Symmetry reduction and exact solutions of the (3+1)-dimensional Zakharov–Kuznetsov equation*

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By means of the classical method, we investigate the (3+1)-dimensional Zakharov–Kuznetsov equation. The symmetry group of the (3+1)-dimensional Zakharov–Kuznetsov equation is studied firstly. And the theorem of group invariant solutions is constructed. Then using the associated vector fields of the obtained symmetry, we give the one-, two-, and three-parameter optimal systems of group-invariant solutions. Based on the optimal system, we derive the reductions and some new solutions of the (3+1)-dimensional Zakharov–Kuznetsov equation.

Keywords: Zakharov–Kuznetsov equation, classical Lie method, explicit solution

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1. Introduction

It is well known that many phenomena are described by (3+1)-dimensional partial differential equations. Thus it is of great importance to find exact solutions of such high dimensional equations.[1−7] But it is very difficult to solve (3+1)-dimensional equations. As is known, the symmetry group technique is one of the powerful tools to solve nonlinear differential equation.[8−15] However, if one uses the symmetry method to solve partial differential equations, independent variables are added. The dimension of the symmetry algebra increases correspondingly. This induces a complication for the reduction of the initial equation by one-dimensional subalgebras. If the former dimension of the symmetry algebra is n and the later dimension of the symmetry algebra is $n+3$, then when one uses the symmetry to reduce the initial equation directly, one will cope with about 2^{n+3} cases instead of 2^n. For the reduction into two- and three-dimensional subalgebras it will be even more difficult. From this, we can see the complication of solving (3+1)-dimensional partial differential equations. So, we need an effective method to solve them.

Symmetry group techniques provide one method for obtaining solutions of partial differential equations. Since Sophus Lie[8,9] set up the theory of point symmetry groups, a standard method has been widely used to find Lie point symmetry algebras and groups for almost all the known differential systems. The adjoint representation of a Lie group on its Lie algebra was known to Lie. Its use in classifying group-invariant solutions appeared in Ovsiannikov.[10] As is known, the symmetry method allows one to determine special classes of exact solutions of the given partial differential equation(s). Furthermore, from the knowledge of the symmetry group one obtains the Lie algebra of the equation(s) under consideration. Via solving the characteristic equation, the reduction of variables in a given equation is obtained. Such reductions of the equations may be obtained from any linear combination of the symmetry field. However, it is usually not feasible to list all of the possible similarity reductions since there is almost an infinite number of such combinations. This fact is important because by means of the adjoint representations of the symmetry group on its Lie algebra one finds invariant solutions which are not related by a transformation in the symmetry group. Therefore, it is sufficient to consider only linear combinations, which lead to reductions that are inequivalent with respect to symmetry transformations. This set of solutions is called an optimal system.[8] Some works on such optimal

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systems have been reported.[16,17] However, two- and three-parameter optimal systems have seldom been applied to (3+1)-dimensional partial differential equations and the results obtained in this paper have not been reported before. Precisely, by introducing the adjoint representation of the Lie algebra, we obtain the following basic fields of an optimal system, from which every other solutions can be derived.

In Ref. [18], the authors investigated an isothermal multicomponent magnetised plasma and firstly derived the (3+1)-dimensional Zakharov–Kuznetsov(ZK) equation

$$u_t + auu_x + bu_{xxx} + u_{xyy} + u_{xzz} = 0, \tag{1}$$

which was also obtained by using the perturbation method in the propagation of dust.[19,20]

This paper is arranged as follows: in Section 2, by using the classical method, the symmetry group of the (3+1)-dimensional ZK equation is obtained. The transformations leaving the solutions invariant are also obtained. In Section 3, by using the equivalent vector of the symmetry, we give the one-, two-, and three-parameter optimal systems of group-invariant solutions. Based on these optimal systems, some reductions and new solutions of the (3+1)-dimensional ZK equation are derived. Finally, some conclusions and discussions are given in Section 4.

