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A new four-dimensional chaotic system*

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A new four-dimensional chaotic system with a linear term and a 3-term cross product is reported. Some interesting figures of the system corresponding different parameters show rich dynamical structures.

Keywords: chaotic system, Lyapunov exponent, attractor

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1. Introduction

In 1963, Lorenz found the first chaotic attractor in a three-dimensional (3D) autonomous system,[1] later Rösslor constructed an even simpler three-dimensional chaotic system.[2] Since then, chaos as an important nonlinear phenomenon has been studied in science, mathematics, engineering communities, and so on.[3−12] As chaos is useful and has great potential applications in many technological disciplines, the discovery and the creation of chaos are important. In the past few years, Chen[4] constructed a 3D chaotic system via a simple state feedback to the second equation in the Lorenz system, followed by a closely related Lü system constructed by Lü,[5] and a unified system[6] that combines Lorenz system, Chen system and Lü system as its special cases. Some other 3D chaotic systems are also constructed. Recently, Qi *et al.*[13−18] proposed a new 3D chaotic system and 4D chaotic system with cubic terms. Here we report a new 4D chaotic system with a linear term and a cubic term, which also takes on good symmetries and similarities.

2. New 4D system and its properties

The new 4D system is described by

$$\begin{aligned}\dot{x}_1 &= ax_1 - b_1x_1x_2x_3,\\ \dot{x}_2 &= bx_2 - b_2x_1x_3x_4,\\ \dot{x}_3 &= cx_3 - b_3x_1x_2x_4,\\ \dot{x}_4 &= cx_4 - b_4x_1x_2x_3.\end{aligned} \tag{1}$$

(i) Symmetry

The system is invariant for the following coordinate transformations:

$$\begin{aligned}&(x_1,x_2,x_3,x_4)\\ \longrightarrow\ &(-x_1,-x_2,-x_3,-x_4),(-x_1,-x_2,x_3,x_4),\\ &(-x_1,x_2,-x_3,x_4),(-x_1,x_2,x_3,-x_4),\\ &(x_1,-x_2,-x_3,x_4),(x_1,-x_2,x_3,-x_4),\\ &(x_1,-x_2,-x_3,-x_4).\end{aligned} \tag{2}$$

So, it is of symmetry.

(ii) Dissipation

Since

$$\bigtriangledown V = \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} + \frac{\partial \dot{x}_4}{\partial x_4} = a+b+c+d, \tag{3}$$

when $a+b+c+d<0$, system (1) is dissipative, with an exponential contraction rate

$$\frac{\mathrm{d}V}{\mathrm{d}t} = a+b+c+d. \tag{4}$$

It means that a volume element V_0 is contracted into a $V_0\,\mathrm{e}^{(a+b+c+d)t}$ at time t. Therefore, orbits near the chaotic attractor are ultimately restricted within a specific fractal-dimensional subspace of zero volume.

(iii) Equilibria

The equilibria of system (1) can be obtained by solving the following equation:

$$ax_1 - b_1x_1x_2x_3 = 0, \quad bx_2 - b_2x_1x_3x_4 = 0,$$

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$$cx_3 - b_3x_1x_2x_4 = 0, \quad cx_4 - b_4x_1x_2x_3 = 0. \tag{5}$$

By calculations one can find 9 real equilibria including zero. Let

$$\begin{aligned} p &= \pm\frac{\sqrt[4]{cdb_1\,bb_3{}^3a^3b_2{}^3b_4{}^3}}{b_4ab_2b_3}, \\ q &= \pm\frac{\sqrt{b_1\,bab_2}p}{b_1\,b}, \\ r &= \pm\frac{\sqrt{cb_1\,b_3\,ap}}{cb_1}, \end{aligned} \tag{6}$$

