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Symmetry Reductions and Exact Solutions of Blaszak–Marciniak Four-Field Lattice Equation*

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Abstract *Applying the Lie group method to the differential-difference equation, the Lie point symmetry of Blaszak–Marciniak four-field Lattice equation is obtained. Using the obtained symmetry, the similarity reduction equations of Blaszak–Marciniak four-field Lattice equation are derived. Solving the reduction, we get the solution of Blaszak–Marciniak four-field Lattice equation which not only recovers one of the solutions obtained by Ma and Hu [J. Math. Phys.* **40** *(1999) 6071] but also has the singularity when we choose the arbitrary constants accurately.*

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1 Introduction

Symmetry group techniques provide one method for obtaining solutions of partial differential equations and has been tested to be one of the powerful tools to solve nonlinear differential equation.[1−6] Since Sophus Lie set up the theory of Lie point symmetry group, a standard method had been widely and successfully used to find Lie point symmetry algebras and groups for almost all the known differential systems. Recently, Levi, Winternitz and others applied the Lie group method into some differential-difference and lattice equations.[7−9]

In Ref. [10], Blaszak and Marciniak applied an r-matrix formalism to the algebra of shift operators and derived the four-field systems:

$$\begin{aligned} &u_t(n) = u(n)(v(n) - v(n-1))\,, \\ &v_t(n) = w(n)u(n+1) - u(n)w(n-1)\,, \\ &w_t(n) = q(n)u(n+2) - u(n)q(n-1)\,, \\ &q_t(n) = u(n+3) - u(n)\,. \end{aligned} \tag{1}$$

The Bäcklund transformations, nonlinear superposition formulas, and soliton solutions for Eq. (1) were obtained by Ma and Hu.[11] In Ref. [12], starting from the bilinear form for the four-field lattice, Hu and Tam found a new lattice, which has soliton solutions. By means of new matrix Lax representation, the corresponding conserved density and the associated flux of Eq. (1) were given by Zhu and co-workers.[13] In Ref. [14], by means of considering a 4×4 discrete isospectral problem, a Blaszak–Marciniak four-field lattice hierarchy was re-derived and the corresponding conserved densities and the associated fluxes were obtained.

This paper is arranged as follows. In Sec. 2, by using the classical method, we obtained the Lie point symmetry of Blaszak–Marciniak four-field lattice equation and consider the commutator relations of the generators associated with the symmetry. In Sec. 3, by using the obtained symmetry, we give the similarity reduction equations of Blaszak–Marciniak four-field lattice equation. Solving the reduction, the solution of Blaszak–Marciniak four-field lattice equation are given. Finally, some conclusions and discussions are given in Sec. 4.

2 Symmetry Group of Blaszak–Marciniak Four-Field Lattice Equation

To Eq. (1), we consider the one-parameter local Lie group of transformations

$$\begin{aligned} &n^* = n + \epsilon\xi_1(n) + o(\epsilon^2)\,, \\ &t^* = t + \epsilon\xi_2(n) + o(\epsilon^2)\,, \\ &u^*_{n^*} = u_n + \epsilon\xi_3(n) + o(\epsilon^2)\,, \\ &v^*_{n^*} = v_n + \epsilon\xi_4(n) + o(\epsilon^2)\,, \\ &w^*_{n^*} = w_n + \epsilon\xi_5(n) + o(\epsilon^2)\,, \\ &q^*_{n^*} = q_n + \epsilon\xi_6(n) + o(\epsilon^2)\,, \end{aligned} \tag{2}$$

where ϵ is group parameter, $\xi_i(n) = \xi_i(n, t, u_n, v_n, w_n, q_n)$, $i = 1, 2, \ldots, 6$ are infinitesimals and the associated infinitesimal generator is

$$\begin{aligned} v = {}&\xi_1(n)\partial_n + \xi_2(n)\partial_t + \xi_3(n)\partial_{u_n} + \xi_4(n)\partial_{v_n} \\ &+ \xi_5(n)\partial_{w_n} + \xi_6(n)\partial_{q_n}\,. \end{aligned} \tag{3}$$

It is required that the set of Eq. (1) be invariant under the transformation (2), and this yields a system of overdetermined, linear equations for the infinitesimals $\xi_i(n)$. Solving these equations, one can have

$$\begin{aligned} &\xi_1(n) = \alpha, \quad \xi_2(n) = -\beta t + \gamma\,, \\ &\xi_3(n) = \frac{4}{3}\beta u_n, \quad \xi_4(n) = \beta v_n\,, \end{aligned}$$

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$$\xi_5(n)=\frac{2}{3}\beta w_n,\quad \xi_6(n)=\frac{1}{3}\beta q_n\,,\tag{4}$$

where α, β, and γ are arbitrary constants. And the infinitesimal generator (3) has the following form:

$$v=\alpha\partial_n+(-\beta t+\gamma)\partial_t+\frac{4}{3}\beta u_n\partial_{u_n}+\beta v_n\partial_{v_n}+\frac{2}{3}\beta w_n\partial_{w_n}+\frac{1}{3}\beta q_n\partial_{q_n}\,.$$

If v_1, v_2, v_3 are written as

$$v_1=\partial_n,\quad v_2=\partial_t\,,$$
$$v_3=-t\partial_t+\frac{4}{3}u_n\partial_{u_n}+v_n\partial_{v_n}+\frac{2}{3}w_n\partial_{w_n}+\frac{1}{3}q_n\partial_{q_n}\,,$$

then the commutator relation satisfies

$$[v_1,v_2]=0,\quad [v_1,v_3]=0,\quad [v_2,v_3]=v_2$$

indicating that the underlying symmetry algebra of Eq. (1) is nilpotent.