2. Symmetry group of the Zakharov–Kuznetsov equation

By applying the classical method we consider the one-parameter group of infinitesimal transformations in (x, y, z, t, u) of Eq. (1) given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x, y, z, t, u) + O(\epsilon^2),\\ y^* &= y + \epsilon\eta(x, y, z, t, u) + O(\epsilon^2),\\ z^* &= z + \epsilon\zeta(x, y, z, t, u) + O(\epsilon^2),\\ t^* &= t + \epsilon\tau(x, y, z, t, u) + O(\epsilon^2),\\ u^* &= u + \epsilon\Psi(x, y, z, t, u) + O(\epsilon^2), \end{aligned} \tag{2}$$

where ϵ is the group parameter. It is required that Eq. (1) be invariant under the transformation (2), and this yields a system of overdetermined, linear equations for the infinitesimals ξ, η, ζ, τ, and Ψ. Solving these equations, one can find

$$\begin{aligned} \xi &= c_1x + c_6at + c_7,\\ \eta &= c_1y + c_3z + c_4,\\ \zeta &= -c_3y + c_1z + c_5,\\ \tau &= 3c_1t + c_2,\\ \Psi &= -2c_1u + c_6, \end{aligned}$$

where the $c_i (i = 1, 2, \ldots, 7)$ are arbitrary constants. The associated vector fields for the one-parameter Lie group of infinitesimal transformations $v_1, v_2, \ldots, v_7$, are given by

$$\begin{aligned} &v_1 = \partial_x,\quad v_2 = \partial_y,\quad v_3 = \partial_z,\quad v_4 = \partial_t,\\ &v_5 = z\partial_y - y\partial_z,\quad v_6 = at\partial_x + \partial_u,\\ &v_7 = x\partial_x + y\partial_y + z\partial_z + 3t\partial_t - 2u\partial_u. \end{aligned} \tag{3}$$

Equations (3) show that the following transformations (defined by $\exp(\epsilon v_i), i = 1, 2, \ldots, 7$) of variables (x, y, t, u) leave the solutions of Eq. (1) invariant:

$$\begin{aligned} &\exp(\epsilon v_1) : (x, y, z, t, u) \mapsto (x + \epsilon, y, z, t, u),\\ &\exp(\epsilon v_2) : (x, y, z, t, u) \mapsto (x, y + \epsilon, z, t, u),\\ &\exp(\epsilon v_3) : (x, y, z, t, u) \mapsto (x, y, z + \epsilon, t, u),\\ &\exp(\epsilon v_4) : (x, y, z, t, u) \mapsto (x, y, z, t + \epsilon, u),\\ &\exp(\epsilon v_5) : (x, y, z, t, u) \mapsto (x, y\cos(\epsilon) + z\sin(\epsilon),\\ &\qquad -y\sin(\epsilon) + z\cos(\epsilon), t, u),\\ &\exp(\epsilon v_6) : (x, y, z, t, u) \mapsto (x + at\epsilon, y, z, t, u + \epsilon),\\ &\exp(\epsilon v_7) : (x, y, z, t, u) \mapsto (xe^{\epsilon}, ye^{\epsilon}, ze^{\epsilon}, te^{3\epsilon}, ue^{-2\epsilon}). \end{aligned} \tag{4}$$

Then the following theorem holds:

Theorem 1 If $\psi = p(x, y, z, t)$ is a solution of the ZK equation, then so are

$$\begin{aligned} \psi^{(1)} &= p(x - \epsilon, y, z, t),\\ \psi^{(2)} &= p(x, y - \epsilon, z, t),\\ \psi^{(3)} &= p(x, y, z - \epsilon, t),\\ \psi^{(4)} &= p(x, y, z, t - \epsilon),\\ \psi^{(5)} &= p(x, y\cos(\epsilon) - z\sin(\epsilon), y\sin(\epsilon) + z\cos(\epsilon), t),\\ \psi^{(6)} &= p(x - at\epsilon, y, z, t) + \epsilon,\\ \psi^{(7)} &= \mathrm{e}^{-2\epsilon}p(x\,\mathrm{e}^{-\epsilon}, y\,\mathrm{e}^{-\epsilon}, z\,\mathrm{e}^{-\epsilon}, t\,\mathrm{e}^{-3\epsilon}). \end{aligned} \tag{5}$$

3. Reductions and solutions of the Zakharov–Kuznetsov equation

By employing the generators $v_i (i = 1, 2, \ldots, 7)$ of the Lie point transformations in Eqs. (3), one can build up exact solutions of Eq. (1) via the symmetry reduction approach. This allows one to lower the number of independent variables of the system of differential equations under consideration using the invariants associated with a given subgroup of the symmetry group. In the following we present some reductions leading to exact solutions of the ZK equation of possible physical interest.

