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Bäcklund Transformations and Explicit Solutions of (2+1)-Dimensional Barotropic and Quasi-Geostrophic Potential Vorticity Equation*

HU Xiao-Rui (胡晓瑞)[1] and CHEN Yong (陈勇)[1,2,†]

[1]Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062, China

[2]Nonlinear Science Center and Department of Mathematics, Ningbo University, Ningbo 315211, China

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Abstract *By the Bäcklund transformation method, an important (2+1)-dimensional nonlinear barotropic and quasi-geostrophic potential vorticity (BQGPV) equation is investigated. Some simple special Bäcklund transformation theorems are proposed and used to get explicit solutions of the BQGPV equation. Furthermore, all solutions of a second order linear ordinary differential equation including an arbitrary function can be used to construct explicit solutions of the (2+1)-dimensional BQGPV equation. Some figures are also given out to describe these solutions.*

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1 Introduction

It is known that most of the models of atmospheric and ocean dynamical systems are nonlinear differential equations. The explicit solutions of these differential equations can well depict and reflect fruitful phenomena in fluids and other physical fields. However, due to the high nonlinearity of the models, it is often difficult to solve these differential equations analytically. Based on the classical Lie symmetry approach,[1–4] Huang[5] investigated the (2+1)-dimensional nonlinear barotropic and quasi-geostrophic potential vorticity (BQGPV) equation without forcing and dissipation on a β-plane channel and obtained its some types of explicit solutions including the ring solitary waves and the breaking soliton type of vorticity solutions. The BQGPV equation reads

$$\frac{\partial}{\partial t}\nabla^2\psi - F\frac{\partial\psi}{\partial t} + J(\psi,\nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = 0\,, \qquad (1)$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ denotes the two-dimensional Laplacian operator, $J(a,b) = a_x b_y - a_y b_x$ is the Jacobian operator, ψ is the dimensional steam function, $F = L^2/R_0^2$ is the square of the ratio of the characteristic horizonal length scale L to the Rossby deformation radius R_0, $\beta = \beta_0(L^2/U)$, and $\beta_0 = (\omega_0/R_0)\cos\phi_0$, in which R_0 is the Earth's radius, ω_0 is the angular frequency of the Earth's rotation, ϕ_0 is the latitude, and U is the characteristic velocity scales. The subscripts x, y represent partial derivatives.

The BQGPV equation (1) as one of the most important models of the atmospheric and ocean dynamical systems can be widely used to study the development of turbulence and coherent structures in atmosphere and in magnetized plasmas.[6–9] Furthermore, Eq. (1) is a potential vorticity conservation and its solutions can indicate the movement state of free ocean depending on its initial conditions. In Ref. [10], based on the standard multi-scale expansion method and the long wave approximation, one possible approximate solution to Eq. (1) was gained by Tang *et al.* and it has been used to explain the life cycle of a blocking system without loss of the long wave property of the Rossby wave. Then Eq. (1) was restudied[11–12] by Huang and Tang *et al.* using the classical Lie symmetry approach again. Tang *et al.* found a new symmetry and obtained two new types of similarity reduction solutions as a note to the results in Ref. [5].

In this paper, Eq. (1) is reinvestigated in an alternative way, i.e. Bäcklund transformation and more exact solutions of Eq. (1) are constructed. In Ref. [13], Lou *et al.* have made use of this simple and essential technique, i.e. Bäcklund transformation, to obtain some types of exact solutions of $(2+1)$-dimensional Euler equation. Here, it is found that the same method in Ref. [13] is also valid for Eq. (1). In Sec. 2, some Bäcklund transformation theorems for the BQGPV equation are given out. With the help of Bäcklund transformation, some types of exact solutions are obtained and some figures are given out in Sec. 3. Section 4 is our conclusions.

2 Bäcklund Transformations of BQGPV Equation

Firstly, for the convenience of our computation, we rewrite Eq. (1) in the following equivalent form

$$q = \psi_{xx} + \psi_{yy} \equiv \nabla^2\psi\,, \qquad (2)$$

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†E-mail: ychen@sei.ecnu.edu.cn

$$q_t - F\psi_t + J(\psi, q) + \beta\psi_x = 0\,. \tag{3}$$

Then we can investigate Eqs. (2) and (3) instead of Eq. (1).

