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Integrability of the modified generalised Vakhnenko equation

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Integrability of the modified generalised Vakhnenko equation is investigated systematically. Based on binary Bell polynomials, its bilinear representation, *N* soliton solutions, bilinear Bäcklund transformation, and Lax pair are succinctly constructed. Moreover, the conservation laws of the modified generalised Vakhnenko equation are discussed by using corresponding Lax pair. Furthermore, the quasiperiodic solution of the modified generalised Vakhnenko equation is presented by applying Hirota direct method and Riemann theta function. The asymptotic behavior of the one periodic wave is analyzed in details. It is shown that the one periodic wave solution tends to the one soliton solution under a small amplitude limit $\lambda \rightarrow 0$. Finally, the new *N* soliton solutions of the standard Vakhnenko equation are presented. It would be specially mentioned that all the results of modified generalised Vakhnenko equation can be reduced to the generalised Vakhnenko equation and standard Vakhnenko equation under the special case of $\alpha = 1$ and $\alpha = 1$, $\beta = 0$, respectively. © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.4764845]

I. INTRODUCTION

In recent years, considerable attention has been paid to the nonlinear evolution equations (NLEEs) which can describe certain nonlinear phenomena realistically. In Ref. 1, Vakhnenko equation (VE) is presented

$$\frac{\partial}{\partial x}\mathcal{D}u + u = 0$$
, with $\mathcal{D} := \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$, (1)

which describes the propagation of shortwave perturbations in a relaxing medium.² It can be demonstrated that VE stems from the short wave limit of the following NLEE:^{3,4}

$$u_t - u_{t,2x} + \alpha u_x + (\beta + 1)uu_x = \beta u_x u_{2x} + u u_{3x}$$
(2)

with $\beta = 2$ and 3, Eq. (2) can be reduced to the Camassa-Holm equation and Degasperis-Procesi equation, respectively.

In fact, introducing the time and space variables *T* and *X* as follows:

$$T = \epsilon t, X = \epsilon^{-1} x, \tag{3}$$

where ϵ is a small positive parameter, and it is easy to get

$$\frac{\partial}{\partial t} = \epsilon \frac{\partial}{\partial T}, \ \frac{\partial}{\partial x} = \epsilon^{-1} \frac{\partial}{\partial X}.$$
(4)

Assuming the expansion

$$u = \epsilon^2 (u_0 + \epsilon u_1 + \ldots) \tag{5}$$

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with $u_i(i = 0, 1, ...)$ being functions with respect to T and X. Substituting (4) and (5) into (2), we obtain the following NLEE for u_0 at the lowest order in ϵ :

$$u_{0,2X,T} - \alpha u_{0,X} + \beta u_{0,X} u_{0,2X} + u_0 u_{0,3X} = 0.$$
 (6)

Based on the original variables x, t, and $u \sim \epsilon^2 u_0$, we can rewrite (6) as

$$u_{t,2x} - \alpha u_x + \beta u_x u_{2x} + u u_{3x} = 0.$$
⁽⁷⁾

Integrating (7) about x and taking the integral constants as 0, we have

$$u_{t,x} - \alpha u + \frac{\beta - 1}{2}u_x^2 + uu_{2x} = 0.$$
 (8)

When $\beta = 3$, (8) can be rewritten as

$$u_{t,x} - \alpha u + \left(\frac{u^2}{2}\right)_{2x} = 0,$$
 (9)

which is also called the reduced Ostrovsky equation,^{5,6} and when $\beta = 3$, $\alpha = -1$, it is just to the VE. The multiple-valued function solutions of VE and an example of a collision of solitons are given.¹ The exact two loop soliton solutions and *N* loop soliton solutions to the VE are also obtained.^{7,8} More recently, the interaction of a soliton with a one-mode periodic wave, the interaction of the solitons and multi-mode periodic waves, and the interaction of the *N* periodic waves of this equation are discussed, respectively.^{9–11}

In Ref. 12, the VE was extended to the generalised Vakhnenko equation (GVE),

$$\frac{\partial}{\partial x}(\mathcal{D}^2 u + \frac{1}{2}u^2 + \beta u) + \mathcal{D}u = 0,$$
(10)

or equivalently

$$\left(\frac{\partial u}{\partial x} + \mathcal{D}\right)\left(\frac{\partial}{\partial x}\mathcal{D}u + u + \beta\right) = 0,\tag{11}$$

where β is an arbitrary constant. If $\beta = 0$, Eq. (11) can be transformed to the VE (1). The *N* soliton solution to GVE is found by a blend of transformations of the independent variables and Hirota's direct method.¹² Bäcklund transformation (BT), Lax pair, and conservation laws for the GVE are discussed and the exact *N* soliton solutions are obtained via the inverse scattering method.¹³

Morrison and Parkes¹⁴ presented the modified generalised Vakhnenko equation (mGVE),

$$\frac{\partial}{\partial x}(\mathcal{D}^2 u + \frac{1}{2}pu^2 + \beta u) + q\mathcal{D}u = 0,$$
(12)

where p, q, and β are arbitrary constants. If p = q = 1, Eq. (12) can be transformed to the GVE (10); if p = q = 1, $\beta = 0$, Eq. (12) can be transformed to the VE (1).

Recently, the Bell polynomials are found to play an important role in the characterization of integrability of NLEEs.^{15–31} As is well known, the Hirota direct method^{32,33} is a powerful tool for investigating integrability of the NLEEs. Once bilinear equation of the NLEE is given, one can construct its corresponding multi-soliton solutions, quasiperiodic solutions, Wronskian solutions, bilinear Bäcklund transformation, etc. However, the construction of the bilinear equation usually requires suitable variable transformation and complex calculation. Lambert and his co-workers link Hitota D-operators to Bell polynomials and provide a quick and natural approach to obtain bilinear formulisms, bilinear Bäcklund transformations, Lax pairs, and Darboux covariant Lax pairs for the NLEEs.¹⁶⁻²¹ More recently, Fan²²⁻²⁵ extend Bell polynomials to variable coefficients NLEEs and supersymmetric equations and propose a approach to construct infinite conservation laws of NLEEs through decoupling binary Bell polynomials into a Riccati type equation and a divergence type equation. We extend this approach to investigate the integrability of generalised Nizhnik-Novikov-Veselov equation, variable coefficient mKdV equation and some (2+1)-dimensional NLEEs.²⁸⁻³¹ The aim of this paper is to extend the Bell polynomials approach to systematically investigate integrability of the mGVE. There are many significant properties, such as bilinear form, Lax pair, Bäcklund transformation, Darboux transformation, Painlevé analysis, infinite conservation laws, Hamiltonian

structure, and infinite symmetries that can characterize integrability of NLEEs. In present paper, the bilinear formulism, bilinear Bäcklund transformation, Lax pair, and infinite conservation laws of the mGVE are investigated systematically by using the binary Bell polynomials. The *N* soliton solutions and quasiperiodic solution of the mGVE are also presented by applying the Hirota direct method and Riemann theta function. Specifically, the new *N* soliton solutions of the standard VE are presented. In addition, the asymptotic analysis shown that the one periodic wave solution tends to the one soliton solution under a small amplitude limit $\lambda \rightarrow 0$. It would be specially mentioned that all the results of mGVE can be reduced to the GVE and standard VE under the special case of $\alpha = 1$ and $\alpha = 1$, $\beta = 0$, respectively.

The organization of this paper is as follows. In Sec. II, we give a brief introduction about the binary Bell polynomials. In Sec. III, we systematically construct bilinear formulism, *N* soliton solutions, bilinear Bäcklund transformation, Lax pair, and infinite conservation laws of the mGVE. In Sec. IV, we present the quasiperiodic solution of mGVE by using Hirota direct method and Riemann theta function. The relation between the one periodic wave solution and the one soliton solution is analyzed in details. Section V will offer the conclusions which contains the new *N* soliton solutions of the standard VE.