3.1. Reductions by one-dimensional subalgebras

In general, to each subgroup of the symmetry group, there will correspond a family of group-invariant solutions of the ZK equation. It is too complicated to list all possible group-invariant solutions. So it is necessary to introduce an optimal system of group-invariant solutions. By using the method presented in Refs. [8] and [10], we can find the optimal system of group-invariant solutions.

Applying the commutator operators $[v_m, v_n] = v_m v_n - v_n v_m$, we get the following table (the entry in row i and the column j representing $[v_i, v_j]$):

Lie	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	0	0	0	0	0	v_1
v_2	0	0	0	0	$-v_3$	0	v_2
v_3	0	0	0	0	v_2	0	v_3
v_4	0	0	0	0	0	av_1	$3v_4$
v_5	0	v_3	$-v_2$	0	0	0	0
v_6	0	0	0	$-av_1$	0	0	$-2v_6$
v_7	$-v_1$	$-v_2$	$-v_3$	$-3v_4$	0	$2v_6$	0

Therefore, there is

Proposition 1 The operators $v_i (i = 1, 2, \ldots, 7)$ form a Lie algebra, which is a seven-dimensional symmetry algebra.

If we set $v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6 + a_7v_7$, applying the formula

$$\begin{aligned} &Ad\,(\exp(\varepsilon v))\, v_0 \\ &= v_0 - \varepsilon\, [v, v_0] + \frac{1}{2!}\varepsilon^2\, [v, [v, v_0]] \\ &-\frac{1}{3!}\varepsilon^3\, [v, [v, [v, v_0]]] + \ldots, \end{aligned} \tag{6}$$

and Proposition 1, one can get the following theorem by detailed computation:

Theorem 2 The operators generate an optimal system S_1 given by

(a) $v_7 + a_5v_5, a_7 \neq 0$;

(b) $v_5 + v_4 + \alpha v_6, a_7 = 0, a_5 \neq 0, a_4 \neq 0$;

(c) $v_5 + v_6, a_4 = a_7 = 0, a_5 \neq 0, a_6 \neq 0$;

(d_1) $v_5 + v_1, a_4 = a_6 = a_7 = 0, a_1 \neq 0, a_5 \neq 0$;

(d_2) $v_5, a_1 = a_4 = a_6 = a_7 = 0, a_5 \neq 0$;

(e) $v_4 + v_3 + \alpha v_6, a_5 = a_7 = 0, a_3 \neq 0, a_4 \neq 0$;

(f) $v_4 + v_2 + \alpha v_6, a_3 = a_5 = a_7 = 0, a_2 \neq 0, a_4 \neq 0$;

(g_1) $v_4 + v_6, a_2 = a_3 = a_5 = a_7 = 0, a_4 \neq 0, a_6 \neq 0$;

(g_2) $v_4, a_2 = a_3 = a_5 = a_6 = a_7 = 0, a_4 \neq 0$;

(h) $v_6 + v_3, a_4 = a_5 = a_7 = 0, a_3 \neq 0, a_6 \neq 0$;

(i_1) $v_6 + v_2, a_3 = a_4 = a_5 = a_7 = 0, a_2 \neq 0, a_6 \neq 0$;

(i_2) $v_6, a_2 = a_3 = a_4 = a_5 = a_7 = 0, a_6 \neq 0$;

(j) $v_3 + \alpha v_1, a_4 = a_5 = a_6 = a_7 = 0, a_3 \neq 0$;

(k) $v_2 + \alpha v_1, a_3 = a_4 = a_5 = a_6 = a_7 = 0, a_2 \neq 0$;

(l) $v_1, a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$.

