Symmetry Reductions and Exact Solutions of the (2+1)-Dimensional Navier-Stokes Equations

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By means of the classical symmetry method, we investigate the (2+1)-dimensional Navier-Stokes equations. The symmetry group of Navier-Stokes equations is studied and its corresponding group invariant solutions are constructed. Ignoring the discussion of the infinite-dimensional subalgebra, we construct an optimal system of one-dimensional group invariant solutions. Furthermore, using the associated vector fields of the obtained symmetry, we give out the reductions by one-dimensional and two-dimensional subalgebras, and some explicit solutions of Navier-Stokes equations are obtained. For three interesting solutions, the figures are given out to show their properties: the solution of stationary wave of fluid (real part) appears as a balance between fluid advection (nonlinear term) and friction parameterized as a horizontal harmonic diffusion of momentum.

Key words: Navier-Stokes Equations; Classical Lie Symmetry Method; Optimal System; Explicit Solution.

1. Introduction

Symmetry group techniques provide one method for obtaining exact solutions of partial differential equations [1 – 4]. Since Sophus Lie [1] set up the theory of Lie point symmetry group, the standard method had been widely used to find Lie point symmetry algebras and groups for almost all the known differential systems. One of the main applications of the Lie theory of symmetry groups for differential equations is to get group-invariant solutions. Via any subgroup of the symmetry group, the original equation can be reduced to an equation with fewer independent variables by solving the characteristic equation. In general, to each s-parameter subgroup of the full symmetry group, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is usually not feasible to list all possible group-invariant solutions to the system. That needs an effective, systematic means of classifying these solutions, leading to an optimal system of group-invariant solutions from which every other such solution can be derived. About the optimal systems, a lot of excellent work has been done by many famous experts [3 – 7] and some examples of optimal systems can also be found in Ibragimov [8]. Up to now, several methods have been developed to construct optimal systems. The adjoint representation of a Lie group on its Lie algebra was also known to Lie. Its use in classifying group-invariant solutions appeared in [3] and [4] which are written by Ovsiannikov and Olver, respectively. The latter reference contains more details on how to perform the classification of subgroup under the adjoint action. Here we will use Olver's method which only depends on fragments of the theory of Lie algebras to construct the optimal system of Navier-Stokes equations.

One of the most important open problems in fluid is the existence and smoothness problem of the Navier-Stokes equations, which has been recognized as the basic equation and the very starting point of all problems in fluid physics [9 – 10]. Therefore solving Navier-Stokes equations becomes very important and valuable but difficult. Here, by means of the classical Lie symmetry method, we investigate the (2+1)-dimensional Navier-Stokes equations:

$$\omega = \psi_{xx} + \psi_{yy}, \tag{1}$$

$$\omega_t + \psi_x \omega_y - \psi_y \omega_x - \gamma(\omega_{xx} + \omega_{yy}) = 0. \tag{2}$$

Since the initial derivation of (1) and (2), many authors have been studying them [11–14]. Substituting (1) into (2), we can get

$$\begin{aligned} &\psi_{xxt} + \psi_{yyt} + \psi_x \psi_{xxy} + \psi_x \psi_{yyy} - \psi_y \psi_{xxx} \\ &-\psi_y \psi_{xyy} - \gamma(\psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}) = 0. \end{aligned} \tag{3}$$

So we can investigate (3) instead of Navier-Stokes equations (1) and (2) in the following sections.

This paper is arranged as follows: In Section 2, by using the classical Lie symmetry method, we get the vector fields of the (2+1)-dimensional Navier-Stokes equation (3). Then the transformations leaving the solutions invariant, i.e. its symmetry groups are obtained. In Section 3, after an optimal system of one-dimensional symmetry group of (3) is constructed, the corresponding one-parameter and some two-parameter reductions are given out. Thanks to the Maple, we can obtain some exact solutions [15–17] of (3). Finally, some conclusions and discussions are given in Section 4.