II. BELL POLYNOMIALS PRELIMINARY

In 1934, Bell¹⁵ proposed three types of exponent-form polynomials which have been successfully applied to combinatorics, statistics, and other fields. Lambert and his co-workers link the third kind Bell polynomials to Hirota *D*-operator, propose a lucid approach to investigate NLEEs. In the following, we give a brief introduction of the Bell polynomials.^{15–21}

Definition 1: With the assumption that f = f(x) is a \mathbb{C}^{∞} function of x and $f_{rx} = \partial_x^r f$, r = 1, 2, ..., n, then

$$Y_{nx}(f) \equiv Y_n(f_x, \dots, f_{nx}) = Y_n(\{f_{rx}(1 \le n)\}) = e^{-f} \partial_x^n e^f, \ f_{0x} \equiv f,$$
(13)

i.e.,

$$Y_x(f) = f_x, Y_{2x}(f) = f_{2x} + f_x^2, Y_{3x}(f) = f_{3x} + 3f_x f_{2x} + f_x^3, \dots$$
(14)

is a polynomial in the derivatives of f with respect to x, which called the one-dimensional Bell polynomials or Y-polynomials.

Based on the one-dimensional Bell polynomials, the multi-dimensional Bell polynomials are defined as follows.

Definition 2: With $f = f(x_1, x_2, ..., x_l)$ be a \mathbb{C}^{∞} function with multi-variables and

$$f_{r_1 x_1, \dots, r_l x_l} = \partial_{x_1}^{r_1} \dots \partial_{x_l}^{r_l} f, \ f_{0 x_i} \equiv f, \tag{15}$$

where l denotes arbitrary integer, then

$$Y_{n_1 x_1, \dots, n_l x_l}(f) \equiv Y_{n_1, \dots, n_l}(\{f_{r_1 x_1, \dots, r_l x_l}(1 \le r_i \le n_i, 0 \le i \le l)\}) = e^{-f} \partial_{x_1}^{n_1} \dots \partial_{x_l}^{n_l} e^f$$
(16)

is a polynomial in the partial derivatives of f with respect to x_1, \ldots, x_l , which called the multidimensional Bell polynomials.

For f = f(x, t), the associated two-dimensional Bell polynomials can be written as

$$Y_{x,t}(f) = f_{x,t} + f_x f_t, Y_{2x,t}(f) = f_{2x,t} + f_{2x} f_t + 2f_{x,t} f_x + f_x^2 f_t, \dots$$
(17)

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Definition 3: By virtue of above multi-dimensional Bell polynomials, the multi-dimensional binary Bell polynomials can be defined as follows:

$$\mathcal{Y}_{n_{1}x_{1},...,n_{l}x_{l}}(v,w) \equiv Y_{n_{1}x_{1},...,n_{l}x_{l}}(f)$$

$$\equiv Y_{n_{1},...,n_{l}(\{f_{r_{1}x_{1},...,r_{l}x_{l}}\})} \bigg|_{f_{r_{1}x_{1},...,r_{l}x_{l}}} \begin{cases} v_{r_{1}x_{1},...,r_{l}x_{l}}, \sum_{i=1}^{l} r_{i} \text{ is odd,} \\ w_{r_{1}x_{1},...,r_{l}x_{l}}, \sum_{i=1}^{l} r_{i} \text{ is even,} \end{cases}$$
(18)

where the vertical line means that the elements on the left-hand side are chosen according to the rule on the right-hand side, v and w are both the \mathbb{C}^{∞} functions of (x_1, x_2, \ldots, x_l) .

For example, the first few lowest order binary Bell Polynomials are

$$\mathcal{Y}_{x}(v) = v_{x}, \mathcal{Y}_{2x}(v, w) = w_{2x} + v_{x}^{2}, \mathcal{Y}_{x,y}(v, w) = w_{x,y} + v_{x}v_{y},$$
(19a)

$$\mathcal{Y}_{3x}(v,w) = v_{3x} + 3v_x w_{2x} + v_x^3, \dots$$
(19b)

Proposition 1: The relations between the binary Bell polynomials and the standard Hirota D-operators can be given by the identity

$$\mathcal{Y}_{n_1 x_1, \dots, n_l x_l}(v = \ln \frac{F}{G}, w = \ln FG) = (F \cdot G)^{-1} D_{x_1}^{n_1} \dots D_{x_l}^{n_l} F \cdot G,$$
(20)

where $\sum_{i=1}^{l} n_i \ge 1$, and Hirota D-operators defined by

$$D_{x_1}^{n_1} \dots D_{x_l}^{n_l} F \cdot G = (\partial_{x_1} - \partial_{x_1'})^{n_1} \dots (\partial_{x_l} - \partial_{x_l'})^{n_l} F(x_1, \dots, x_l) G(x_1', \dots, x_l') \Big|_{x_1' = x_1, \dots, x_l' = x_l}$$
(21)

In the particular case of F = G, the formula (20) can be rewritten as

$$F^{-2}D_{x_{1}}^{n_{1}}\dots D_{x_{l}}^{n_{l}}F \cdot F = \mathcal{Y}_{n_{1}x_{1},\dots,n_{l}x_{l}}(0, Q = w - v = 2\ln F) = \begin{cases} 0, \sum_{i=1}^{l} n_{i} \text{ is odd,} \\ P_{n_{1}x_{1},\dots,n_{l}x_{l}}(Q), \sum_{i=1}^{l} n_{i} \text{ is even,} \end{cases}$$
(22)

which is also called P-polynomials

$$P_{n_1x_1,\dots,n_lx_l}(Q) = \mathcal{Y}_{n_1x_1,\dots,n_lx_l}(0, Q = 2\ln F),$$
(23)

where they vanish unless $\sum_{i=1}^{l} n_i$ is even.

For example, the first few P-polynomials are

$$P_{2x}(Q) = Q_{2x}, P_{x,t}(Q) = Q_{x,t}, P_{3x,t}(Q) = Q_{3x,t} + 3Q_{x,t}Q_{2x}, \dots$$
(24)

It has been found that formulas (20) and (22) play an important role in connecting NLEEs with their corresponding bilinear equations, i.e., once a NLEE can be expressed as a linear combination of the *P*-polynomials, then its bilinear equation can be established directly.

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Proposition 2: The binary Bell polynomials $\mathcal{Y}_{n_1x_1,...,n_lx_l}(v, w)$ can be written as the combination of P-polynomials and Y-polynomials

$$(F \cdot G)^{-1} D_{x_{1}}^{n_{1}} \dots D_{x_{l}}^{n_{l}} F \cdot G = \mathcal{Y}_{n_{1}x_{1},\dots,n_{l}x_{l}}(v, w) \Big|_{v=\ln F/G, w=\ln FG}$$

$$= \mathcal{Y}_{n_{1}x_{1},\dots,n_{l}x_{l}}(v, v+Q) \Big|_{v=\ln F/G, Q=2\ln G}$$

$$= \sum_{n_{1}+\dots+n_{l}=\text{even}} \sum_{r_{1}=0}^{n_{1}} \dots \sum_{r_{l}=0}^{n_{l}} \prod_{i=1}^{l} \binom{n_{i}}{r_{i}} P_{n_{1}x_{1},\dots,n_{l}x_{l}}(Q) Y_{(n_{1}-r_{1})x_{1},\dots,(n_{l}-r_{l})x_{l}}(v).$$
(25)

Remark 1: In order to obtain the Lax pairs of corresponding NLEEs, we introduce the Hopf-Cole transformation $v = \ln \psi$, i.e., $\psi = F/G$, then the Y-polynomials can be written as

$$Y_{n_1 x_1, \dots, n_l x_l}(v)\Big|_{v=\ln \psi} = \frac{\psi_{n_1 x_1, \dots, n_l x_l}}{\psi},$$
(26)

which provides the shortest way to the associated Lax systems of NLEEs.