Theorem 1 If $\{q_0, \psi_0\}$ is a solution of Eqs. (2) and (3), so is $\{q_1, \psi_1\}$ under the definition

$$q_1 = q_0 + \omega\,, \quad \psi_1 = \psi_0 + p\,, \tag{4}$$

where $\{\omega, p\}$ is a solution of

$$\omega = \nabla^2 p\,, \tag{5}$$

$$\omega_t - Fp_t + J(p, \omega) + J(\psi_0, \omega) + J(p, q_0) + \beta p_x = 0\,. \tag{6}$$

Proof Direct calculations.

From Bäcklund transformation (4), one can find new exact solutions $\{q_1, \psi_1\}$ from known ones $\{q_0, \psi_0\}$ when $\{\omega, p\}$ are solved from Eqs. (5) and (6). Furthermore, we put a constraint between ω and p as general as

$$\omega = Q(p)\,, \tag{7}$$

where $Q(p)$ is an arbitrary function of p. Due to this ansatz, we may have a much simplified Bäcklund transformation theorem:

Theorem 2 If $\{q_0, \psi_0\}$ is a solution of Eqs. (2) and (3), so is $\{q_1, \psi_1\}$ under the definition

$$q_1 = q_0 + Q(p)\,, \quad \psi_1 = \psi_0 + p\,, \tag{8}$$

where $Q(p)$ is an arbitrary function of p and p is a solution of the over-determined equation system

$$\nabla^2 p = Q(p)\,, \tag{9}$$

$$p_t + J(\psi_0, p) = 0\,, \tag{10}$$

$$-Fp_t + J(p, q_0) + \beta p_x = 0\,. \tag{11}$$

Proof As for Eq. (9), it is just a simple combination of Eqs. (5) and (7). Substituting the ansatz (7) into Eq. (6) of Theorem 1, we have

$$(p_t + J(\psi_0, p))Q_p - Fp_t + J(p, q_0) + \beta p_x = 0\,, \tag{12}$$

where $Q_p \equiv dQ/dp$. One can see that Eq. (12) is correct for arbitrary $Q(p)$ if and only if Eqs. (10) and (11) are satisfied simultaneously. Theorem 2 is proven.

However, if the known solutions $\{q_0, \psi_0\}$ are very complicated, it is still very difficult to construct some exact solutions by solving Eqs. (9), (10), and (11). So to make sure Eqs. (10) and (11) be solved easily, we can select the seed solution for a very special and simple form

$$q_0 = c\,, \tag{13}$$

i.e. q_0 is an arbitrary constant. Firstly, we should solve another seed solution ψ_0 by substituting $q_0 = c$ into Eqs. (2) and (3)

$$\psi_{0xx} + \psi_{0yy} = q_0\,, \tag{14}$$

$$q_{0t} - F\psi_{0t} + J(\psi_0, q_0) + \beta\psi_{0x} = 0\,. \tag{15}$$

For the assumption $q_0 = c$, Eqs. (14) and (15) can be solved readily. One can see that the corresponding general solution of Eqs. (14) and (15) has the form of ($i \equiv \sqrt{-1}$)

$$\psi_0 = \frac{c}{4}\Big[y^2 + \Big(x + \frac{\beta}{F}t\Big)^2\Big] + f_1\Big[y + \Big(x + \frac{\beta}{F}t\Big)i\Big] + f_2\Big[y - \Big(x + \frac{\beta}{F}t\Big)i\Big], \tag{16}$$

where

$$f_1 \equiv f_1\Big[y + \Big(x + \frac{\beta}{F}t\Big)i\Big]\,, \quad f_2 \equiv f_2\Big[y - \Big(x + \frac{\beta}{F}t\Big)i\Big]$$

are arbitrary functions of the indicated variables. Furthermore, ψ_0 is assured to be real on condition that f_2 is a complex conjugate of f_1 and vice verse. In order to get explicit solutions of Eqs. (2) and (3), here we select

$$f_1 + f_2 = m_1 y + m_2\Big(x + \frac{\beta}{F}t\Big)\,, \tag{17}$$

i.e.