Based on S_1, we construct the two-parameter optimal system S_2 and three-parameter optimal system S_3 by using the method of Ref. [10], which are listed in the Appendix. Making use of Theorem 2, one obtains the reductions and solutions of Eq. (1), which are listed in Table 1.

Table 1. Solutions and reductions of (3+1)-dimensional Zakharov–Kuznetsov equation.

case	invariant parameters	solution and reduction of Eq. (1)
(a)	$X = xt^{-1/3}$, $Y = \left[\sin\left(\frac{\alpha}{3}\ln(t)\right) y + \cos\left(\frac{\alpha}{3}\ln(t)\right) z\right] t^{-1/3}$, $Z = \left[\cos\left(\frac{\alpha}{3}\ln(t)\right) y - \sin\left(\frac{\alpha}{3}\ln(t)\right) z\right] t^{-1/3}$	$u = F(X, Y, Z)\, t^{-2/3}$, $-2F - (XF_X + YF_Y + ZF_Z) + \alpha(ZF_Y - YF_Z) + 3aFF_X + 3bF_{XXX}$ $+ 3(F_{YY} + F_{ZZ})_X = 0$
(b)	$X = x + \frac{a\alpha}{2}\arctan\left(\frac{z}{y}\right)^2 + a\alpha t\arctan\left(\frac{z}{y}\right)$, $Y = y^2 + z^2$, $Z = t + \arctan\left(\frac{z}{y}\right)$	$u = -\alpha\arctan\left(\frac{z}{y}\right) + F(X, Y, Z)$, $YF_Z + aYFF_X + a\alpha(F + 2ZF_Z)_{XX} + \left(a^2\alpha^2Z^2 + bY\right) F_{XXX}$ $+ \left(4Y^2F_{YY} + 4YF_y + F_{ZZ}\right)_X = 0$

Table 1 (*continued*)

case	invariant parameters	solution and reduction of Eq. (1)
(c)	$X=x+at\arctan\left(\frac{z}{y}\right)$, $Y=y^2+z^2,\ Z=t$	$u=-\arctan\left(\frac{z}{y}\right)+F(X,Y,Z)$, $YF_Z+aYFF_X+\left(a^2Z^2+bY\right)F_{XXX}+4Y\left(YF_Y\right)_{XY}=0$
(d$_1$)	$X=x-\arctan\left(\frac{y}{z}\right)$, $Y=y^2+z^2,\ Z=t$	$u=F(X,Y,Z),\ YF_Z+aYFF_X+(bY+1)F_{XXX}+4Y\left(YF_Y\right)_{XY}=0$
(d$_2$)	$X=x,\ Y=y^2+z^2,\ Z=t$	$u=F(X,Y,Z),\ F_Z+aFF_X+bF_{XXX}+4\left(YF_Y\right)_{XY}=0$
(e)	$X=x-\frac{a\alpha}{2}t^2,\ Y=y,\ Z=z-t$	$u=\alpha t+F(X,Y,Z),\ \alpha-F_Z+aFF_X+bF_{XXX}+\left(F_{YY}+F_{ZZ}\right)_X=0$
(f)	$X=x-\frac{a\alpha}{2}t^2,\ Y=y-t,\ Z=z$	$u=\alpha t+F(X,Y,Z),\ \alpha-F_Y+aFF_X+bF_{XXX}+\left(F_{YY}+F_{ZZ}\right)_X=0$
(g$_1$)	$X=x-\frac{a}{2}t^2,\ Y=y,\ Z=z$	$u=t+F(X,Y,Z),\ 1+aFF_X+bF_{XXX}+\left(F_{YY}+F_{ZZ}\right)_X=0$
(g$_2$)	$X=x,\ Y=y,\ Z=z$	$u=F(X,Y,Z),\ aFF_X+bF_{XXX}+\left(F_{YY}+F_{ZZ}\right)_X=0$
(h)	$X=x-azt,\ Y=y,\ Z=t$	$u=z+F(X,Y,Z),\ F_Z+aFF_X+\left(b+a^2Z^2\right)F_{XXX}+F_{XYY}=0$
(i$_1$)	$X=x-ayt,\ Y=z,\ Z=t$	$u=y+F(X,Y,Z),\ F_Z+aFF_X+\left(b+a^2Z^2\right)F_{XXX}+F_{XYY}=0$
(i$_2$)	$X=y,\ Y=z,\ Z=t$	$u=\frac{x}{at}+F(X,Y,Z),\ F+ZF_Z=0$
(j)	$X=x-\alpha z,\ Y=y,\ Z=t$	$u=F(X,Y,Z),\ F_Z+aFF_X+\left(b+\alpha^2\right)F_{XXX}+F_{XYY}=0$
(k)	$X=x-\alpha y,\ Y=z,\ Z=t$	$u=F(X,Y,Z),\ F_Z+aFF_X+\left(b+\alpha^2\right)F_{XXX}+F_{XYY}=0$
(l)	$X=y,\ Y=z,\ Z=t$	$u=F(X,Y,Z),\ F_Z=0$