III. *N* SOLITON SOLUTION, BILINEAR BÄCKLUND TRANSFORMATION, LAX PAIR, AND INFINITE CONSERVATION LAWS OF THE MGVE

A. The transformation process of the mGVE

On the investigation of integrability for mGVE, the key step is to introduce the following transformations:⁷

$$x = \theta(X, T) := T + Z(X, T) + x_0, t = X, Z = \int_{-\infty}^{X} U(X', T) dX',$$
(27)

where x_0 is a constant, u(x, t) = U(X, T).

We next introduce W(X, T) defined by

$$W_X = U, (28)$$

where W tends to a constant and its derivatives vanish as $|X| \rightarrow \infty$. The dependent variable transformation (28) means that

$$u(x,t) = U(X,T) = W_X.$$
 (29)

From (27) and (28), it is easy to get

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial T} = \phi(X, T) \frac{\partial}{\partial x}, \quad (30)$$

where

$$\phi(X,T) = 1 + \int_{-\infty}^{X} U_T dX' = 1 + W_T.$$
(31)

By virtue of (27)–(31), mGVE (12) is transformed into the following version of the shallow water wave equation:

$$U_{2X,T} + pUU_T + qU_X \int_{-\infty}^X U_T(X',T) dX' + qU_X + \beta U_T = 0.$$
 (32)

Some special cases of equation (32) have been studied in the literature.^{14,34–36} In present paper, we focus on the case $p = q = \alpha$ and β an arbitrary non-zero constant, namely,

$$U_{2X,T} + \alpha U U_T + \alpha U_X \int_{-\infty}^X U_T(X',T) dX' + \alpha U_X + \beta U_T = 0$$
(33)

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or equivalently

$$W_{2X,T} + \alpha W_X(W_T + 1) + \beta W_T = 0, \qquad (34)$$

which corresponding to the mGVE,^{36,37}

$$\frac{\partial}{\partial x}(\mathcal{D}^2 u + \frac{1}{2}\alpha u^2 + \beta u) + \alpha \mathcal{D} u = 0.$$
(35)

Remark 2: For the case of $\alpha = 1$, *Eq. (34) reduced to the GVE*,¹³

$$W_{2X,T} + W_X(W_T + 1) + \beta W_T = 0.$$
(36)

For the case of $\alpha = 1$, $\beta = 0$, Eq. (34) reduced to the standard VE,^{7,8}

$$W_{2X,T} + W_X(W_T + 1) = 0. (37)$$

B. Bilinearization

Theorem 1: Under the following transformation:

.

$$W = \frac{6}{\alpha} (\ln F)_X + \Phi(T), \qquad (38)$$

the mGVE (34) can be bilinearized into

$$G(D_X, D_T) = \left\{ D_X^3 D_T + \alpha [1 + \phi(T)] D_X^2 + \beta D_X D_T + \frac{\alpha \beta}{3} \phi(T) \right\} F \cdot F = 0,$$
(39)

where $\alpha \neq 0$, $\phi(T) = \Phi'(T)$.

Proof: We introduce a dimensionless potential field Q by setting

$$W = CQ_X + \Phi(T), \tag{40}$$

with *C* a parameter to be determined.

Substituting (40) into mGVE (34), then mGVE (34) can be rewritten as

$$H(Q) = C Q_{3X,T} + \alpha C^2 Q_{2X} Q_{X,T} + \alpha C \phi(T) Q_{2X} + \alpha C Q_{2X} + \beta C Q_{X,T} + \beta \phi(T) = 0, \quad (41)$$

where $\phi(T) = \Phi'(T)$.

According to the formula (23), the *C* can be identified as $\frac{3}{\alpha}$ and thus (41) can be cast into the combination of *P*-polynomials

$$H(Q) = P_{3X,T}(Q) + \alpha [1 + \phi(T)] P_{2X}(Q) + \beta P_{X,T} + \frac{\alpha \beta}{3} \phi(T) = 0.$$
(42)

By virtue of dependent variable transformation

$$Q = 2 \ln F$$
, i.e., $W = \frac{3}{\alpha} Q_X + \Phi(T) = \frac{6}{\alpha} (\ln F)_X + \Phi(T)$, (43)

and the identity (20), the bilinear representation of mGVE (34) can be obtained directly as follows:

$$G(D_X, D_T) = \left\{ D_X^3 D_T + \alpha [1 + \phi(T)] D_X^2 + \beta D_X D_T + \frac{\alpha \beta}{3} \phi(T) \right\} F \cdot F = 0.$$
(44)

Based on (29), (38), and (44) the solutions of the mGVE (35) can be given under the relations between (x, t) and (X, T),

$$x = T + \frac{6}{\alpha} (\ln F)_X + \Phi(T) + x_0, t = X, u(x, t) = U(X, T).$$
(45)

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Remark 3: For the case of $\alpha = 1$ *, bilinear representation (44) of mGVE reduced to the ones of the GVE,*

$$G(D_X, D_T) = \left\{ D_X^3 D_T + [1 + \phi(T)] D_X^2 + \beta D_X D_T + \frac{\beta}{3} \phi(T) \right\} F \cdot F = 0.$$
(46)

For the case of $\alpha = 1$, $\beta = 0$, bilinear representation (44) of mGVE reduced to the ones of the standard VE,

$$G(D_X, D_T) = \left\{ D_X^3 D_T + [1 + \phi(T)] D_X^2 \right\} F \cdot F = 0.$$
(47)

C. N soliton solutions

In this section, we give the *N* soliton solution of the mGVE by using the Hirota direct method. For convenience to derived the *N* soliton solution, taking $\phi(T) = 0$, which indicates that $\Phi(T)$ to be arbitrary constant C_1 , then bilinear equation (39) can be rewritten as

$$G(D_X, D_T) = \left[D_X^3 D_T + \alpha D_X^2 + \beta D_X D_T \right] F \cdot F = 0.$$
(48)

Based on Hirota direct method, we expand F(X, T) as exponential functions

 $F(X,T) = 1 + F^{(1)}\epsilon + F^{(2)}\epsilon^2 + \ldots + F^{(j)}\epsilon^j + \ldots,$ (49)

where ϵ is a constant called the perturbation parameter.

Inserting the expression (49) into (48) and by using (29), the N soliton solution of the mGVE (35) can be obtained as follows:

$$\begin{cases} x = T + \frac{3\omega_1}{\alpha} \Big[1 + \tanh \frac{\eta_1}{2} \Big] + x_1, \text{ with } x_1 = C_1 + x_0, \\ u(x,t) = \frac{6}{\alpha} \Big[\ln \left(\sum_{\mu=0,1} e^{\sum_{j=1}^n \mu_j \eta_j + \sum_{1 \le j < l}^n \mu_j \mu_l A_{jl}} \right) \Big]_{2X}, \iota_j = -\frac{\alpha\omega_j}{\beta + \omega_j^2}, \beta + \omega_j^2 \neq 0, \end{cases}$$
(50)

where

$$e^{A_{jl}} = \frac{(\omega_j - \omega_l)^2 (\omega_j^2 - \omega_j \omega_l + \omega_l^2 + 3\beta)}{(\omega_j + \omega_l)^2 (\omega_j^2 + \omega_j \omega_l + \omega_l^2 + 3\beta)}, \eta_j = \omega_j X + \iota_j T + \delta_j, (j < l, j, l = 1, 2, 3, \ldots),$$
(51)

and $\sum_{\mu=0,1}$ indicates the summation over all possible combination of $\mu_j = 0, 1 (j = 1, 2, ...)$.