$$f_1 = \frac{m_1 - m_2 i}{2}\Big[y + \Big(x + \frac{\beta}{F}t\Big)i\Big]\,,$$

$$f_2 = \frac{m_1 + m_2 i}{2}\Big[y - \Big(x + \frac{\beta}{F}t\Big)i\Big]\,,$$

where m_1 and m_2 are arbitrary real constants. Hence formula (16) becomes

$$\psi_0 = \frac{c}{4}\Big[y^2 + \Big(x + \frac{\beta}{F}t\Big)^2\Big] + m_1 y + m_2\Big(x + \frac{\beta}{F}t\Big)\,. \tag{18}$$

Then substituting the seed solution $\{q_0, \psi_0\}$ with the form of Eq. (13) and (18) into Eqs. (10) and (11), one can get

$$p = P(c((Fx + \beta t)^2 + F^2y^2) + 4F^2(m_2x + m_1y) + 4F\beta(m_2t - y))\,. \tag{19}$$

For convenience, we denote

$$\xi = c[(Fx + \beta t)^2 + F^2y^2] + 4F^2(m_2x + m_1y) + 4F\beta(m_2t - y)\,. \tag{20}$$

Substituting $p = P(\xi)$ with Eq. (20) into Eq. (9), we can obtain

$$\Big[m_2^2F^2 + (Fm_1 - \beta)^2 + \frac{c\xi}{4}\Big]\frac{d^2P(\xi)}{d\xi^2} + \frac{c}{4}\frac{dP(\xi)}{d\xi} = Q(P(\xi))\,.$$

The above results can be summarized as the following theorem:

Theorem 3 If $P(\xi) \equiv P$ is a solution of the second order ordinary differential equation

$$\Big[m_2^2F^2 + (Fm_1 - \beta)^2 + \frac{c}{4}\xi\Big]\frac{d^2P(\xi)}{d\xi^2} + \frac{c}{4}\frac{dP(\xi)}{d\xi} = Q(P)\,, \tag{21}$$

with $\xi = c[(Fx + \beta t)^2 + F^2y^2] + 4F^2(m_2x + m_1y) + 4F\beta(m_2t - y)$, $Q(P)$ being an arbitrary function of P and c, m_1, m_2 being arbitrary constants, then ψ given by

$$\psi = \frac{c}{4}\Big[y^2 + \Big(x + \frac{\beta}{F}t\Big)^2\Big] + m_1 y + m_2\Big(x + \frac{\beta}{F}t\Big) + P(\xi) \tag{22}$$

is a solution of the BQGPV equation (1).

3 Some Special Explicit Solutions from Bäcklund Transformations

From Theorem 3, one can get fruitful explicit solutions of Eq. (1) only by solving a second order ordinary differential equation with an arbitrary function $Q(P)$. In this section, we select some special $Q(P)$ and give out the corresponding explicit solutions of Eq. (1). For the arbitrary constant c in Theorem 3, there are two circumstances, i.e.

$c=0$ and $c\neq 0$. Then for these two cases, we will study them separately.

3.1 *Special Explicit Solutions from $c\neq 0$ in Theorem 3*

For the parameter $c\neq 0$ in Eq. (21), we give out some explicit solutions of Eq. (1) for the special selections of $Q(P)$.

(i) For the selection $Q(P)=c_0$, Eq. (21) becomes

$$\Big[m_2^2F^2+(Fm_1-\beta)^2+\frac{c}{4}\xi\Big]\frac{d^2P(\xi)}{d\xi^2}+\frac{c}{4}\frac{dP(\xi)}{d\xi}-c_0=0\,,\tag{23}$$

where c_0 is also an arbitrary constant. Solving Eq. (23), one can get

$$P(\xi)=\Big[\frac{k_1}{c}-\frac{16c_0}{c^2}(m_2^2F^2+(Fm_1-\beta)^2)\Big]\ln(4m_2^2F^2+(Fm_1-\beta)^2+c\xi)+\frac{4c_0}{c}\xi+k2\,,$$

with k_1 and k_2 being two arbitrary constants. Hence, one can obtain one explicit solution of Eq. (1) via (22) directly, there being