Solving such reduction equations, one obtains the solutions of (3+1)-dimensional ZK equation. For example, the reduction equation in case (d$_2$) has the solution

$$F=\frac{C_1X+G(Y)+C_3}{aC_1Z+C_2},$$

from which one obtains the following solution of Eq. (1),

$$u=\frac{C_1x+G(y^2+z^2)+C_3}{aC_1t+C_2}, \tag{7}$$

where $C_i(i=1,2,3)$ are arbitrary constants, and $G(y^2+z^2)$ is an arbitrary function of y^2+z^2. The corresponding projective structure figures are plotted along $z=0$, $y=0$ and $x=0$ in Figs. 1(a), 1(b) and 1(c), respectively. It can be seen that along the y and z directions, there is a soliton; while along the x direction, it is shown a single-soliton excitation.

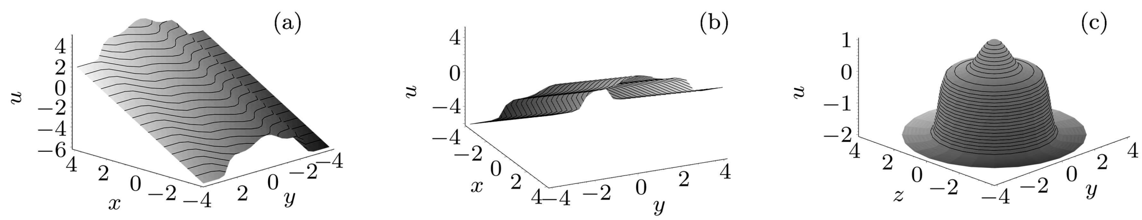

Fig. 1. The projective structure of Eq. (7) with $a=C_1=1$, $C_2=C_3=0$, $G(y^2+z^2)=-\tanh(y^2+z^2)-\tanh(y^2+z^2-6)$ at time $t=1$ along: (a) $z=0$; (b) $y=0$; (c) $x=0$.

If $C_3/(C_1^2(\alpha^2+b)+C_2^2>0)$, the solution of the reduction equation in case (j) has the form

$$F=-\frac{3C_3}{a}^2\sec\left(\frac{1}{2}\sqrt{\frac{C_3}{C_1^2(\alpha^2+b)+C_2^2}}(C_1X+C_2Y+C_1C_3Z+C_4)\right)^2,$$

from which we have the solution of Eq. (1)

$$u=-\frac{3C_3}{a}^2\sec\left(\frac{1}{2}\sqrt{\frac{C_3}{C_1^2(\alpha^2+b)+C_2^2}}(C_1(x-\alpha z)+C_2y+C_1C_3t+C_4)\right)^2,$$

where $C_i(i=1,2,3,4)$ are arbitrary constants. Otherwise, it leads to a solution covering that in Ref. [20]

$$u=-\frac{3{C_3}^2}{a}\operatorname{sech}\left(\frac{1}{2}\sqrt{-\frac{C_3}{C_1^2(\alpha^2+b)+C_2^2}}(C_1(x-\alpha z)+C_2y+C_1C_3t+C_4)\right)^2,$$

where $C_i(i=1,2,3,4)$ are arbitrary constants.