For example, the single soliton solution takes the form

$$\begin{cases} x = T + \frac{3\omega_1}{\alpha} \left\{ 1 + \tanh\left[\frac{\omega_1 t}{2} - \frac{\alpha\omega_1}{2(\beta + \omega_1^2)}T + \frac{\delta_1}{2}\right] \right\} + x_1, \\ u = \frac{3\omega_1^2}{2\alpha} \operatorname{sech}^2 \left[\frac{\omega_1 t}{2} - \frac{\alpha\omega_1}{2(\beta + \omega_1^2)}T + \frac{\delta_1}{2}\right], \beta + \omega_1^2 \neq 0. \end{cases}$$
(52)

Taking $x_1 = -\frac{3\omega_1}{\alpha}$, the single soliton solution (52) can be rewritten as

$$\begin{cases} x = T + \frac{3\omega_1}{\alpha} \tanh\left[\frac{\omega_1 t}{2} - \frac{\alpha\omega_1}{2(\beta + \omega_1^2)}T + \frac{\delta_1}{2}\right],\\ u = \frac{3\omega_1^2}{2\alpha} \operatorname{sech}^2\left[\frac{\omega_1 t}{2} - \frac{\alpha\omega_1}{2(\beta + \omega_1^2)}T + \frac{\delta_1}{2}\right], \beta + \omega_1^2 \neq 0. \end{cases}$$
(53)

Fixing $\alpha = 1$, t = 0, $\delta_1 = 0$, and by choosing different values for parameter β , we get three shape types of single-soliton solution which are shown in Fig. 1,

$$\beta = -11, \omega_1 = -4, \tag{54a}$$

$$\beta = -11, \,\omega_1 = -5,\tag{54b}$$

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FIG. 1. The single soliton solution (53) with $\alpha = 1$, $t = \delta_1 = 0$, and (a) (54a); (b) (54b); (c) (54c); (d) (54d); (e) (54e); and (f) (54f). And these figures demonstrates that the parameter β have strong impacts on the shapes of the soliton solution of the mGVE. The parameter β determines the shape types of the single solution, namely, loop-like soliton solution, cusp-like soliton solution, and hump-like soliton solution, while the soliton solution obviously varies in the amplitudes with the different ω_1 .

$$\beta = 10, \,\omega_1 = -4.5, \tag{54c}$$

$$\beta = 10, \,\omega_1 = -4,\tag{54d}$$

$$\beta = 20, \, \omega_1 = -3.5, \tag{54e}$$

$$\beta = 20, \omega_1 = -2.$$
 (54f)

Two soliton solution is

$$\begin{cases} x = T + \frac{6}{\alpha} \left(\ln F \right)_X + x_2, x_2 = C_1 + x_0, \\ u = \frac{6}{\alpha} \left(\ln F \right)_{2X}, \end{cases}$$
(55)

where $F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}}$, $\eta_j = \omega_j X + \iota_j T + \delta_j$, $\beta + \omega_j^2 \neq 0$ (j = 1, 2), and $e^{A_{12}}$ can be obtained directly from (50) and (51).

By setting $\alpha = 1$, t = 2, $x_2 = \delta_1 = \delta_2 = 0$, and choosing different values for parameters ω_1 , ω_2 , and β , six shapes of the two soliton solutions are shown in Fig. 2 and the interaction process of cusp-loop-like and loop-loop-like soliton solutions are shown in Fig. 3,

$$\omega_1 = 2, \, \omega_2 = 5, \, \beta = 2, \tag{56a}$$

$$\omega_1 = 2, \, \omega_2 = 5, \, \beta = 4,$$
 (56b)

$$\omega_1 = 2, \, \omega_2 = 2.1, \, \beta = 5, \tag{56c}$$

$$\omega_1 = 5, \, \omega_2 = 6, \, \beta = 6, \tag{56d}$$

$$\omega_1 = 2, \, \omega_2 = 5, \, \beta = 15, \tag{56e}$$

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FIG. 2. The two soliton solution (55) with $\alpha = 1, t = 2, x_2 = \delta_1 = \delta_2 = 0$, and (a) cusp-loop-like (56a); (b) hump-loop-like (56b); (c) hump-hump-like (56c); (d) loop-loop-like (56d); (e) hump-cusp-like (56e); and (f) cusp-cusp-like (56f).

$$\omega_1 = 5, \, \omega_2 = 6, \, \beta = 16. \tag{56f}$$

Three soliton solution is

$$\begin{cases} x = T + \frac{6}{\alpha} \left(\ln F \right)_X + x_3, x_3 = C_1 + x_0, \\ u = \frac{6}{\alpha} \left(\ln F \right)_{2X}, \end{cases}$$
(57)



FIG. 3. The interaction process of cusp-loop-like and loop-loop-like soliton solutions. (a) The time t from left to right are t = -2, -0.5, 2. (b) The time t from left to right are t = -1.5, 0.2, 2. The two figures show that both of cusp-loop-like and loop-loop-like soliton would collide but preserve their individual shapes and speeds. In fact, the other shape types soliton solutions including hump-loop-like, hump-cusp-like, cusp-cusp-like, hump-hump-like soliton solutions also have the characteristic.

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FIG. 4. The three soliton solution (57) with $\alpha = 1$, t = 3, $x_3 = \delta 1 = \delta 2 = \delta 3 = 0$, and (a) cusp-loop-loop-like (58a); (b) loop-loop-loop-like (58b); (c) cusp-cusp-cusp-like (58c); and (d) hump-hump-like (58d).

where $F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}} + e^{\eta_1 + \eta_3 + A_{13}} + e^{\eta_2 + \eta_3 + A_{23}} + e^{\eta_1 + \eta_2 + \eta_3 + A_{13} + A_{23}}, \beta + \omega_j^2 \neq 0$ and $\eta_j, e^{A_{jl}} (j < l, j, l = 1, 2, 3)$ can also be obtained directly from (50) and (51).

By setting $\alpha = 1$, t = 3, $x_3 = \delta_1 = \delta_2 = \delta_3 = 0$, and choosing different values for parameters $\omega_1, \omega_2, \omega_3$, and β , four shapes of the three soliton solutions are shown in Fig. 4 and the interaction process of loop-loop-cusp-like soliton solution is shown in Fig. 5,

$$\omega_1 = 2, \, \omega_2 = 5, \, \omega_3 = 3, \, \beta = 1.9, \tag{58a}$$

$$\omega_1 = 5, \, \omega_2 = 5.4, \, \omega_3 = 5.2, \, \beta = 5, \tag{58b}$$

$$\omega_1 = 5, \, \omega_2 = 5.4, \, \omega_3 = 5.2, \, \beta = 15, \tag{58c}$$

$$\omega_1 = 3, \, \omega_2 = 3.5, \, \omega_3 = 4, \, \beta = 22.$$
 (58d)

Remark 4: The transformation $X \to \frac{1}{\alpha}X$, $\beta \to \alpha^2\beta$ with $\alpha \neq 0$ reduces Eq. (34) to the GVE (36). Therefore, for the corresponding figures, it is necessary to consider the case with $\alpha = 1$ only.



FIG. 5. The interaction process of loop-loop-cusp-like soliton solutions. The time t from left to right are t = -3, -0.5, 4. This figure shows that loop-loop-cusp-like soliton collide but preserve their individual shapes and speeds. As similar to the two soliton solutions, the other shape types three soliton solutions including loop-loop-loop-like, cusp-cusp-like, hump-hump-like, and so on also have the characteristic.