2. Symmetry Group of Navier-Stokes Equations

To (3), by applying the classical Lie symmetry method, we consider the one-parameter group of infinitesimal transformations in (x,y,t,ψ) given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,y,t,\psi) + o(\varepsilon^2), \\ y^* &= y + \varepsilon\eta(x,y,t,\psi) + o(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x,y,t,\psi) + o(\varepsilon^2), \\ \psi^* &= \psi + \varepsilon\Psi(x,y,t,\psi) + o(\varepsilon^2), \end{aligned} \tag{4}$$

where ε is the group parameter. It is required that the set of equations in (3) be invariant under the transformations (4), and this yields a system of overdetermined, linear equations for the infinitesimals ξ, η, τ, and Ψ. Solving these equations, one can have

$$\begin{aligned} \xi &= \frac{c_1 x}{2} - c_3 yt - c_4 y + f(t), \\ \eta &= \frac{c_1 y}{2} + c_3 xt + c_4 x + g(t), \\ \tau &= c_1 t + c_2, \\ \Psi &= g'(t)x - f'(t)y + h(t) + \frac{c_3(x^2+y^2)}{2}, \end{aligned}$$

where $c_i (i = 1,2,3,4)$ are arbitrary constants and $f(t)$, $g(t)$, and $h(t)$ are arbitrary functions of t. And the associated vector fields for the one-parameter Lie group of infinitesimal transformations are $v_1, v_2, \cdots, v_7$ given by

$$\begin{aligned} &v_1 = \frac{x}{2}\partial_x + \frac{y}{2}\partial_y + t\partial_t, \quad v_2 = \partial_t, \\ &v_3 = -yt\partial_x + xt\partial_y + \frac{x^2+y^2}{2}\partial_\psi, \quad v_4 = -y\partial_x + x\partial_y, \\ &v_5 = f(t)\partial_x - f'(t)y\partial_\psi, \\ &v_6 = g(t)\partial_y + g'(t)x\partial_\psi, \quad v_7 = h(t)\partial_\psi. \end{aligned} \tag{5}$$

Equations (5) show that the following transformations (given by $\exp(\varepsilon v_i), i = 1,2,\cdots,7$) of variables (x,y,t,ψ) leave the solutions of (3) invariant:

$$\begin{aligned} &\exp(\varepsilon v_1) : (x,y,t,\psi) \mapsto (x e^{\frac{\varepsilon}{2}}, y e^{\frac{\varepsilon}{2}}, t e^{\varepsilon}, \psi), \\ &\exp(\varepsilon v_2) : (x,y,t,\psi) \mapsto (x,y,t+\varepsilon,\psi), \\ &\exp(\varepsilon v_3) : (x,y,t,\psi) \mapsto \\ &\quad \bigg(x\cos(t\varepsilon) - y\sin(t\varepsilon), x\sin(t\varepsilon) \\ &\qquad + y\cos(t\varepsilon), t, \psi + \frac{x^2+y^2}{2}\varepsilon \bigg), \\ &\exp(\varepsilon v_4) : (x,y,t,\psi) \mapsto \\ &\quad (x\cos(\varepsilon) - y\sin(\varepsilon), x\sin(\varepsilon) + y\cos(\varepsilon), t, \psi), \\ &\exp(\varepsilon v_5) : (x,y,t,\psi) \mapsto \\ &\quad (x + f(t)\varepsilon, y, t, \psi - f'(t)y\varepsilon), \\ &\exp(\varepsilon v_6) : (x,y,t,\psi) \mapsto \\ &\quad (x, y + g(t)\varepsilon, t, \psi + g'(t)x\varepsilon), \\ &\exp(\varepsilon v_7) : (x,y,t,\psi) \mapsto (x,y,t,\psi + h(t)\varepsilon). \end{aligned} \tag{6}$$

And the following theorem holds:

Theorem 1: If $\psi = p(x,y,t)$ is a solution of (3), so are the functions:

$$\begin{aligned} \psi^{(1)} &= p\left(x e^{-\frac{\varepsilon}{2}}, y e^{-\frac{\varepsilon}{2}}, t e^{-\varepsilon}\right), \\ \psi^{(2)} &= p(x,y,t-\varepsilon), \\ \psi^{(3)} &= p\Big(x\cos(t\varepsilon) + y\sin(t\varepsilon), \\ &\qquad -x\sin(t\varepsilon) + y\cos(t\varepsilon), t\Big) + \frac{x^2+y^2}{2}\varepsilon, \\ \psi^{(4)} &= p\Big(x\cos(\varepsilon) + y\sin(\varepsilon), -x\sin(\varepsilon) + y\cos(\varepsilon), t\Big), \\ \psi^{(5)} &= p(x - f(t)\varepsilon, y, t) - f'(t)y\varepsilon, \\ \psi^{(6)} &= p(x, y - g(t)\varepsilon, t) + g'(t)x\varepsilon, \\ \psi^{(7)} &= p(x,y,t) + h(t)\varepsilon. \end{aligned}$$