then the equilibria except zero can be denoted by

$$S = \left(p, q, r, -\frac{b_4pqr}{d}\right). \tag{7}$$

Let

$$\begin{aligned} p_1 &= \frac{\sqrt[4]{cdb_1\,bb_3{}^3a^3b_2{}^3b_4{}^3}}{b_4ab_2b_3}, \\ p_2 &= -\frac{\sqrt[4]{cdb_1\,bb_3{}^3a^3b_2{}^3b_4{}^3}}{b_4ab_2b_3}, \\ q_{11} &= \frac{\sqrt{b_1\,bab_2}p_1}{b_1\,b}, \quad q_{12} = -\frac{\sqrt{b_1\,bab_2}p_1}{b_1\,b}, \\ q_{21} &= \frac{\sqrt{b_1\,bab_2}p_2}{b_1\,b}, \quad q_{22} = -\frac{\sqrt{b_1\,bab_2}p_2}{b_1\,b}, \\ r_{11} &= \frac{\sqrt{cb_1\,b_3\,ap_1}}{cb_1}, \quad r_{12} = -\frac{\sqrt{cb_1\,b_3\,ap_1}}{cb_1}, \\ r_{21} &= \frac{\sqrt{cb_1\,b_3\,ap_2}}{cb_1}, \quad r_{22} = -\frac{\sqrt{cb_1\,b_3\,ap_2}}{cb_1}, \end{aligned} \tag{8}$$

the equilibria can be denoted as follows:

$$\begin{aligned} S_0 &= (0,0,0,0), \\ S_1 &= \left(p_1, q_{11}, r_{11}, -\frac{b_4p_1q_{11}r_{11}}{d}\right), \\ S_2 &= \left(p_1, q_{11}, r_{12}, -\frac{b_4p_1q_{11}r_{12}}{d}\right), \\ S_3 &= \left(p_1, q_{12}, r_{11}, -\frac{b_4p_1q_{12}r_{11}}{d}\right), \\ S_4 &= \left(p_1, q_{12}, r_{12}, -\frac{b_4p_1q_{12}r_{12}}{d}\right), \\ S_5 &= \left(p_2, q_{21}, r_{21}, -\frac{b_4p_2q_{21}r_{21}}{d}\right), \\ S_6 &= \left(p_2, q_{21}, r_{22}, -\frac{b_4p_2q_{21}r_{22}}{d}\right), \\ S_7 &= \left(p_2, q_{22}, r_{21}, -\frac{b_4p_2q_{22}r_{21}}{d}\right), \\ S_8 &= \left(p_2, q_{22}, r_{22}, -\frac{b_4p_2q_{22}r_{22}}{d}\right). \end{aligned} \tag{9}$$

It can be seen that S_1 and S_2 are symmetric with respect to plane x_1–x_2, S_1 and S_3 are symmetric with respect to plane x_1–x_3, S_1 and S_4 are symmetric with respect to plane x_1–x_4, S_1 and S_5 are symmetric with respect to (0,0,0,0), S_1 and S_6 are symmetric with respect to plane x_3–x_4, S_1 and S_7 are symmetric with respect to plane x_2–x_4, S_1 and S_8 are symmetric with respect to plane x_2–x_3.

(iv) Jacobian matrix

By linearizing system (1) at $S_i = (x_1, x_2, x_3, x_4)$, one can obtain the Jacobian as follows:

$$A_i = \begin{bmatrix} a & b_1x_3x_4 & b_1x_2x_4 & b_1x_2x_3 \\ b_2x_3x_4 & b & b_2x_1x_4 & b_2x_1x_3 \\ b_3x_2x_4 & b_3x_1x_4 & c & b_3x_1x_2 \\ b_4x_2x_3 & b_4x_1x_3 & b_4x_1x_2 & d \end{bmatrix}. \tag{10}$$

If $i = 0, x_1 = x_2 = x_3 = x_4 = 0$, then the eigenvalues of matrix A_0 are

$$\lambda_{01} = a, \quad \lambda_{02} = b, \quad \lambda_{03} = c, \quad \lambda_{04} = d. \tag{11}$$

Therefore, when $abcd < 0$, the equilibrium S_0 is a saddle point. Given the values of x_1, x_2, x_3, x_4 at S_i $(i = 1, \ldots, 8)$, one can calculate the eigenvalues of A_i. By calculating, it is seen that A_i $(i = 1, \ldots, 8)$ have the same eigenvalues.

3. Observation of new chaotic attractor

By choosing the parameters from system (1), a great deal of dynamics can be observed, which is listed together with some discoveries as follows:

(I) $a = -35, b = 10, c = -1, d = -10, b_1 = 1, b_2 = -1, b_3 = 1$ and $b_4 = 1$.