$$\begin{aligned}\psi&=\Big(4c_0F^2+\frac{c}{4}\Big)\Big(x^2+y^2+\frac{\beta^2}{F^2}t^2\Big)+\Big(8c_0\beta F+\frac{c\beta}{2F}\Big)xt+\Big(m_2+\frac{16c_0m_2F^2}{c}\Big)x\\&\quad+\Big(m_1-\frac{16c_0\beta F}{c}+\frac{16c_0m_1F^2}{c}\Big)y+\Big(\frac{m_2\beta}{F}+\frac{16c_0m_2\beta F}{c}\Big)t+k_2+\Big[\frac{k_1}{c}-\frac{16c_0}{c^2}(m_2^2F^2+(Fm_1-\beta)^2)\Big]\\&\quad\times\ln(c^2((Fx+\beta t)^2+F^2y^2)+4Fc(Fm_2x+(Fm_1-\beta)y+\beta m_2t)+4(Fm_1-\beta)^2+4m_2^2F^2)\,.\end{aligned}\tag{24}$$

When $c_0=0$ and $k_1=0$ in Eq. (24), we obtain a polynomial solution of Eq. (1). In other cases, the solution obtained in the form of Eq. (24) will be with singularities due to the logarithm function.

Figures 1(a) and 1(b) display the structure for the solution (24) under the parameter selections

$$\beta=1\,,\quad F=1\,,\quad c=0.1\,,\quad m_1=1\,,\quad m_2=0\,,\quad k_1=-10\,,\quad k_2=0\,,\tag{25}$$

with $c_0=1/16$ and $c_0=1$ at time $t=0$, respectively. Both of the singularities are $(0,0)$.

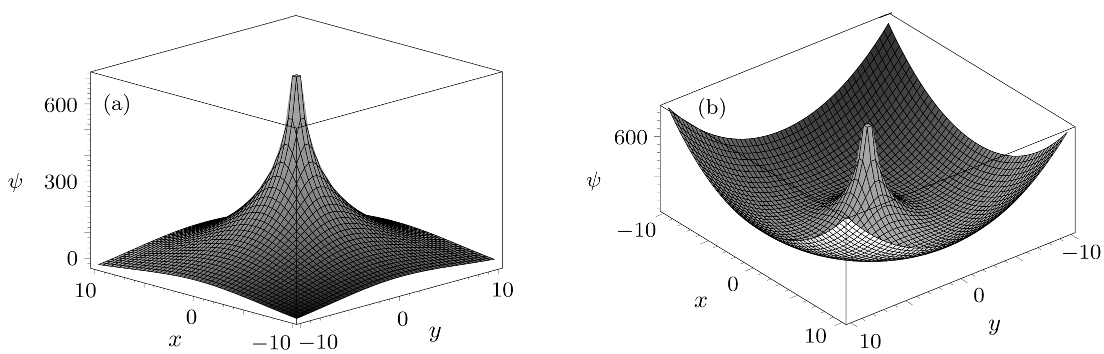

Fig. 1 The special solution ψ given by Eq. (24) with Eq. (25) and (a) $c_0=1/16$, (b) $c_0=1$, both at time $t=0$.

(ii) Selecting $Q(P)=mP(\xi)$ in Eq. (21), one can obtain

$$\Big[m_2^2F^2+(Fm_1-\beta)^2+\frac{c}{4}\xi\Big]\frac{d^2P(\xi)}{d\xi^2}+\frac{c}{4}\frac{dP(\xi)}{d\xi}-mP(\xi)=0\,,\tag{26}$$

where m is an arbitrary constant. Solving $P(\xi)$ in Eq. (26) and substituting it into Eq. (22), we get the second type of explicit solution of Eq. (1)

$$\begin{aligned}\psi&=\frac{c}{4}\Big[y^2+\Big(x+\frac{\beta}{F}t\Big)^2\Big]+m_1y+m_2\Big(x+\frac{\beta}{F}t\Big)+l_1\mathrm{J}\Big\{0,4\sqrt{-m}\Big[F^2\Big(x^2+y^2+\frac{\beta^2}{F^2}t^2\Big)\\&\quad+2\beta Fxt+\frac{4m_2F^2}{c}x+\frac{4F(m_1F-\beta)}{c}y+\frac{4m_2\beta F}{c}t+\frac{4(m_2^2F^2+(Fm_1-\beta)^2)}{c^2}\Big]^{1/2}\Big\}\\&\quad+l_2\mathrm{Y}\Big\{0,4\sqrt{-m}\Big[F^2\Big(x^2+y^2+\frac{\beta^2}{F^2}t^2\Big)+2\beta Fxt+\frac{4m_2F^2}{c}x\\&\quad+\frac{4F(m_1F-\beta)}{c}y+\frac{4m_2\beta F}{c}t+\frac{4(m_2^2F^2+(Fm_1-\beta)^2)}{c^2}\Big]^{1/2}\Big\}\,,\end{aligned}\tag{27}$$