3.2. Reductions by two-dimensional and three-dimensional subalgebras

For the case $(v_7+\alpha v_5, v_6)$ in S_2, taking $\alpha=0$, from $xu_x+yu_y+zu_z+3tu_t+2u=0$ and $atu_x-1=0$, we have $u=x/at+(1/t^{2/3})F(y/t^{1/3},z/t^{1/3})$. And the reduction equation of Eq. (1) is

$$XF_X+YF_Y-F=0,$$

where $X=y/t^{1/3}$ and $Y=z/t^{1/3}$. Solving the above equation, one obtains $F=XG(Y/X)$, where $G(Y/X)$ is an arbitrary function of Y/X. So we have calculated the solution

$$u=\frac{x}{at}+\frac{y}{t}G\left(\frac{z}{y}\right) \tag{8}$$

of Eq. (1). The corresponding projective structure figures of Eq. (8) are plotted along $z=0$, $y=0$ and $x=0$ in Figs. 2(a), 2(b) and 2(c), respectively.

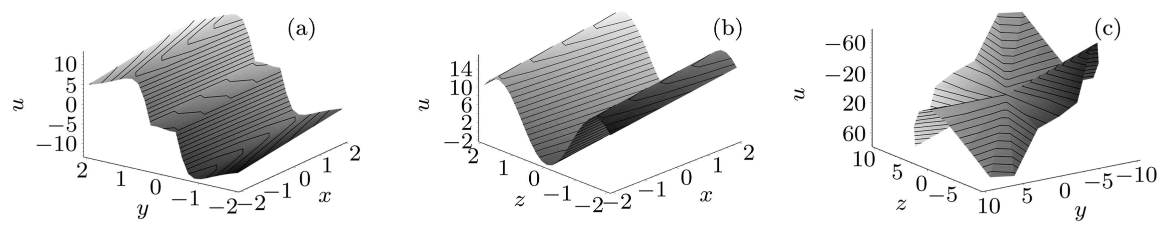

Fig. 2. The projective structure of Eq. (8) with $a=1, G(z/y)=15\sin(z/y)^2$ at time $t=1$ along: (a) $z=1$; (b) $y=1$; (c) $x=0$.

For the case (v_5+v_1, v_6) in S_2, solving $u_x+zu_y-yu_z=0$ and $atu_x-1=0$, one can have $u=(x+\arctan(z/y))/at+F(y^2+z^2,t)$. The substitution of it into Eq. (1) leads to

$$F+tF_t=0.$$

Solving this equation, the solution of Eq. (1) has the form

$$u=\frac{x+\arctan\left(\dfrac{z}{y}\right)}{at}+\frac{G(y^2+z^2)}{t}, \tag{9}$$

where $G(y^2+z^2)$ is an arbitrary function of y^2+z^2. Figures 3(a), 3(b) and 3(c) exhibit the corresponding projective structure figures of Eq. (9) along $z=0$, $y=0$ and $x=0$, respectively.

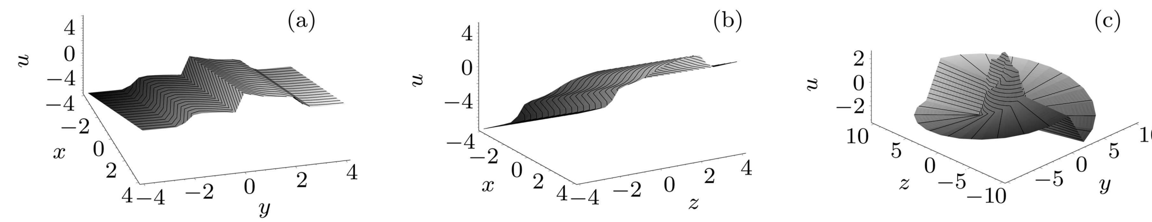

Fig. 3. The projective structure of Eq. (9) with $a=1, G(y^2+z^2)=-\tanh(y^2+z^2)-\tanh(y^2+z^2-6)$ at time $t=1$ along: (a) $z=1$; (b) $y=1$; (c) $x=0$.