In [18], Clarkson and Kruskal (CK) introduced a direct method to derive symmetry reductions of a nonlinear system without using any group theory. For many types of nonlinear systems, the method can be used to find all the possible similarity reductions. Then Lou and Ma modified CK's direct method [19 – 22] to find out the generalized Lie and non-Lie symmetry groups of differential equations by an ansatz reading

$$u(x,y,t) = \alpha(x,y,t) + \beta(x,y,t)U(\xi,\eta,\tau), \quad (7)$$

where ξ,η,τ are all functions of x,y,t. (7) also points that if $U(x,y,t)$ is a solution of the original differential equation, so is $u(x,y,t)$. Actually, instead of the ansatz (7), the general one-parameter group of symmetries can be obtained by considering linear combination $c_1v_1+c_2v_2+c_3v_3+c_4v_4+c_5v_5+c_6v_6+c_7v_7$ of the given vector fields. But the explicit formulae for the above transformations are very complicated. Factually, it can be represented uniquely in the form

$$g = \exp(\varepsilon_1 v_1)\exp(\varepsilon_2 v_2)\exp(\varepsilon_3 v_3)\exp(\varepsilon_4 v_4) \cdot \exp(\varepsilon_5 v_5)\exp(\varepsilon_6 v_6)\exp(\varepsilon_7 v_7). \quad (8)$$

Thus making use of group transformations (8), the most general solution obtainable from a given solution $p(x,y,t)$ is in the form (for simplicity, one can do it by computer algebra):

$$\begin{aligned}\psi = &-\frac{a_4}{2}(x^2+y^2) - (a_4a_5f(t)+a_6g'(t))x \\ &+(-a_4a_6g(t)+a_5f'(t))y - \frac{1}{2}a_4{a_5}^2f(t)^2 \\ &-\frac{1}{2}a_4{a_6}^2g(t)^2 + a_5a_6f'(t)g(t) - a_7h(t) \\ &+p(X,Y,T),\end{aligned}$$

$$\begin{aligned}X = a_1\Big[&(\cos(a_4t)\sqrt{1-{a_3}^2} - a_3\sin(a_4t))x \\ &-(\sin(a_4t)\sqrt{1-{a_3}^2} - a_3\cos(a_4t))y \\ &+\sqrt{1-{a_3}^2}(a_5f(t)\cos(a_4t) - a_6g(t)\sin(a_4t)) \\ &-a_3(a_5f(t)\sin(a_4t)+a_6g(t)\cos(a_4t))\Big],\end{aligned}$$

$$\begin{aligned}Y = a_1\Big[&(\sin(a_4t)\sqrt{1-{a_3}^2} + a_3\cos(a_4t))x \\ &+(\cos(a_4t)\sqrt{1-{a_3}^2} - a_3\sin(a_4t))y \\ &+\sqrt{1-{a_3}^2}(a_5f(t)\sin(a_4t) + a_6g(t)\cos(a_4t)) \\ &+a_3(a_5f(t)\cos(a_4t)-a_6g(t)\sin(a_4t))\Big],\end{aligned}$$

$$T = {a_1}^2(t+a_2),$$

where $a_1,a_2,\cdots,a_6$ are arbitrary constants.

3. Reductions and Solutions of Navier-Stokes Equations

By exploiting the generators v_i of the Lie-point transformations in (5), one can build up exact solutions of (3) via the symmetry reduction approach. This allows one to lower the number of independent variables of the system of differential equations under consideration using the invariants associated with a given subgroup of the symmetry group. In the following we present some reductions leading to exact solutions of the Navier-Stokes equations of possible physical interest.

Firstly, we construct an optimal system to classify the group-invariant solutions of (3). As it is said in [4], the problem of classifying group-invariant solutions reduces to the problem of classifying subgroups of the full symmetry group under conjugation. And the problem of finding an optimal of subgroups is equivalent to that of finding an optimal system of subalgebras. Here, by using the method presented in [3 – 4], we will construct an optimal system of one-dimensional subalgebras of (3).