where l_1, l_2 are arbitrary constants, J and Y are the Bessel functions of the first and the second kind, respectively. Then we give out Fig. 2 to display the property of the solution (27). The parameter selections in Figs. 2(a) and 2(b) are respectively

$$\beta = 1\,,\quad F = 1\,,\quad c = 0.1\,,\quad m_1 = 1\,,\quad m_2 = 0\,,\quad m = -1/16\,,\quad l_1 = 15\,,\quad l_2 = 0\,, \tag{28}$$

$$\beta = 1\,,\quad F = 1\,,\quad c = 0.1\,,\quad m_1 = 0\,,\quad m_2 = 1\,,\quad m = -1/16\,,\quad l_1 = 10\,,\quad l_2 = 0\,, \tag{29}$$

at time $t = 0$.

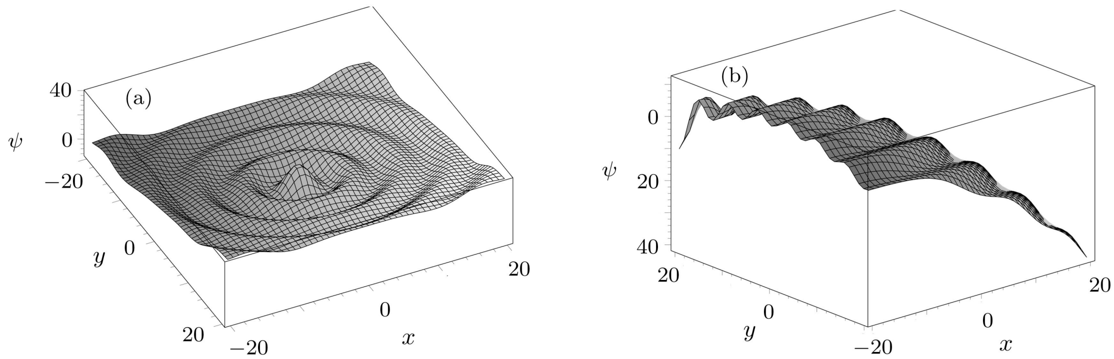

Fig. 2 (a) and (b) describe the steam function ψ given by Eq. (27) with Eqs. (28) and (29) separately at time $t = 0$.

(iii) Selecting $Q(P) = \exp(P)$, Eq. (21) becomes

$$\Big[m_2^2F^2 + (Fm_1 - \beta)^2 + \frac{c}{4}\xi\Big]\frac{d^2P(\xi)}{d\xi^2} + \frac{c}{4}\frac{dP(\xi)}{d\xi} - \exp(P) = 0\,. \tag{30}$$

The explicit solution of Eq. (30) is

$$P(\xi) = \ln\Big[\frac{(a_1 - c^2)\text{sech}^2[(1/2)(\sqrt{c^2 - a_1}/c)(\ln(4m_2^2F^2 + 4(m_1F - \beta)^2 + c\xi) - a_2))]}{8(4m_2^2F^2 + 4(m_1F - \beta)^2 + c\xi)}\Big]\,, \tag{31}$$

where a_1 and a_2 are two arbitrary constants.

Hence from $P(\xi)$ in Eq. (31) with ξ in Eq. (20), one can gain the explicit solution of Eq. (1) from Eq. (22). Because the expression of special solution is so complicated that we do not write it here.