In this case, $a+b+c+d = -35+10-1-10 = -36$, so the system is dissipative, and the eigenvalues of the Jacobian matrix at S_0 are

$$\lambda_{01} = -35, \ \lambda_{02} = 10, \ \lambda_{03} = -1, \ \lambda_{04} = -10, \tag{12}$$

and one can easily find $\lambda_{02} = 10 > 0$, implying that S_0 is a saddle point. By calculating with Maple, we can obtain the eigenvalues of the Jacobian matrix at S_i $(i = 1, \ldots, 8)$ as

$$\begin{aligned} \lambda_{i1} &= 3.4903 + 12.1911\mathrm{i}, \\ \lambda_{i2} &= 3.4903 - 12.1911\mathrm{i}, \\ \lambda_{i3} &= -3.9551, \quad \lambda_{i4} = -44.0254, \end{aligned} \tag{13}$$

the real part of $\lambda_{i1}, \lambda_{i2}$ is $3.49026006 > 0$, so S_i $(i = 1, \ldots, 8)$ is also a saddle point. By calculating with Matlab, the Lyapunov exponents of this system with these parameters are obtained to be

$$l_1 = 4.3614, \quad l_2 = 0.0000,$$

$$l_3 = -3.47442, \quad l_4 = -36.8879. \tag{14}$$

We can easily find that the maximum Lyapunov exponent is positive, so the system is chaotic. Figure 1 shows numerical results for projections on different phase planes and phase spaces. Especially, we can obtain two chaotic attractors when we choose different initial values, which can be seen in Figs. 1(a)–1(k).