From (5), ignoring the discussion of the infinite-dimensional subalgebra, one can get the following four operators:

$$\begin{aligned}&v_1 = \frac{x}{2}\partial x + \frac{y}{2}\partial y + t\partial t, \quad v_2 = \partial t, \\ &v_3 = -yt\partial x + xt\partial y + \frac{x^2+y^2}{2}\partial\psi, \\ &v_4 = -y\partial x + x\partial y.\end{aligned}$$

Applying the commutator ~~operators~~ $[v_m,v_n] = v_mv_n - v_nv_m$, we get the following table (the entry in row i and the column j representing $[v_i,v_j]$):

	v_1	v_2	v_3	v_4
v_1	0	$-v_2$	v_3	0
v_2	v_2	0	v_4	0
v_3	$-v_3$	$-v_4$	0	0
v_4	0	0	0	0

Therefore, there is

Proposition 1: The operators $v_i(i=1,2,3,4)$ form a Lie algebra, which is a four-dimensional symmetry algebra.

To compute the adjoint representation, we use the Lie series in conjunction with the above commutator table. Applying the formula

$$\mathrm{Ad}(\exp(\varepsilon v))v_0 = v_0 - \varepsilon[v,v_0] + \frac{1}{2}\varepsilon^2[v,[v,v_0]] - \cdots,$$

we can construct the following table:

Ad	v_1	v_2	v_3	v_4
v_1	v_1	$\exp(\varepsilon)v_2$	$\exp(-\varepsilon)v_3$	v_4
v_2	$v_1-\varepsilon v_2$	v_2	$v_3-\varepsilon v_4$	v_4
v_3	$v_1+\varepsilon v_3$	$v_2+\varepsilon v_4$	v_3	v_4
v_4	v_1	v_2	v_3	v_4

with the (i,j)-th entry indicating $\mathrm{Ad}(\exp(\varepsilon v_i))v_j$.

Following Ovsiannikov [3], one calls two subalgebras v_2 and v_1 of a given Lie algebra equivalent if one can find an element g in the Lie group so that $Adg(v_1) = v_2$, where Adg is the adjoint representation of g on v. Given a nonzero vector

$$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4,$$

our task is to simplify as many of the coefficients a_i as possible though judicious applications of adjoint maps to v. In this way, omitting the detailed computation, one can get the following theorem by the complicated computation:

Theorem 2: The operators generate an optimal system S

(a) $v_1 + a_4v_4, a_1 \neq 0$;

(b1) $v_3, a_1 = 0, a_3 \neq 0$;

(b2) $v_3 + v_2, a_1 = 0, a_3 \neq 0$;

(b3) $v_3 - v_2, a_1 = 0, a_3 \neq 0$;

(c) $v_2, a_1 = a_3 = 0, a_2 \neq 0$;

(d) $v_4, a_1 = a_2 = a_3 = 0$.

Making use of Theorem 2, we will discuss the reductions and solutions of (3).

3.1. Reductions by One-Dimensional Subalgebras

For case (a), from $(v_1 + a_4v_4)(\psi) = 0$, i. e.

$$\frac{x}{2}\psi_x + \frac{y}{2}\psi_y + t\psi_t + a_4(-y\psi_x + x\psi_y) = 0,$$

one can get $\psi = F(\xi,\eta)$, where $\xi = \frac{\sin(a_4\ln(t))}{\sqrt{t}}x - \frac{\cos(a_4\ln(t))}{\sqrt{t}}y$, and $\eta = \frac{\cos(a_4\ln(t))}{\sqrt{t}}x + \frac{\sin(a_4\ln(t))}{\sqrt{t}}y$. Then (3) is reduced to

$$\begin{aligned}
&2\gamma(F_{\xi\xi}+F_{\eta\eta})_{\xi\xi} + 2\gamma(F_{\xi\xi}+F_{\eta\eta})_{\eta\eta} \\
&+ \xi(F_{\xi\xi}+F_{\eta\eta})_{\xi} + \eta(F_{\xi\xi}+F_{\eta\eta})_{\eta} \\
&+ 2a_4\xi(F_{\xi\xi}+F_{\eta\eta})_{\eta} - 2a_4\eta(F_{\xi\xi}+F_{\eta\eta})_{\xi} \\
&+ 2(F_{\xi\xi}+F_{\eta\eta}) - 2F_{\xi}(F_{\xi\xi}+F_{\eta\eta})_{\eta} \\
&+ 2F_{\eta}(F_{\xi\xi}+F_{\eta\eta})_{\xi} = 0.
\end{aligned}$$