(iv) For the $Q(P) = \xi^2\sin(\xi)$, Eq. (21) becomes

$$\Big[m_2^2F^2 + (Fm_1 - \beta)^2 + \frac{c}{4}\xi\Big]\frac{d^2P(\xi)}{d\xi^2} + \frac{c}{4}\frac{dP(\xi)}{d\xi} - \xi^2\sin(\xi) = 0\,. \tag{32}$$

By solving $P(\xi)$ in Eq. (32), one can readily check that

$$\begin{aligned}\psi = \frac{1}{c}\{&\text{Ci}\,(\zeta + k)[8k\sin(k) + (4k^2 - 8)\cos(k)]\\ &+ \text{Si}\,(\zeta + k)[(4k^2 - 8)\sin(k) - 8k\cos(k)]\\ &+ 4[(k-1)\zeta\sin(\zeta) + (k-3)\cos(\zeta)]\\ &+ b_1\ln(c(\zeta + k)) + \frac{c}{4F^2}(\zeta + 4F\beta y + 4b_2F^2)\}\end{aligned} \tag{33}$$

is the explicit solution of Eq. (1). In Eq. (33), Si, Ci are respectively the Sine integral and the Cosine integral and b_1, b_2 are two arbitrary constants. Here for convenience, we denote

$$\zeta = cF^2\Big(x^2 + y^2 + \frac{\beta^2}{F^2}t^2\Big) + 2c\beta Fxt + 4F(F - \beta)y\,,$$

$$k = \frac{4(\beta - F)^2}{c}\,. \tag{34}$$

3.2 *Special Exact Solutions from $c = 0$ in Theorem 3*

In this section, we will give out some another explicit solutions of Eq. (1) for the parameter $c = 0$ in Eq. (21). In this case, theorem 3 will be more special and simplified. Here we rewrite it as follows.

Theorem 4 If $P(\xi) \equiv P$ is a solution of the two-order ordinary differential equation

$$(m_2^2F^2 + (Fm_1 - \beta)^2)\frac{d^2P(\xi)}{d\xi^2} = Q(P)\,, \tag{35}$$

with

$$\xi = F(m_2x + m_1y) + \beta(m_2t - y)\,,$$

$Q(P)$ being an arbitrary function of P and m_1, m_2 being arbitrary constants, then ψ given by

$$\psi = m_1y + m_2\Big(x + \frac{\beta}{F}t\Big) + P(\xi) \tag{36}$$

is a solution of the BQGPV equation (1).

For $Q(P)$ in Eq. (35), we can not list all of them and Eq. (35) is not solved easily for every $Q(P)$. Here we only list some special selections which are of interest to us. We will select $Q(P)$ as P^2, $-P^3$, $\exp(P)$, $\sin(P)$, $\tanh(\xi)$, $\xi\sin(\xi)$, and $-\xi P$. Substituting them into Eq. (35), the corresponding solutions can be solved readily. We list them in the following.

$$P_1(\xi) = 6R^2\mathrm{P}\,(\xi + c_1, 0, c_2)\,, \quad P_2(\xi) = c_3\mathrm{sn}\left[c_3\Big(\frac{\xi}{\sqrt{2}R} + c_4\Big), i\right],$$
$$P_3(\xi) = -\ln(2) + \ln\left[-c_5\mathrm{sech}^2\Big(\frac{\sqrt{c_5}(\xi + c_6)}{2R}\Big)\right],$$
$$P_4(\xi) = 2\arctan\left[\mathrm{cn}\left(\frac{\sqrt{c_7R^2+2}}{2R}(\xi + c_8), \frac{2}{\sqrt{c_7R^2+2}}\right), \mathrm{sn}\left(\frac{\sqrt{c_7R^2+2}}{2R}(\xi + c_8), \frac{2}{\sqrt{c_7R^2+2}}\right)\right],$$
$$P_5(\xi) = -\frac{1}{2R^2}[\xi^2 + \mathrm{dilog}\,(1 + \exp(2\xi))] + \Big(c_9 - \frac{\ln 2}{R^2}\Big)\xi + c_{10}\,,$$
$$P_6(\xi) = -\frac{1}{R^2}[\xi\sin(\xi) + 2\cos(\xi)] + c_{11}\xi + c_{12}\,, \quad P_7(\xi) = c_{13}\mathrm{Ai}\,(-R^{-2/3}\xi) + c_{14}\mathrm{Bi}\,(-R^{-2/3}\xi)\,.$$

Here,

$$\mathrm{dilog}\,(x) = \int_1^x \frac{\ln z}{1-z}dz\,,$$

Ai and Bi are Airy wave function,

$$R = \sqrt{m_2^2F^2 + (Fm_1 - \beta)^2}\,, \quad i \equiv \sqrt{-1}$$

and $c_k\,(k = 1, \ldots, 14)$ are arbitrary constants.