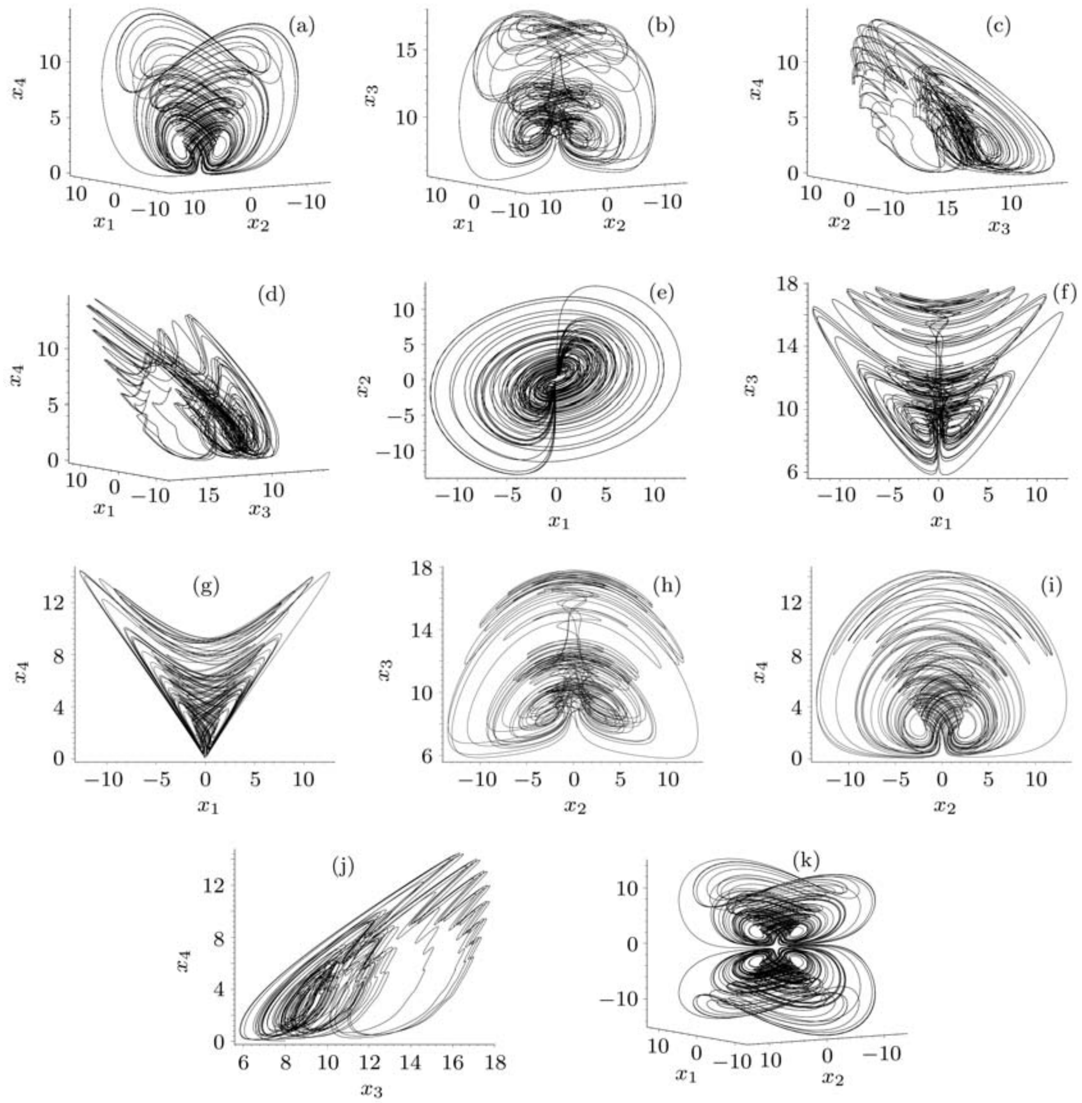

Fig. 1. Chaos system projections on different phase planes and phase spaces with the parameters: $a = -35, b = 10, c = -1, d = -10, b_1 = 1, b_2 = -1, b_3 = 1$ and $b_4 = 1$. (a) 3D view in the x_1–x_2–x_4 space; (b) 3D view in the x_1–x_2–x_3 space; (c) 3D view in the x_2–x_3–x_4 space; (d) 3D view in the x_1–x_3–x_4 space; (e) Projection in the x_1–x_2 plane; (f) Projection in the x_1–x_3 plane; (g) Projection in the x_1–x_4 plane; (h) Projection in the x_2–x_3 plane; (i) Projection in the x_2–x_4 plane; (j) Projection in the x_3–x_4 plane; (k) two coexisting chaotic attractor.

When a varies from –35 to –20, the Lyapunov exponents are

$$\begin{aligned} &l_1 = 0.0022, \quad l_2 = -2.2532, \\ &l_3 = -4.7042, \quad l_4 = -14.0448. \end{aligned} \tag{15}$$

The maximum Lyapunov equals zero, implying that the system has a periodic orbit. Figure 2 shows numerical results for projections on different phase spaces.

(II) $a = -15, b = 5, c = -1, d = -9, b_1 = 1, b_2 = -1, b_3 = 1$ and $b_4 = 1$.

Similar to the first case, the system by choosing the above parameters is also chaotic. Here, only the 3D view figure (see Fig. 3) in the x_1–x_2–x_4 space is given, the other figures are omitted for the sake of concision.