By solving the above equation, one can obtain

$$F(\xi,\eta) = F(\xi \pm \eta \mathrm{i}),$$

where $\mathrm{i}^2 = -1$ and F is an arbitrary functions of the corresponding variable.

In case (b1), solving

$$-yt\psi_x + xt\psi_y - \frac{x^2+y^2}{2} = 0,$$

it follows

$$\psi = -\frac{x^2+y^2}{2t}\arctan\left(\frac{x}{y}\right) + F(\xi,\eta),$$

where $\xi = x^2+y^2$ and $\eta = t$. Substituting them into (3), and integrating the reduced equation once about ξ, one can have

$$4\gamma\xi\eta(\xi F_{\xi\xi\xi} + 2F_{\xi\xi}) + \xi^2F_{\xi\xi} - \xi\eta F_{\xi\eta} - F = 0.$$

In case (b2) and (b3), solving

$$-yt\psi_x + xt\psi_y - \frac{x^2+y^2}{2} + \varepsilon\psi_t = 0,$$

it follows

$$\psi = \frac{t}{2\varepsilon}(x^2+y^2) + F(\xi,\eta),$$

where $\xi = x^2+y^2$, $\varepsilon = \pm 1$ and $\eta = 2\varepsilon\arctan(\frac{x}{y}) + t^2$. And the reduced equation is

$$\begin{aligned}
&8\varepsilon^5\gamma F_{\eta\eta\eta\eta} + 8\varepsilon^4(\xi F_{\xi}F_{\eta\eta\eta} - \xi F_{\eta}F_{\xi\eta\eta} + F_{\eta}F_{\eta\eta}) \\
&+ 8\varepsilon^3\gamma(2\xi^2F_{\xi\xi\eta\eta} + F_{\eta\eta}) \\
&+ 8\varepsilon^2\xi^2(\xi F_{\xi}F_{\xi\xi\eta} - \xi F_{\eta}F_{\xi\xi\xi} + F_{\xi}F_{\xi\eta} - 2F_{\eta}F_{\xi\xi}) \\
&+ 8\varepsilon\gamma\xi^2(\xi^2F_{\xi\xi\xi\xi} + 4\xi F_{\xi\xi\xi} + 2F_{\xi\xi}) - \xi^2 = 0.
\end{aligned}$$

For case (c), from $\psi_t = 0$, one can get $\psi = F(x,y)$, which ~~indicate~~ a stationary fluid. Then (3) is cast into the reduced form

$$\begin{aligned}
&F_x(F_{xx}+F_{yy})_y - F_y(F_{xx}+F_{yy})_x \\
&-\gamma(F_{xx}+F_{yy})_{xx} - \gamma(F_{xx}+F_{yy})_{yy} = 0.
\end{aligned}$$

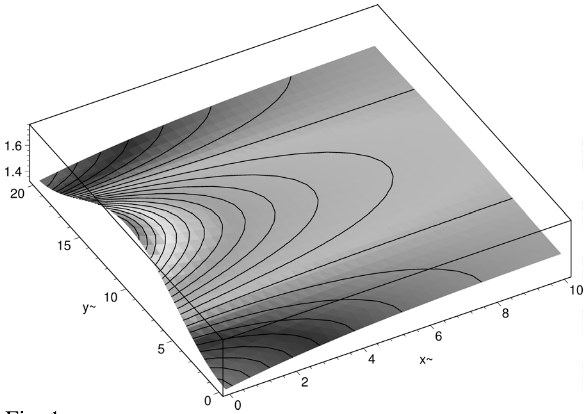

Fig. 1.

