta_{2i}) \end{pmatrix}. \tag{A1}$$

The associated bi-vector of $\mathcal{J}$ is

$$\begin{aligned}
\Theta_{\mathcal{J}} &= \frac{1}{2}\int \theta\wedge\mathcal{J}\theta\,\mathrm{d}x = \frac{1}{2}\sum_{j=1}^{s}\int \theta_{1j}\wedge M_j + \theta_{2j}\wedge N_j\,\mathrm{d}x \\
&= \frac{1}{2}\sum_{i,j=1}^{s}\int \theta_{1j}\wedge(m_j\partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) + m_i\partial^{-1}(m_j\theta_{1i} - n_i\theta_{2j})) \\
&\quad + \theta_{2j}\wedge(-n_j\partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) - n_i\partial^{-1}(m_i\theta_{1j} - n_j\theta_{2i}))\,\mathrm{d}x \\
&= \frac{1}{2}\sum_{i,j=1}^{s}\int (m_j\theta_{1j} - n_j\theta_{2j})\wedge\partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) + (m_j\theta_{1i} - n_i\theta_{2j})\wedge\partial^{-1}(m_i\theta_{1j} - n_j\theta_{2i})\,\mathrm{d}x\,,
\end{aligned}$$

where we have substituted the expressions of M_i, N_i $(i = 1, 2, \ldots, s)$ and used the skew-symmetry of the operator ∂^{-1}.

By direct calculation, we have the prolongation

$$\begin{aligned}
\mathrm{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}}) &= \sum_{i,j=1}^{s}\int (\theta_{1j}\wedge M_j - \theta_{2j}\wedge N_j)\wedge\partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) + (\theta_{1i}\wedge M_j - \theta_{2j}\wedge N_i)\wedge\partial^{-1}(m_i\theta_{1j} - n_j\theta_{2i})\,\mathrm{d}x \\
&= \sum_{i,j,k=1}^{s}\int [\theta_{1j}\wedge(m_j\partial^{-1}(m_k\theta_{1k} - n_k\theta_{2k}) + m_k\partial^{-1}(m_j\theta_{1k} - n_k\theta_{2j})) \\
&\quad + \theta_{2j}\wedge(n_j\partial^{-1}(m_k\theta_{1k} - n_k\theta_{2k}) + n_k\partial^{-1}(m_k\theta_{1j} - n_j\theta_{2k}))]\wedge\partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) \\
&\quad + [\theta_{1i}\wedge(m_j\partial^{-1}(m_k\theta_{1k} - n_k\theta_{2k}) + m_k\partial^{-1}(m_j\theta_{1k} - n_k\theta_{2j})) \\
&\quad + \theta_{2j}\wedge(n_i\partial^{-1}(m_k\theta_{1k} - n_k\theta_{2k}) + n_k\partial^{-1}(m_k\theta_{1i} - n_i\theta_{2k}))]\wedge\partial^{-1}(m_i\theta_{1j} - n_j\theta_{2i})\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j,k=1}^{s} \int (m_j\theta_{1j} + n_j\theta_{2j}) \wedge \partial^{-1}(m_k\theta_{1k} - n_k\theta_{2k}) \wedge \partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) \\
&\quad + (m_j\theta_{1k} + n_k\theta_{2j}) \wedge \partial^{-1}(m_k\theta_{1j} - n_j\theta_{2k}) \wedge \partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) \\
&\quad + (m_j\theta_{1i} + n_i\theta_{2j}) \wedge \partial^{-1}(m_k\theta_{1k} - n_k\theta_{2k}) \wedge \partial^{-1}(m_i\theta_{1j} - n_j\theta_{2i}) \\
&\quad + m_k\theta_{1i} \wedge \partial^{-1}(m_j\theta_{1k} - n_k\theta_{2j}) \wedge \partial^{-1}(m_i\theta_{1j} - n_j\theta_{2i}) \\
&\quad + n_k\theta_{2j} \wedge \partial^{-1}(m_k\theta_{1i} - n_i\theta_{2k}) \wedge \partial^{-1}(m_i\theta_{1j} - n_j\theta_{2i})\,\mathrm{d}x \\
&= \sum_{i,j,k=1}^{s} \int m_k\theta_{1i} \wedge \partial^{-1}(m_j\theta_{1k} - n_k\theta_{2j}) \wedge \partial^{-1}(m_i\theta_{1j} - n_j\theta_{2i}) \\
&\quad + n_k\theta_{2j} \wedge \partial^{-1}(m_k\theta_{1i} - n_i\theta_{2k}) \wedge \partial^{-1}(m_i\theta_{1j} - n_j\theta_{2i})\,\mathrm{d}x\,, \qquad \text{(A2)}
\end{aligned}
$$

where we have used integration by parts and the skew-symmetry of the operator ∂^{-1}. Afterwards, expanding the two terms in Eq. (A2) into eight terms, we get