FIG. 6. One periodic wave-(96) of Eq. (12) with parameters: $p = q = \alpha = 1$, $\beta = 2$, $\kappa = 0.025$, $x_0 = \gamma = 0$. (a) Perspective view of the wave; (b) Overhead view of the wave, with contour plot shown. The bright lines are crests and the dark lines are troughs; (c) Wave propagation pattern of the wave along the x axis.

(c)

0 x

-200

200

400

600

D. Bilinear Bäcklund transformation and associated Lax pair

-0.01

-600

-400

Bilinear Bäcklund transformation and Lax pair are two important tools in soliton theory and integrable systems. The next couple of problems are searching for the bilinear Bäcklund transformation and associated Lax pair of mGVE (34).

Theorem 2: If F is a known solution of the bilinear equation (39), then G satisfying

$$\left\{D_X^3 + \beta D_X - \tau\right\}F \cdot G = 0,$$
(59a)

$$\left\{3D_X D_T - 3\varsigma D_X + \alpha [1 + \phi(T)]\right\} F \cdot G = 0$$
(59b)

is a novel solution of mGVE (34), where τ and ς are arbitrary parameters. Thus system (59) is called a bilinear Bäcklund transformation for mGVE (34).

Proof: First, with the assumption that Q' and Q be two different solutions of Eq. (42), and introduce two new variables

$$v = \frac{Q' - Q}{2} = \ln \frac{F}{G}, w = \frac{Q' + Q}{2} = \ln FG,$$
(60)

which means that function H invariant under the two-field Q' and Q,

$$H(Q') - H(Q) = H(w + v) - H(w - v)$$

= $2 \{ v_{3X,T} + 3w_{2X}v_{X,T} + 3w_{X,T}v_{2X} + \alpha[1 + \phi(T)]v_{2X} + \beta v_{X,T} \}$ (61)
= $2\partial_T [\mathcal{Y}_{3X}(v, w) + \beta \mathcal{Y}_X(v, w) - \tau] + \mathcal{R}(v, w) = 0,$

where $\mathcal{R}(v, w) = 6$ Wronskian $\left\{ \mathcal{Y}_{X,T}(v, w) + \frac{\alpha}{3} [1 + \phi(T)], \mathcal{Y}_X(v) \right\}$ and τ is an arbitrary parameter.

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In order to obtain the bilinear Bäcklund transformation of mGVE (34), it is essential to rewrite $\mathcal{R}(v, w)$ as the *T*-derivative of a combination of \mathcal{Y} -polynomials or 0. A suitable constraint may be

$$\mathcal{Y}_{X,T}(v,w) + \frac{\alpha}{3} [1 + \phi(T)] - \varsigma \mathcal{Y}_X(v,w) = 0,$$
(62)

where ς is an arbitrary parameter. Under constraint condition (62), a coupled system of \mathcal{Y} -polynomials can be obtained as follows:

$$\mathcal{Y}_{3X}(v,w) + \beta \mathcal{Y}_X(v,w) - \tau = 0, \tag{63a}$$

$$\mathcal{Y}_{X,T}(v,w) + \frac{\alpha}{3}[1+\phi(T)] - \varsigma \mathcal{Y}_X(v) = 0,$$
 (63b)

which is very useful to construct conservation laws.

By virtue of the identity (20), the system (63) immediately leads to the bilinear Bäcklund transformation as follows:

$$\left[D_X^3 + \beta D_X - \tau\right] F \cdot G = 0, \left\{3D_X D_T - 3\zeta D_X + \alpha [1 + \phi(T)]\right\} F \cdot G = 0.$$
(64)

Remark 5: For the case of $\alpha = 1$ *, bilinear BT (59) of mGVE reduced to the ones of the GVE,*

$$\left\{D_X^3 + \beta D_X - \tau\right\} F \cdot G = 0, \left\{3D_X D_T - 3\varsigma D_X + \alpha \phi(T)\right\} F \cdot G = 0.$$
(65)

For the case of $\alpha = 1$, $\beta = 0$, bilinear BT (59) of mGVE reduced to the ones of the VE,

$$\left\{D_X^3 - \tau\right\}F \cdot G = 0, \left\{3D_X D_T - 3\varsigma D_X + \alpha \phi(T)\right\}F \cdot G = 0.$$
(66)

Bilinear BT can be used to construct new exact solutions of NLEEs. Next, we apply the system (59) to deduce the associated Lax paix of mGVE (34).

Theorem 3: Under the conditions (63), mGVE (34) admits a Lax pair

$$\psi_{3X} + \alpha W_X \psi_X + \beta \psi_X - \tau \psi = 0, \tag{67a}$$

$$\psi_{X,T} + \frac{\alpha}{3}(W_T + 1)\psi - \varsigma\psi_X = 0,$$
 (67b)

where W is a solution of the mGVE(34).

Proof: By means of Hopf-Cole transformation $v = \ln \psi$, (25) and (26), we obtain

$$\mathcal{Y}_{X}(v,w) = \frac{\psi_{X}}{\psi}, \quad \mathcal{Y}_{X,T}(v,w) = \frac{\psi_{X,T}}{\psi} + Q_{X,T}, \quad \mathcal{Y}_{3X}(v,w) = \frac{\psi_{3X}}{\psi} + \frac{3Q_{2X}\psi_{X}}{\psi}, \quad (68)$$

which make the system (63) linearized into a pair of equations with double parameters τ and ς ,

$$\mathcal{L}_1 \psi = (\partial_X^3 + 3Q_{2X}\partial_X + \beta \partial_X - \tau)\psi = 0, \tag{69a}$$

$$\mathcal{L}_2 \psi = [\partial_X \partial_T + Q_{X,T} + \frac{\alpha}{3} + \frac{\alpha}{3} \phi(T) - \varsigma \partial_X] \psi = 0,$$
(69b)

which is equivalent to

$$\psi_{3X} + \alpha W_X \psi_X + \beta \psi_X - \tau \psi = 0, \ \psi_{X,T} + \frac{\alpha}{3} (W_T + 1) \psi - \varsigma \psi_X = 0$$
(70)

with $W = \frac{3}{\alpha}Q_X + \Phi(T)$.

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It is easy to check that the integrability condition of Eq. (69) yields the condition

$$\left[W_{2X,T} + \alpha W_X(W_T + 1) + \beta W_T\right]_X = 0$$
(71)

or

$$W_{2X,T} + \alpha W_X(W_T + 1) + \beta W_T + h(T) = 0, \tag{72}$$

where h(T) is an arbitrary function of T.

In Refs. 10 and 38, Vakhnenko and Parkes investigated the VE by using the inverse scattering method and indicated that the following system:

$$\psi_{3X} + W_X \psi_X - \lambda \psi = 0, \\ 3\psi_{X,T} + (W_T + 1)\psi = 0$$
(73)

can be considered as the Lax pair for the VE (37). Correspondingly, according to the idea in Refs. 10 and 38, the similar processing steps for the mGVE indicate that h(T) is to be identically zero as $|X| \rightarrow \infty$. Thus, the pair of equations (69) can be considered as the Lax pair for the mGVE (34). \Box

Remark 6: For the case of $\alpha = 1$, Lax pair (67) of mGVE reduced to the ones of the GVE,

$$\psi_{3X} + W_X \psi_X + \beta \psi_X - \tau \psi = 0, \ \psi_{XT} + \frac{1}{3}(W_T + 1)\psi - \varsigma \psi_X = 0, \ \text{with} \ W = 3Q_X + \Phi(T).$$
(74)

For the case of $\alpha = 1$, $\beta = 0$, Lax pair (67) of mGVE reduced to the ones of the standard VE,

$$\psi_{3X} + W_X \psi_X - \tau \psi = 0, \ \psi_{XT} + \frac{1}{3}(W_T + 1)\psi - \varsigma \psi_X = 0, \ \text{with } W = 3Q_X + \Phi(T).$$
(75)

E. Conservation laws of mGVE

Conservation laws play an important role in mathematical physics, such as it describes the conservation of fundamental physical quantities, provides a method to study quantitative and qualitative properties of equations and their solutions, verifies complete integrability of NLEEs. There are many methods used to obtain the infinite conservation laws or conserved quantities for both continuous system and discrete system. For instance, Satsuma *et al.* developed a systematic way to derive the higher conservation laws for several NLEEs through BT.^{39–42}

The infinite sequence of conservation laws usually taking the following form:

$$I_{n,T} + F_{n,X} = 0, n = 1, 2, 3, \dots,$$
 (76)

which provides a corresponding sequences of integrals of motion given by the functions $\int I_n dx$. In this section, we show that mGVE (34) also has an infinite sequence of conserved quantities based on its Lax paix (67).