The above equation has the solution

$$F(x,y) = C_3 + C_4 \tanh(C_1 + C_2 x + C_2 y\mathrm{i}) + C_5 \tanh^2(C_1 + C_2 x + C_2 y\mathrm{i}),$$

where $\mathrm{i}^2 = -1$, and C_i $(i = 1,2,3,4,5)$ are arbitrary constants. When the growth rate of fluid (imagine part of the solution) tends to be zero when $C_5 = C_4\left(\frac{E^2-1}{2(E^2+1)} + \frac{2E^2\cos(C_2 y)^2}{E^4-1}\right)$, $E = \mathrm{e}^{C_1+C_2 x}$, the solution of stationary wave of fluid (real part) appears as a balance between fluid advection (nonlinear term) and friction parameterized as a horizontal harmonic diffusion of momentum with coefficient γ. Figure 1 shows a stationary interior ocean circulation with $C_1 = 0.8$, $C_2 = 0.15$, $C_3 = C_4 = 1$, which looks like an anticyclonic subtropical gyre in a closed ocean basin, two cyclonic tropical and subpolar lows at the north and south, respectively [23].

In case (d), solving $-y\psi_x + x\psi_y = 0$, we obtain $\psi = F(\xi,\eta)$, where $\xi = x^2 + y^2$ and $\eta = t$. Substituting it into (3) and integrating the reduced equation twice about ξ, one can get

$$F_\eta - 4\gamma(\xi F_\xi)_\xi = 0.$$

Solving the above equation, we have the solution of (3)

$$\psi = \exp(4C_1\gamma t)\left(C_2 \mathrm{BesselJ}\left(0, 2\sqrt{-C_1(x^2+y^2)}\right) + C_3 \mathrm{BesselY}\left(0, 2\sqrt{-C_1(x^2+y^2)}\right)\right), \quad (9)$$

where C_i $(i = 1,2,3)$ are arbitrary constants.

Figure 2 exhibits the plot of ψ in (9) with

$$\gamma = 1,\ C_1 = -1,\ C_2 = 10,\ C_3 = 0,$$

and the time $t = 1$.

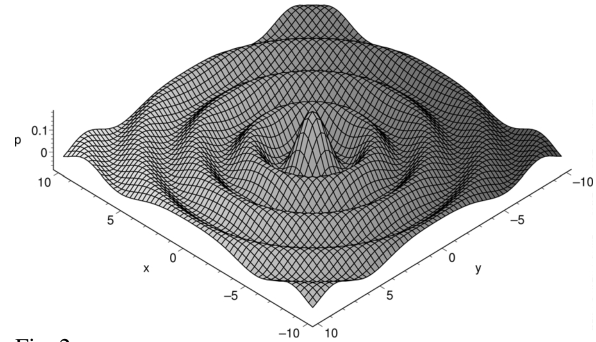

Fig. 2.

3.2. Reductions by Two-Dimensional Subalgebras

Case 1: $\{v_1, v_2\}$. From $\frac{x}{2}\psi_x + \frac{y}{2}\psi_y + t\psi_t = 0$ and $\psi_t = 0$, we have $\psi = F(\frac{x}{y})$. Substituting it into (3), one can get

$$\gamma\Big[(\xi^2+1)^2 F_{\xi\xi\xi\xi} + 12\xi(\xi^2+1)F_{\xi\xi\xi} + 12(3\xi^2+1)F_{\xi\xi} + 24\xi F_\xi\Big] + 2(\xi^2+1)F_\xi F_{\xi\xi} + 4\xi F_\xi^2 = 0,$$

where $\xi = \frac{x}{y}$.

Case 2: $\{v_1, v_3\}$. Solving $\frac{x}{2}\psi_x + \frac{y}{2}\psi_y + t\psi_t = 0$ and $-yt\psi_x + xt\psi_y - \frac{x^2+y^2}{2} = 0$, it follows $\psi = -\frac{x^2+y^2}{2t}\arctan(\frac{x}{y}) + F(\frac{x^2+y^2}{t})$. Substituting it into (3), it follows

$$4\gamma(\xi^2 F_{\xi\xi\xi\xi} + 4\xi F_{\xi\xi\xi} + 2F_{\xi\xi}) + 2\xi^2 F_{\xi\xi\xi} + 5\xi F_{\xi\xi} = 0.$$