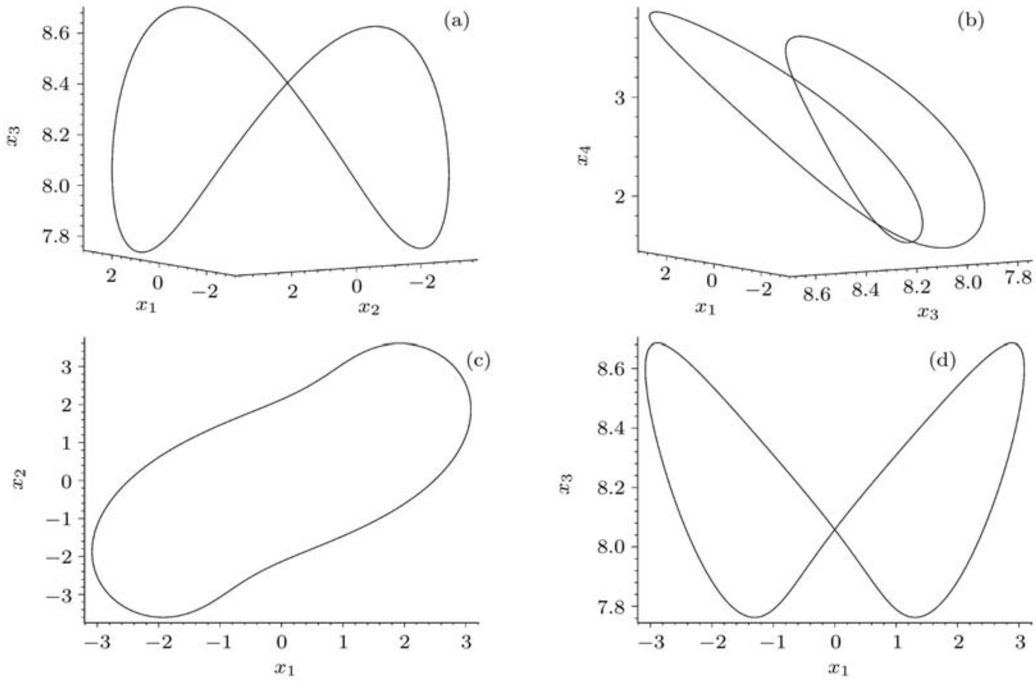

Fig. 2. Chaos system projections on different phase planes and phase spaces with the parameters: $a = -20, b = 10, c = -1, d = -10, b_1 = 1, b_2 = -1, b_3 = 1$ and $b_4 = 1$. (a) 3D view in the x_1–x_2–x_3 space; (b) 3D view in the x_1–x_3–x_4 space; (c) Projection in the x_1–x_2 plane; (d) Projection in the x_1–x_3 plane.

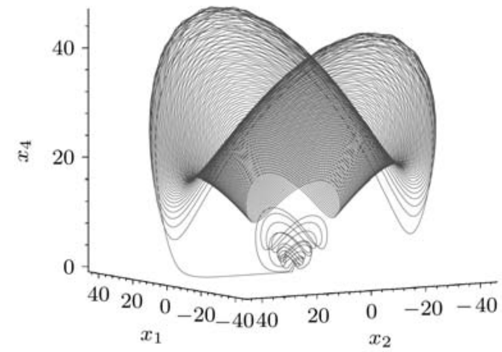

Fig. 3. 3D view in the x_1–x_2–x_4 space with parameters: $a = -15, b = 5, c = -1, d = -9, b_1 = 1, b_2 = -1, b_3 = 1$ and $b_4 = 1$.

(III) $a = -10, b = 3, c = -1, d = -2, b_1 = 1, b_2 = -1, b_3 = 1$ and $b_4 = 1$.

Similar to the first case, this system under the above parameters is chaotic. The 3D view figures (see Fig. 4) in the x_1–x_2–x_4 space with one initial value and two initial values are given, the other figures are omitted for the sake of concision.

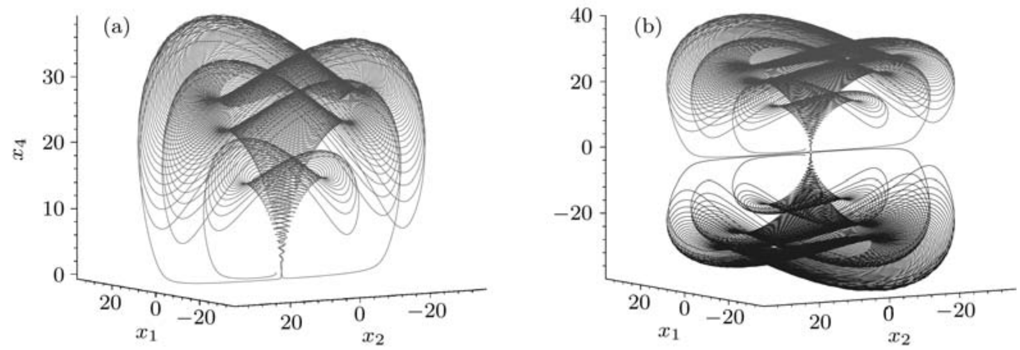

Fig. 4. 3D view in the x_1–x_2–x_4 space with parameters: $a = -10, b = 3, c = -1, d = -2, b_1 = 1, b_2 = -1, b_3 = 1$ and $b_4 = 1$: (a) one initial value; (b) two initial values.

In summary, we first constructed a new 4D chaotic system and studied its properties. Some interesting figures are given, in which one can see that the new system possesses very rich dynamical structures. Hopf bifurcation, Poincaré map, synchronization and so on will be our further study.

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