From the Lax pair (67) and mGVE (34), it can be shown that

$$3\tau\psi_T + \alpha(1+W_T)\psi_{2X} - \alpha W_{X,T}\psi_X + (\alpha\beta - 3\tau\varsigma)\psi = 0.$$
⁽⁷⁷⁾

Thus we have the following system:

$$\psi_{3X} + \alpha W_X \psi_X + \beta \psi_X - \tau \psi = 0, \tag{78a}$$

$$3\tau\psi_T + \alpha(1+W_T)\psi_{2X} - \alpha W_{X,T}\psi_X + (\alpha\beta - 3\tau\varsigma)\psi = 0.$$
(78b)

Recalling that

$$v = \frac{Q' - Q}{2}, v = \ln \psi, (Q' - Q)_X = \frac{\alpha}{3}(W' - W),$$
(79)

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we have

$$\psi_X = \frac{\alpha}{6} (W' - W)\psi. \tag{80}$$

In order to derive the infinite conservation laws of mGVE (34), the transformation from W' to W can be given as

$$W' = W + \kappa + \sum_{n=1}^{\infty} \frac{I_n(W, W_X, \ldots)}{\kappa^n},$$
(81)

which reduce

$$W'_{X} = W_{X} + \sum_{n=1}^{\infty} \frac{I_{n,X}(W_{X}, W_{2X} \dots)}{\kappa^{n}}.$$
(82)

Inserting (80) into (78), we have

$$(W' - W)_{2X} + \frac{\alpha}{2}(W' - W) \Big[W' + (2\alpha - 1)W \Big]_X + \frac{\alpha^2}{36}(W' - W)^3 + \beta(W' - W) - \frac{6\tau}{\alpha} = 0,$$
(83a)

$$(W' - W)_T + \frac{\alpha}{3\tau} \left\{ (1 + W_T) \left[(W' - W)_X + \frac{\alpha}{6} (W' - W)^2 \right] - W_{X,T} (W' - W) \right\}_X = 0.$$
(83b)

Based on (81) and (82) and equating the coefficients for power of $\frac{1}{\kappa}$ in (83a), we obtain the recursion relations to calculate I'_n in an explicit form

$$I_{1} = -\frac{12W_{X}}{\alpha} - \frac{12\beta}{\alpha^{2}}, I_{2} = \frac{72W_{2X}}{\alpha^{2}}, I_{3} = -\frac{288W_{3X}}{\alpha^{3}}, I_{4} = \frac{864W_{4X}}{\alpha^{4}},$$
(84a)

$$\mathbf{I}_{n} = -\frac{12}{\alpha^{2}}\mathbf{I}_{n-2,2X} - \frac{6}{\alpha}\mathbf{I}_{n-1,X} - \frac{12W_{X}}{\alpha}\mathbf{I}_{n-2} - \frac{6}{\alpha}\sum_{j=1}^{n-2}\mathbf{I}_{j}\mathbf{I}_{n-j-2,X} - \sum_{j=1}^{n-1}\mathbf{I}_{j}\mathbf{I}_{n-j-1}$$
(84b)
$$-\frac{1}{3}\sum_{j=1}^{n-4}\sum_{l=1}^{n-j-3}\mathbf{I}_{j}\mathbf{I}_{l}\mathbf{I}_{n-j-l-2} - \frac{12\beta}{\alpha^{2}}\mathbf{I}_{n-2}, n = 3, 4, \dots,$$

Due to Eq. (83b), all these quantities do satisfy the conservation law (76) for appropriate F_n . Equation (83b) shows that $\int (W' - W - \kappa) dX$ is a conserved quantity when suitable boundary conditions are imposed on W' and W.

IV. QUASIPERIODIC SOLUTION AND ASYMPTOTIC ANALYSIS

A. Riemann theta function and quasiperiodic solution

As is well known, exact solutions play an important role in the study of nonlinear mathematical physics, such as soliton, peakon, cuspon, rational and quasiperiodic solutions.^{43–56} Moreover, investigating relations among different exact solutions is also a very interesting topic. Since these relations not only provide an approach to deforming exact solutions, but also help us to investigate the structures and properties of some complicated forms of the solutions such as quasiperiodic solutions.

Nakamura presents a direct method to multi-periodic wave solutions for NLEEs based on Riemann theta function.^{57,58} Recently, this method was extended to investigate the discrete Toda lattice, Nizhnik-Novikov-Veselov equation, (2 + 1)-dimensions Bogoyavlenskii's breaking soliton equation, supersymmetric Ito's equation, and difference equations, etc.^{29,59–64}

In the following, combine the Riemann theta function and Hirota direct method, we give the one periodic solution of the mGVE and discuss the relation between one periodic solution and one soliton solution.

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Now, we consider the Riemann theta function solution of mGVE,

$$F = \sum_{n=-\infty}^{\infty} \nu(\xi, \tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n \xi + \pi i n^2 \tau},$$
(85)

where $n \in \mathbb{Z}$, $\tau \in \mathbb{C}$, $\text{Im}\tau > 0$, and $\xi = \kappa X + \rho T + \gamma$, with κ and ρ are constants to be determined, γ is a constant.

Taking $\phi(T) = \frac{3c}{\alpha\beta}$, then (39) can be rewritten as

$$G(D_X, D_T) = \left[D_X^3 D_T + (\frac{3c}{\beta} + \alpha) D_X^2 + \beta D_X D_T + c \right] F \cdot F = 0,$$
(86)

where *c* is a constant that must not be dropped.

Remark 7: The D-operators have good property when acting on exponential functions which plays a key role in the construction of the periodic wave solutions

$$D_X^m D_T^n e^{\xi_1} \cdot e^{\xi_2} = (\kappa_1 - \kappa_2)^m (\iota_1 - \iota_2)^n e^{\xi_1 + \xi_2},$$
(87)

where $\xi_j = \kappa_j X + \iota_j T + \xi_j^{(0)}, j = 1, 2.$ More generally, we have

$$G(D_X, D_T)e^{\xi 1} \cdot e^{\xi 2} = G(\kappa_1 - \kappa_2, \iota_1 - \iota_2)e^{\xi 1 + \xi 2}.$$
(88)

Substituting (85) into (86), we have

$$GF \cdot F = G(D_X, D_T) \sum_{n=-\infty}^{\infty} e^{2\pi i n\xi + \pi i n^2 \tau} \sum_{m=-\infty}^{\infty} e^{2\pi i m\xi + \pi i m^2 \tau}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G(D_X, D_T) e^{2\pi i n\xi + \pi i n^2 \tau} \cdot e^{2\pi i m\xi + \pi i m^2 \tau}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G[2\pi i (n-m)\kappa, 2\pi i (n-m)\rho] e^{2\pi i (n+m)\xi + \pi i (n^2 + m^2)\tau}$$

$$= \sum_{p=-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} G[2\pi i (2n-p)\kappa, 2\pi i (2n-p)\rho] e^{\pi i [n^2 + (p-n)^2]\tau} \right\} e^{2\pi i p\xi}$$

$$= \sum_{p=-\infty}^{\infty} \bar{G}(p) e^{2\pi i p\xi}.$$
(89)