Case 3: $\{v_1, v_4\}$. From $\frac{x}{2}\psi_x + \frac{y}{2}\psi_y + t\psi_t = 0$ and $-y\psi_x + x\psi_y = 0$, one can get $\psi = F(\frac{x^2+y^2}{t})$. Substituting it into (3), we have

$$F'(Z) + 8\gamma F''(Z) + 3ZF''(Z) + 16\gamma ZF'''(Z) + Z^2F'''(Z) + 4\gamma Z^2 F''''(Z) = 0,$$

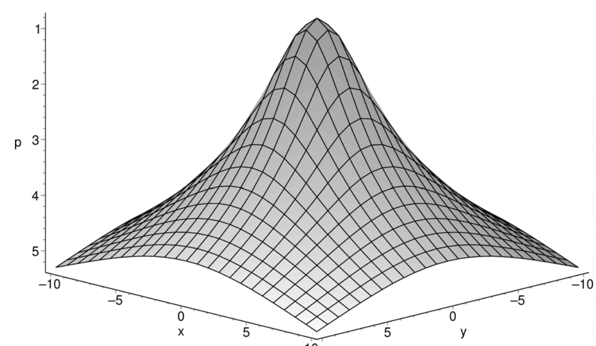

Fig. 3.

where $Z = \frac{x^2+y^2}{t}$. Solving the above equation, it follows

$$\psi = C_1 + C_2 \ln Z + C_3 \mathrm{Ei}\left(1, \frac{Z}{4\gamma}\right) - 3 \cdot 2^{\frac{2}{3}} C_4 \Gamma\left(\frac{2}{3}\right) \int Z e^{-\frac{Z^3}{4\gamma}} \Gamma\left(\frac{1}{3}, -\frac{Z^3}{4\gamma}\right) \mathrm{d}Z + 12 C_4 \gamma^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right) \ln^2 Z + \frac{6}{5} 2^{\frac{1}{3}} \sqrt{3} C_4 \pi \gamma^{\frac{1}{3}} Z e^{-\frac{Z^3}{8\gamma}} \cdot \mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{Z^3}{4\gamma}\right) + 8 \cdot 2^{\frac{1}{3}} \sqrt{3} C_4 \pi \gamma^{\frac{4}{3}} Z^{-2} e^{-\frac{Z^3}{8\gamma}} \cdot \mathrm{WhittakerM}\left(\frac{4}{3}, \frac{5}{6}, \frac{Z^3}{4\gamma}\right) + 12 C_4 \gamma^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right) \ln Z,$$

where $C_i (i = 1,2,3,4)$ are arbitrary constants.

Figure 3 exhibits the plot of ψ in (10) with $\gamma = 1$, $C_1 = 0$, $C_2 = C_3 = 1$, $C_4 = 0$, and the time $t = 0$, appearing an atmospheric subtropical high or monopole anti-cyclonic blocking in the Northern Hemisphere.

Case 4: $\{v_2, v_4\}$. The solution of $\psi_t = 0$ and $-y\psi_x + x\psi_y = 0$ has the form $\psi = F(x^2+y^2)$. Then the reduced equation of (3) is

$$\xi^2 F_{\xi\xi\xi\xi} + 4\xi F_{\xi\xi\xi} + 2F_{\xi\xi} = 0,$$

which has the solution

$$F = C_1 + C_2\xi + C_3 \ln(\xi) + C_4 \xi \ln(\xi),$$

where $C_i (i = 1,2,3,4)$ are arbitrary constants and $\xi = x^2 + y^2$.

4. Conclusions

In summary, we investigate the symmetry of the Navier-Stokes equations by means of the classical Lie symmetry method. The symmetry algebras and groups of (3) are obtained. Specially, the most general one-parameter group of symmetries is given out as the composition of transforms in the seven various one-subgroups $\exp(\varepsilon v_1), \exp(\varepsilon v_2), \cdots, \exp(\varepsilon v_7)$ and the most general solution obtainable from a given solution $p(x,y,t)$ is gained. Next we have classified one-dimensional subalgebras of a Lie algebra of (3). Then the reductions and some solutions of Navier-Stokes equations by using the associated vector fields of the obtained symmetry are given out. By one-dimensional subalgebras, (3) is reduced to some (1+1)-dimensional equations and by two-dimensional subalgebras, (3) is reduced to some ordinary equations. For three interesting explicit solutions of (3), we also give out figures to show their properties.

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