For the case $(v_6 + v_3, v_3 + \beta v_1)$ in S_2, from $atu_x + u_z - 1 = 0$ and $u_z + \beta u_x = 0$, it leads to $u = (x - \beta z)/(at - \beta) + F(y, t)$. And Eq. (1) is reduced to

$$aF + (at - \beta)F_t = 0.$$

Solving this equation, one obtains $F = G(y)/(at - \beta)$, where $G(y)$ is an arbitrary function of y. So the solution of Eq. (1) follows

$$u = \frac{x - \beta z + G(y)}{at - \beta}.$$

For the case $(v_6 + v_2, v_2 + \beta v_1)$ in S_2, solving $atu_x + u_y - 1 = 0$ and $u_y + \beta u_x = 0$, the solution is $u = (x - \beta y)/(at - \beta) + F(z, t)$. Then Eq. (1) is written as

$$aF + (at - \beta)F_t = 0.$$

Solving this equation, one obtains the solution of Eq. (1)

$$u = \frac{x - \beta y + G(z)}{at - \beta},$$

where $G(z)$ is an arbitrary function of z.

For the case $(v_4 + v_2 + \alpha v_6, v_3 + \beta v_1, v_2 + \gamma v_1)$ in S_3, from $a\alpha tu_x + u_y + u_t - \alpha = 0$, $u_z + \beta u_x = 0$, and $u_y + \gamma u_x = 0$, we have $u = \alpha t + F(x - \gamma y - \beta z + \gamma t - (a\alpha/2)t^2)$. Substituting it into Eq. (1), we derive the condition on F which reads

$$(b + \beta^2 + \gamma^2)F''' + aFF' + \gamma F' + \alpha = 0,$$

where $F = F(Z)$ and $Z = x - \gamma y - \beta z + \gamma t - (a\alpha/2)t^2$. Integrating the above equation once on Z, one gets the first Painlevé equation

$$(b + \beta^2 + \gamma^2)F'' + \frac{a}{2}F^2 + \gamma F + \alpha Z + C_0 = 0,$$

where C_0 is an arbitrary constant. Given a solution of the above equation, one can derive the solution of Eq. (1). For the other cases in S_2 and S_3 one may also consider a reduction of Eq. (1).

4. Summary and discussion

In summary, we investigated the symmetry of (3+1)-dimensional ZK equation by means of the classic method. The symmetry group of (3+1)-dimensional ZK equation is obtained. Using this symmetry group, the theorem of group invariant solutions is constructed. Then we give the one-, two-, and three-parameter optimal systems by using the associated vector fields of the obtained symmetry. In some cases of the optimal systems, we obtain the reductions and some new solutions of (3+1)-dimensional ZK equation. To the best of our knowledge, the two- and three-parameter optimal systems have been seldom applied to the (3+1)-dimensional partial differential equations and the results obtained in this paper have not been reported before. To better find the localised structures of the solutions obtained in this paper, three types of three dimensional structures are shown. The projective figures clearly show that they are localised along two coordinates but not along the third. Some rich structures of the solutions can be shown only by choosing the arbitrary functions included in the solutions. Recently, a simple direct method presented by Clarkson and Kruskal[11,12] (CK) is used to find all the possible similarity reductions of a nonlinear system without using any group theory. Lou and Ma[7,13,14] modified CK's direct method to find the generalised Lie and non-Lie symmetry groups for the well-known nonlinear equations. The expressions for the exact finite transformations of the Lie groups are much simpler than those obtained via the standard approaches. In the next work, we will try to combine the classical group method with the modified CK's direct method to investigate high dimensional equations.

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Appendix

Here, we list the two-parameter optimal system S_2 and three-parameter optimal system S_3.