Noting that

$$\bar{G}(p) = \sum_{n=-\infty}^{\infty} G\Big[2\pi i(2n-p)\kappa, 2\pi i(2n-p)\rho\Big] e^{\pi i[n^2 + (p-n)^2]\tau}$$

$$= \sum_{h=-\infty}^{\infty} G\Big[A\kappa, A\rho\Big] e^{\pi i[h^2 + (p-h-2)^2]\tau} \cdot e^{2\pi i(p-1)\tau}$$

$$= \bar{G}(p-2)e^{2\pi i(p-1)\tau},$$
(90)

where $A = 2\pi i [2h - (p - 2)], p = m + n$. Based on (90) and by induction method, we can get

$$\bar{G}(p) = \begin{cases} \bar{G}(0)e^{\pi i n p \tau}, \ p = 2n, \\ \bar{G}(1)e^{\pi i (2n+2n^2)(p+1)\tau}, \ p = 2n+1. \end{cases}$$
(91)

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In this way, we may let

$$\bar{G}(0) = \sum_{n=-\infty}^{\infty} \left\{ 16n^2 \pi^2 \beta \kappa (16n^2 \pi^2 \kappa^2 - \beta)\rho + (\beta - 48n^2 \pi^2 \kappa^2)c - 16n^2 \pi^2 \kappa^2 \alpha \beta \right\} e^{2i\pi n^2 \tau} = 0,$$

$$\bar{G}(1) = \sum_{n=-\infty}^{\infty} \left\{ 4(2n-1)^2 \pi^2 \beta \kappa [16\pi^2 \kappa^2 (n^2 - n) + 4\pi^2 \kappa^2 - \beta]\rho + [48\pi^2 \kappa^2 (n - n^2) + \beta - 12\pi^2 \kappa^2]c - (16n^2 \pi^2 \kappa^2 \alpha \beta + 4\pi^2 \kappa^2 \alpha \beta - 16\pi^2 \kappa^2 n \alpha \beta) \right\} e^{(2n^2 - 2n + 1)\pi i \tau} = 0.$$
(92)

For the sake of convenience, we denote that $\lambda = e^{\pi i \tau}$, thus we have

$$a_{11} = \sum_{n=-\infty}^{\infty} 16n^2 \pi^2 \beta \kappa (16n^2 \pi^2 \kappa^2 - \beta) \lambda^{2n^2}, a_{12} = \sum_{n=-\infty}^{\infty} (\beta - 48n^2 \pi^2 \kappa^2) \lambda^{2n^2},$$
(93a)

$$a_{21} = \sum_{n=-\infty}^{\infty} 4(2n-1)^2 \pi^2 \beta \kappa (16n^2 \pi^2 \kappa^2 - 16n \pi^2 \kappa^2 + 4\pi^2 \kappa^2 - \beta) \lambda^{2n^2 - 2n + 1}, \qquad (93b)$$

$$a_{22} = \sum_{n=-\infty}^{\infty} (48n\pi^2\kappa^2 + \beta - 48n^2\pi^2\kappa^2 - 12\pi^2\kappa^2)\lambda^{2n^2 - 2n + 1},$$
(93c)

$$b_1 = \sum_{n=-\infty}^{\infty} 16n^2 \pi^2 \kappa^2 \alpha \beta \lambda^{2n^2},$$
(93d)

$$b_2 = \sum_{n=-\infty}^{\infty} (16n^2 \pi^2 \kappa^2 \alpha \beta + 4\pi^2 \kappa^2 \alpha \beta - 16\pi^2 \kappa^2 n \alpha \beta) \lambda^{2n^2 - 2n + 1},$$
(93e)

then system (92) can be written as matrix form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \rho \\ c \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$
(94)

Solving this system, we obtain

$$\rho = \frac{a_{12}b_2 - a_{22}b_1}{a_{11}a_{22} - a_{12}^2}, c = \frac{a_{12}b_2 - a_{12}b_1}{a_{11}a_{22} - a_{12}^2}.$$
(95)

Thus, we obtain the one periodic wave solution of mGVE (35),

$$x = T + \frac{6}{\alpha} (\ln F)_X + \frac{3cT}{\alpha\beta} + x_0, t = X, u = \frac{6}{\alpha} (\ln F)_{XX},$$
(96)

where F and ρ , c are given by (85) and (95), respectively.

Remark 8: Figure 6 shows the periodic wave solution (96) for one choice of the parameters.

B. Asymptotic property of one periodic wave solution

It is of interest to investigate the asymptotic behaviour of the periodic wave solutions of mGVE. First of all, we expand the system (94) into power series of λ ,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A_0 + A_1 \lambda + A_2 \lambda^2 + \dots,$$
 (97a)

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$$\binom{b_1}{b_2} = B_0 + B_1 \lambda + B_2 \lambda^2 + \dots,$$
(97b)

$$\binom{\rho}{c} = X_0 + X_1 \gamma + X_2 \gamma^2 + \dots$$
(97c)

Substituting (97) into (94), we have the following recursion relations:

$$A_0 X_0 = B_0, \tag{98a}$$

$$A_0 X_1 + A_1 X_0 = B_1, (98b)$$

$$A_0 X_2 + A_2 X_0 + A_1 X_1 = B_2, (98c)$$

$$A_0X_n + A_1X_{n-1} + \ldots + A_nX_0 = B_n.$$
 (98e)

If the matrix A_0 is reversible, solving (98) leads to

$$X_0 = A_0^{-1} B_0, X_n = A_0^{-1} (B_n - \sum_{j=1}^n A_j B_{n-1}), n = 1, 2, \dots$$
 (99)

If A_0 and A_1 are not inverse, but they take the following form:

$$A_{0} = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, A_{1} = \begin{pmatrix} 0 & 0 \\ 32\pi^{4}\kappa^{3}\beta - 8\beta^{2}\pi^{2}\kappa & -24\pi^{2}\kappa^{2} + 2\beta \end{pmatrix},$$
 (100)

solving relations (98) leads to

$$X_{0} = \begin{pmatrix} \frac{B_{1}^{II} - (2\beta - 24\pi^{2}\kappa^{2})B_{0}^{I}}{32\pi^{4}\kappa^{3}\beta - 8\beta^{2}\pi^{2}\kappa} \\ B_{0}^{I} \end{pmatrix},$$
(101a)

$$X_{1} = \begin{pmatrix} \frac{(B_{2} - A_{2}X_{0})^{II} - (2\beta - 24\pi^{2}\kappa^{2})B_{1}^{I}}{32\pi^{4}\kappa^{3}\beta - 8\beta^{2}\pi^{2}\kappa} \\ B_{1}^{I} \end{pmatrix},$$
(101b)

$$X_{n} = \begin{pmatrix} \frac{(B_{n+1} - \sum_{j=2}^{n+1} A_{j} X_{n+1-j})_{1}^{lI} - (2\beta - 24\pi^{2}\kappa^{2})(B_{n+1} - \sum_{j=2}^{n} A_{j} X_{n-j})^{I}}{32\pi^{4}\kappa^{3}\beta - 8\beta^{2}\pi^{2}\kappa} \\ (B_{n+1} - \sum_{j=2}^{n} A_{j} X_{n-j})^{I} \end{pmatrix}, n = 2, 3, \dots,$$
(101c)

where V^{I} and V^{II} denote the first and second component of a two dimensional vector V, respectively.

Thus, the relation between the one periodic wave solution (96) and the one soliton solution (52) can be established as follows.