$$
\begin{aligned}
\mathrm{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}}) &= \sum_{i,j,k=1}^{s} \int m_k\theta_{1i} \wedge \partial^{-1}m_j\theta_{1k} \wedge \partial^{-1}m_i\theta_{1j} - m_k\theta_{1i} \wedge \partial^{-1}m_j\theta_{1k} \wedge \partial^{-1}n_j\theta_{2i} \\
&\quad - m_k\theta_{1i} \wedge \partial^{-1}n_k\theta_{2j} \wedge \partial^{-1}m_i\theta_{1j} + m_k\theta_{1i} \wedge \partial^{-1}n_k\theta_{2j} \wedge \partial^{-1}n_j\theta_{2i} \\
&\quad + n_k\theta_{2j} \wedge \partial^{-1}m_k\theta_{1i} \wedge \partial^{-1}m_i\theta_{1j} - n_k\theta_{2j} \wedge \partial^{-1}m_k\theta_{1i} \wedge \partial^{-1}n_j\theta_{2i} \\
&\quad - n_k\theta_{2j} \wedge \partial^{-1}n_i\theta_{2k} \wedge \partial^{-1}m_i\theta_{1j} + n_k\theta_{2j} \wedge \partial^{-1}n_i\theta_{2k} \wedge \partial^{-1}n_j\theta_{2i}\,\mathrm{d}x \\
&= \sum_{i,j,k=1}^{s} \int -m_k\theta_{1i} \wedge \partial^{-1}m_j\theta_{1k} \wedge \partial^{-1}n_j\theta_{2i} - m_k\theta_{1i} \wedge \partial^{-1}n_k\theta_{2j} \wedge \partial^{-1}m_i\theta_{1j} \\
&\quad + m_k\theta_{1i} \wedge \partial^{-1}n_k\theta_{2j} \wedge \partial^{-1}n_j\theta_{2i} + n_k\theta_{2j} \wedge \partial^{-1}m_k\theta_{1i} \wedge \partial^{-1}m_i\theta_{1j} \\
&\quad - n_k\theta_{2j} \wedge \partial^{-1}m_k\theta_{1i} \wedge \partial^{-1}n_j\theta_{2i} - n_k\theta_{2j} \wedge \partial^{-1}n_i\theta_{2k} \wedge \partial^{-1}m_i\theta_{1j}\,\mathrm{d}x \\
&= \sum_{i,j,k=1}^{s} \int -m_k\theta_{1i} \wedge \partial^{-1}m_j\theta_{1k} \wedge \partial^{-1}n_j\theta_{2i} + m_i\theta_{1j} \wedge \partial^{-1}m_k\theta_{1i} \wedge \partial^{-1}n_k\theta_{2j} \\
&\quad + n_j\theta_{2i} \wedge \partial^{-1}n_k\theta_{2j} \wedge \partial^{-1}m_k\theta_{1i} - n_k\theta_{2j} \wedge \partial^{-1}n_i\theta_{2k} \wedge \partial^{-1}m_i\theta_{1j}\,\mathrm{d}x = 0\,. \qquad \text{(A3)}
\end{aligned}
$$

In the above, we have dropped the terms which only contain m_i or n_i using the integration by parts and the skew-symmetry of the operator ∂^{-1}, which are also applied to the remaining terms.

From Eq. (3), we know $\mathcal{J}$ is Hamiltonian.

Secondly, we will show the compatibility of $\mathcal{K}$ and $\mathcal{J}$, i.e., the equality (16).

Notice that

$$
\mathcal{K}\theta = \begin{pmatrix} \theta_{2x} + \theta_2 \\ \theta_{1x} - \theta_1 \end{pmatrix}, \qquad \text{(A4)}
$$

we calculate

$$
\begin{aligned}
\mathrm{PrV}_{\mathcal{K}\theta}(\Theta_{\mathcal{J}}) &= \sum_{i,j=1}^{s} \int [\theta_{1j} \wedge (\theta_{2jx} + \theta_{2j}) - \theta_{2j} \wedge (\theta_{1jx} - \theta_{1j})] \wedge \partial^{-1}(m_i\theta_{1i} - n_i\theta_{2i}) \\
&\quad + [\theta_{1i} \wedge (\theta_{2jx} + \theta_{2j}) - \theta_{2j} \wedge (\theta_{1ix} - \theta_{1i})] \wedge \partial^{-1}(m_i\theta_{1j} - n_j\theta_{2i})\,\mathrm{d}x \\
&= \sum_{i,j=1}^{s} \int -\theta_{1j} \wedge \theta_{2j} \wedge (m_i\theta_{1i} - n_i\theta_{2i}) - \theta_{1i} \wedge \theta_{2j} \wedge (m_i\theta_{1j} - n_j\theta_{2i})\,\mathrm{d}x = 0\,, \qquad \text{(A5)}
\end{aligned}
$$

which implies the operators $\mathcal{K}$ and $\mathcal{J}$ are compatible Hamiltonian operators.

Thus, we complete the proof of the theorem.

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