S_2: $(v_7 + \alpha v_5, v_4)$, $(v_7 + \alpha v_5, v_6)$, $(v_7 + \alpha v_5, v_1)$, (v_7, v_5), $(v_7, v_3 + \beta v_1)$, $(v_7, v_2 + \beta v_1)$, $(v_5 + v_4 + \alpha v_6, v_5 + v_1)$, $(v_5 + v_4 + \alpha v_6, v_1)$, $(v_5, v_4 + \beta v_6)$, $(v_5 + v_6, v_5 + v_1)$, $(v_5 + v_6, v_1)$, (v_5, v_6), $(v_5 + v_1, v_4 + v_6)$, $(v_5 + v_1, v_4)$, $(v_5 + v_1, v_6)$, (v_5, v_1), $(v_4 + v_3 + \alpha v_6, v_4 + v_2 + \alpha v_6)$, $(v_4 + v_3 + \alpha v_6, v_3 + \beta v_1)$, $(v_4 + v_3 + a_4 v_6, v_2 + \beta v_1)$, $(v_4 + v_3 + a_4 v_6, v_1)$, $(v_4 + v_2 + \alpha v_6, v_3 + \beta v_1)$, $(v_4 + v_2 + \alpha v_6, v_2 + \beta v_1)$, $(v_4 + v_2 + \alpha v_6, v_1)$, $(v_4 + v6, v_3 + \beta v_1)$, $(v_4 + v_6, v_2 + \beta v_1)$, $(v_4 + v_6, v_1)$, $(v_4, v_3 + \beta v_1)$, $(v_4, v_2 + \beta v_1)$, (v_4, v_1), $(v_6 + v_3, v_6 + v_2)$, $(v_6 + v_3, v_3 + \beta v_1)$, $(v_6 + v_3, v_2 + \beta v_1)$, $(v_6 + v_3, v_1)$, $(v_6 + v_2, v_3 + \beta v_1)$, $(v_6 + v_2, v_2 + \beta v_1)$, $(v_6 + v_2, v_1)$, $(v_6, v_3 + \beta v_1)$, $(v_6, v_3 + \beta v_1)$, (v_6, v_1), $(v_3 + \alpha v_1, v_2 + \beta v_1)$, (v_3, v_1), (v_2, v_1).

S_3: $(v_7+\alpha v_5, v_4, v_3+\gamma v_1)$, $(v_7+\alpha v_5, v_4, v_2+\gamma v_1)$, $(v_7+\alpha v_5, v_4, v_1)$, $(v_7+\alpha v_5, v_6, v_3+\gamma v_1)$, $(v_7+\alpha v_5, v_6, v_2+\gamma v_1)$, $(v_7+\alpha v_5, v_6, v_1)$, (v_7, v_5, v_4), (v_7, v_5, v_6), (v_7, v_5, v_1), $(v_7, v_3+\beta v_1, v_2+\gamma v_1)$, (v_7, v_3, v_1), (v_7, v_2, v_1), $(v_5, v_4+\beta v_6, v_1)$, (v_5, v_6, v_1), $(v_4+v_3+\alpha v_6, v_4+v_2+\alpha v_6, v_3+\gamma v_1)$, $(v_4+v_3+\alpha v_6, v_4+v_2+\alpha v_6, v_2+\gamma v_1)$, $(v_4+v_3+\alpha v_6, v_4+v_2+\alpha v_6, v_1)$, $(v_4+v_3+\alpha v_6, v_3+\beta v_1, v_2+\gamma v_1)$, $(v_4+v_3+\alpha v_6, v_2, v_1)$, $(v_4+\alpha v_6, v_3, v_1)$, $(v_4+v_2+\alpha v_6, v_3+\beta v_1, v_2+\gamma v_1)$, $(v_4+v_2+\alpha v_6, v_3, v_1)$, $(v_4+\alpha v_6, v_2, v_1)$, $(v_4+v_6, v_3+\beta v_1, v_2+\gamma v_1)$, $(v_4, v_3+\beta v_1, v_2+\gamma v_1)$, $(v_6+v_3, v_6+v_2, v_3+\gamma v_1)$, $(v_6+v_3, v_6+v_2, v_2+\gamma v_1)$, (v_6+v_3, v_6+v_2, v_1), $(v_6+v_3, v_3+\beta v_1, v_2+\gamma v_1)$, (v_6+v_3, v_2, v_1), (v_6, v_3, v_1), $(v_6+v_2, v_3+\beta v_1, v_2+\gamma v_1)$, (v_6+v_2, v_3, v_1), (v_6, v_2, v_1), $(v_6, v_3+\beta v_1, v_2+\gamma v_1)$, (v_3, v_2, v_1).

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