Then the corresponding explicit solutions ψ_k of Eq. (1) are obtained by Eq. (36), i.e.

$$\psi_k = m_1y + m_2\Big(x + \frac{\beta}{F}t\Big) + P_k(\xi)\,, \quad (k = 1, \ldots, 7)\,, \quad (37)$$

with $\xi = F(m_2x + m_1y) + \beta(m_2t - y)$.

Some figures are given out to depict these solutions $\psi_k\,(k = 2, 3, 4, 6, 7)$.

Figure 3 shows the real part of complex solution ψ_2 and the corresponding parameters are

$$\beta = 1\,, \quad F = 1\,, \quad m_1 = 0\,,$$
$$m_2 = 1\,, \quad c_3 = 1\,, \quad c_4 = 1\,, \quad (38)$$

with $t = 1$.

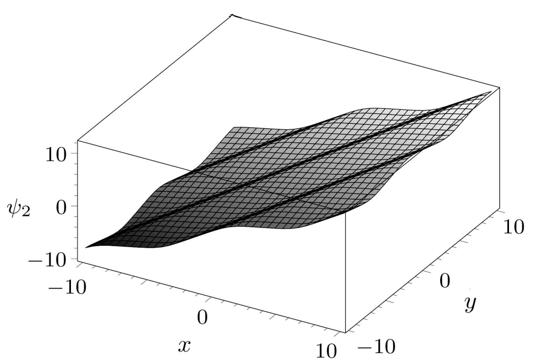

Fig. 3 The real part of ψ_2 given by Eq. (37) with Eq. (38) at $t = 1$.

Figure 4 depicts ψ_3 with the parameters

$$\beta = 1\,, \quad F = 1\,, \quad m_1 = 0\,,$$
$$m_2 = -1\,, \quad c_5 = -5\,, \quad c_6 = 1\,, \quad (39)$$

at time $t = 1$.

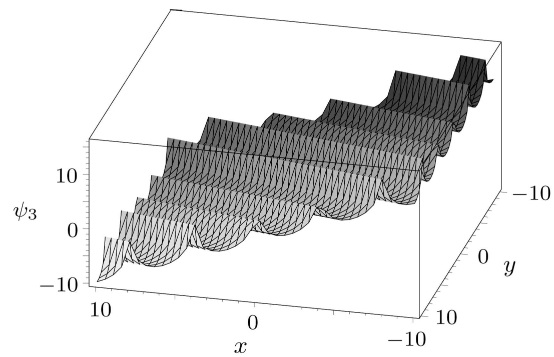

Fig. 4 ψ_3 in Eq. (37) with Eq. (39) at $t = 1$.

Figure 5 describes the solution ψ_4 in Eq. (37). The parameter selections of ψ_4 in Figs. 5(a) and 5(b) are respectively

$$\beta = 1\,, \quad F = 1\,, \quad m_1 = 0\,,$$
$$m_2 = 0\,, \quad c_7 = 1\,, \quad c_8 = 1\,, \quad (40)$$
$$\beta = 1\,, \quad F = 1\,, \quad m_1 = 0\,,$$
$$m_2 = 1\,, \quad c_7 = 1\,, \quad c_8 = 1\,, \quad (41)$$

both at time $t = 1$.

Figure 6 describes the solution ψ_6 in Eq. (37) at $t = 0$. The parameter selections in ψ_6 are

$$\beta = -1\,, \quad F = 1\,, \quad m_1 = 0\,,$$
$$m_2 = -1\,, \quad c_{11} = 0.1\,, \quad c_{12} = 0\,. \quad (42)$$

Figure 7 depicts ψ_7 in Eq. (37) with the parameters

$$\beta = 0.1\,, \quad F = 2\,, \quad m_1 = -1\,,$$
$$m_2 = 1\,, \quad c_{13} = 10\,, \quad c_{14} = 0\,, \quad (43)$$

at $t = 0$.