Theorem 4: Suppose that the vector $(\rho, c)^T$ is a solution of the system (94), and for the one periodic wave solution (96), we let

$$\kappa = \frac{\omega_1}{2\pi i}, \gamma = \frac{\delta_1 - \pi \tau i}{2\pi i},\tag{102}$$

where ω_1 , δ_1 are the same as those in (52). Then we have the following asymptotic properties:

$$c \to 0, \xi \to \frac{\eta_1 - \pi \tau i}{2\pi i}, \nu(\xi, \tau) \to 1 + e^{\eta_1}, \text{ as } \lambda \to 0.$$
 (103)

In other words, the one periodic solution (96) tends to the one soliton solution (52) under $\lambda \rightarrow 0$.

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Proof: By using (93), we write functions a_{ij} , b_j , (i, j = 1, 2) as the series about λ and in term of (97), we have

$$A_{0} = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, A_{1} = \begin{pmatrix} 0 & 0 \\ 32\pi^{4}\kappa^{3}\beta - 8\beta^{2}\pi^{2}\kappa & -24\pi^{2}\kappa^{2} + 2\beta \end{pmatrix},$$
 (104a)

$$A_{2} = \begin{pmatrix} 32\pi^{2}\beta\kappa(16\pi^{2}\kappa^{2} - \beta) & 2(\beta - 48\pi^{2}\kappa^{2}) \\ 0 & 0 \end{pmatrix},$$
 (104b)

$$A_5 = \begin{pmatrix} 0 & 0\\ 72\pi^2 \beta \kappa (36\pi^2 \kappa^2 - \beta) & 0 \end{pmatrix}, A_3 = A_4 = 0, \dots,$$
(104c)

$$B_1 = \begin{pmatrix} 0\\ 8\pi^2\kappa^2\alpha\beta \end{pmatrix}, B_2 = \begin{pmatrix} 32\pi^2\kappa^2\alpha\beta\\ 0 \end{pmatrix},$$
(104d)

$$B_5 = \begin{pmatrix} 0 \\ 72\pi^2 \kappa^2 \alpha \beta \end{pmatrix}, B_1 = B_3 = B_4 = 0, \dots$$
(104e)

Substituting (104) into formulas (101), we then obtain

$$X_{0} = \begin{pmatrix} \frac{\kappa\alpha}{4\pi^{2}\kappa^{2}-\beta} \\ 0 \end{pmatrix}, X_{2} = \begin{pmatrix} \frac{8\kappa\alpha(\beta-12\pi^{2}\kappa^{2})(16\pi^{2}\kappa^{2}-\beta)}{(4\pi^{2}\kappa^{2}-\beta)^{2}} \\ \frac{32\pi^{2}\kappa^{2}\alpha\beta(16\pi^{2}\kappa^{2}-\beta)}{\beta-4\pi^{2}\kappa^{2}} \end{pmatrix}, X_{1} = 0, \dots,$$
(105)

and thus

$$\rho = \frac{\kappa\alpha}{4\pi^2\kappa^2 - \beta} + \frac{8\kappa\alpha(\beta - 12\pi^2\kappa^2)(16\pi^2\kappa^2 - \beta)}{(4\pi^2\kappa^2 - \beta)^2}\lambda^2 + \dots,$$
 (106a)

$$c = \frac{32\pi^{2}\kappa^{2}\alpha\beta(16\pi^{2}\kappa^{2} - \beta)}{\beta - 4\pi^{2}\kappa^{2}}\lambda^{2} + \dots,$$
 (106b)

which exactly implies that

$$c \to 0.$$
 (107)

It remains to show that the one periodic wave (96) degenerates to the one soliton solution (52) under the limit $\lambda \to 0$. For this purpose, we first expand the periodic function $\nu(\xi)$ in the form

$$\nu(\xi,\tau) = 1 + (e^{2\pi i\xi} + e^{-2\pi i\xi})\lambda + (e^{4\pi i\xi} + e^{-4\pi i\xi})\lambda^4 + \dots$$
(108)

By using the transformation (102), it follows that

$$\nu(\xi,\tau) = 1 + e^{\bar{\xi}} + (e^{-\bar{\xi}} + e^{2\bar{\xi}})\lambda^2 + (e^{-2\bar{\xi}} + e^{3\bar{\xi}})\lambda^6 + \dots \to 1 + e^{\bar{\xi}}, as\lambda \to 0,$$
(109)

where

$$\bar{\xi} = 2\pi i\xi + \pi i\tau = \omega_1 X + 2\pi i\rho T + \delta_1.$$
(110)

Combining (102), (106), and (110) deduces that

$$\bar{\xi} = \omega_1 X - \frac{\omega_1 \alpha}{\omega_1^2 + \beta} T + \delta_1 = \omega_1 X + \iota_1 T + \delta_1, \text{ as } \lambda \to 0,$$
(111)

or equivalently,

$$\xi \to \frac{\eta_1 - \pi i \tau}{2\pi i}$$
, as $\lambda \to 0$. (112)

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Again (109) and (111) immediately leads to

$$\nu(\xi, \tau) \to 1 + e^{\eta_1}, \text{ as } \lambda \to 0.$$
(113)

Therefore, we conclude that the one periodic solution (96) just goes to the one soliton solution (52) as the amplitude $\lambda \to 0$.

V. CONCLUSIONS

In this paper, we investigate the integrability of the mGVE. The significant properties, such as bilinear formulism, N soliton solutions, quasiperiodic solution, bilinear Bäcklund transformation, Lax pair, and conservation laws that can characterize integrability of mGVE are discussed. Moreover, the asymptotic analysis shows that the one periodic wave solution tends to the one soliton solution under a small amplitude limit $\lambda \rightarrow 0$. Furthermore, all the results of mGVE can reduced to the GVE and standard VE under the special case of $\alpha = 1$ and $\alpha = 1$, $\beta = 0$, respectively.

The result demonstrates that binary Bell polynomial is a quick and succinct method in investigating the integrabilities of NLEEs.

It would be specially mentioned that the transformation (38) with an arbitrary function in T, which means we may obtain new N soliton solutions of mGVE as well as GVE and VE.

For instance, by virtue of the transformation $W = 6(\ln F)_X + \Phi(T)$, bilinear equation of the VE is

$$\left[D_X^3 D_T + D_X^2 + \phi(T) D_X^2\right] F \cdot F = 0,$$
(114)

which reduced to the known result as $\phi(T) = 0.^{12}$

Based on the Hirota direct method, the N soliton solutions of the VE can be obtained as follows:

$$U = 6 \left[\ln \left(\sum_{\mu=0,1} e^{\sum_{j=1}^{n} \mu_j \eta_j + \sum_{1 \le j < l}^{n} \mu_j \mu_l A_{jl}} \right) \right]_{2X}, \eta_j = k_j X - \frac{\Phi(T)}{k_j} - \frac{T}{k_j} + c_j,$$
(115)

where

$$e^{A_{jl}} = \frac{(k_j - k_l)^2 (k_j^2 - k_j k_l + k_l^2)}{(k_j + k_l)^2 (k_j^2 + k_j k_l + k_l^2)}, (j < l, j, l = 1, 2, 3, \ldots),$$
(116)

and $\sum_{\mu=0,1}$ indicates the summation over all possible combination of $\mu_j = 0, 1 (j = 1, 2, ...)$.

As is well known, VE describes the propagation of shortwave perturbations in a relaxing medium. It is similar to the Kadomtsev-Petviashvili equation is a generalisation of the one-dimensional Korteweg-de Vries equation. Therefore, another interesting question is how to construct the (2 + 1)-dimensional extensions of the VE as well as its generalisations and investigate their potential physical meaning. We will study in the near future.

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