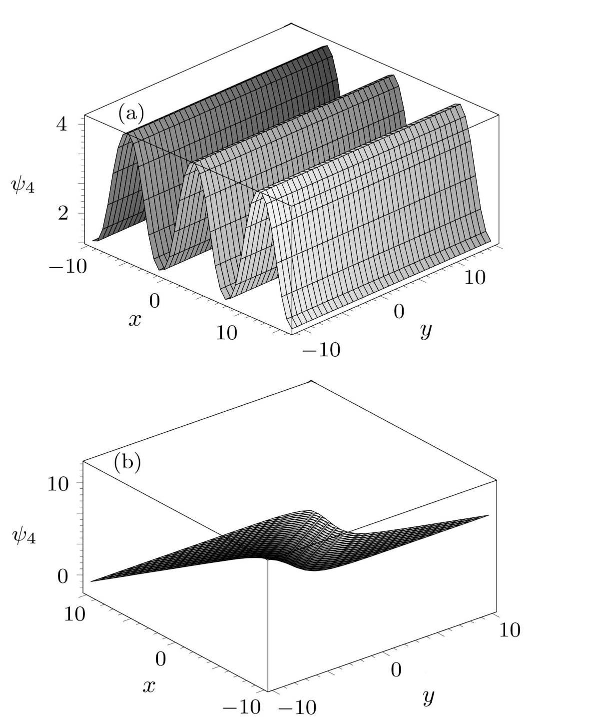

Fig. 5 (a) ψ_4 given by Eq. (37) with Eq. (40) and (b) ψ_4 given by Eq. (37) with Eq. (41) both at $t = 1$.

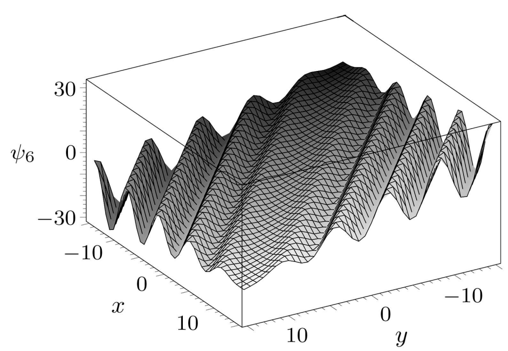

Fig. 6 ψ_6 in Eq. (37) with Eq. (42) at $t = 0$.

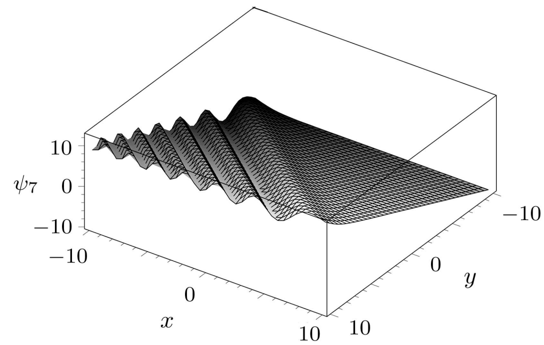

Fig. 7 ψ_7 in Eq. (37) with Eq. (43) at $t = 0$.

4 Conclusions

For the BQGPV equation (1) being one of the most important models of the atmospheric and ocean dynamical systems, to obtain its explicit solutions is very meaningful. Lou *et al.* have gained some types of explicit solutions using the classical symmetry. But solving some of the reduced equations of Lie group is still difficult. To gain more explicit solutions of Eq. (1), the Bäcklund transformation method is used in this paper. Firstly, some simple special Bäcklund transformation theorems are proposed. It is shown that all solutions of a second order linear ordinary differential equation can be used to construct exact solutions of $(2+1)$-dimensional BQGPV equation. For eleven special circumstances of the ordinary differential equation, we give out their solutions. Based that the corresponding explicit solutions of Eq. (1), which are expressed by special functions, are obtained. Furthermore, some figures are given out to depict these solutions. These figures show rich structures of solutions of BQGPV equation, which may be interesting in studying atmospheric and ocean dynamical systems. However, some explicit solutions we obtained have singularities. How to explain these singularities and to search for more exact solutions with no singularities of the BQGPV equation are left to be investigated later.

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