3 Reductions and Solutions of Blaszak–Marciniak Four-Field Lattice Equation

To get the similarity variable and the similarity transformation of Eq. (1) associated with the above set of symmetries $(\xi_1(n),\xi_4(n),\xi_3(n),\xi_4(n),\xi_5(n),\xi_6(n))$, which are given in Eq. (4), one needs to solve the characteristic equation

$$\frac{dn}{\xi_1(n)}=\frac{dt}{\xi_2(n)}=\frac{du_n}{\xi_3(n)}=\frac{dv_n}{\xi_4(n)}=\frac{dw_n}{\xi_5(n)}=\frac{dq_n}{\xi_6(n)}\,.$$

We will consider the similarity variable and the similarity transformation in the following two cases.

Case 1 $\beta\neq 0$. Then the similarity variable η and the similarity transformations $f_i(\eta)$, $i=1,2,3,4$ are

$$\eta=n+\frac{\alpha}{\beta}\ln(-\beta t+\gamma)\,,$$
$$f_1(\eta)=(-\beta t+\gamma)^{4/3}u_n,\quad f_2(\eta)=(-\beta t+\gamma)v_n\,,$$
$$f_3(\eta)=(-\beta t+\gamma)^{2/3}w_n,\quad f_4(\eta)=(-\beta t+\gamma)^{1/3}q_n\,,$$

and so the Blaszak–Marciniak four-field lattice equation is reduced into

$$\frac{4}{3}\beta f_1(\eta)-\alpha\frac{df_1(\eta)}{d\eta}=f_1(\eta)(f_2(\eta)-f_2(\eta-1))\,,$$
$$\beta f_2(\eta)-\alpha\frac{df_2(\eta)}{d\eta}=f_3(\eta)f_1(\eta+1)-f_1(\eta)f_3(\eta-1)\,,$$
$$\frac{2}{3}\beta f_3(\eta)-\alpha\frac{df_3(\eta)}{d\eta}=f_4(\eta)f_1(\eta+2)-f_1(\eta)f_4(\eta-1)\,,$$
$$\frac{1}{3}\beta f_4(\eta)-\alpha\frac{df_4(\eta)}{d\eta}=f_1(\eta+3)-f_1(\eta)\,.\tag{5}$$

Case 2 $\beta=0$ and $\gamma\neq 0$. In this case, the similarity variable η, the similarity transformations $f_i(\eta)$, $i=1,2,3,4$ and the reduced equations are

$$\eta=n-\frac{\alpha}{\gamma}t,\quad f_1(\eta)=u_n\,,$$
$$f_2(\eta)=v_n,\quad f_3(\eta)=w_n,\quad f_4(\eta)=q_n\,,\tag{6}$$
$$-\frac{\alpha}{\gamma}\frac{df_1(\eta)}{d\eta}=f_1(\eta)(f_2(\eta)-f_2(\eta-1))\,,$$
$$-\frac{\alpha}{\gamma}\frac{df_2(\eta)}{d\eta}=f_3(\eta)f_1(\eta+1)-f_1(\eta)f_3(\eta-1)\,,$$
$$-\frac{\alpha}{\gamma}\frac{df_3(\eta)}{d\eta}=f_4(\eta)f_1(\eta+2)-f_1(\eta)f_4(\eta-1)\,,$$
$$-\frac{\alpha}{\gamma}\frac{df_4(\eta)}{d\eta}=f_1(\eta+3)-f_1(\eta)\,.\tag{7}$$

Solving the above reduction equations (7), one can get

$$f_1(\eta)=\frac{(C_2\alpha)^{4/3}\exp(C_2)(1+C_1\exp(C_2(\eta+1)+C_0))(1+C_1\exp(C_2(\eta-1)+C_0))}{\gamma^{4/3}(\exp(3C_2)-1)^{1/3}(\exp(C_2)-1)(1+C_1\exp(C_2\eta+C_0))^2}\,,$$
$$f_2(\eta)=-\frac{C_1C_2\alpha(\exp(C_2(\eta+1)+C_0)-\exp(C_2\eta+C_0))}{\gamma(1+C_1\exp(C_2(\eta+1)+C_0))(1+C_1\exp(C_2\eta+C_0))}\,,$$
$$f_3(\eta)=\frac{C_1(C_2\alpha)^{2/3}(\exp(3C_2)-1)^{1/3}(\exp(C_2)-1)\exp(C_2(\eta+1)+C_0)}{\gamma^{2/3}\exp(C_2)(1+C_1\exp(C_2(\eta+2)+C_0))(1+C_1\exp(C_2\eta+C_0))}\,,$$
$$f_4(\eta)=-\frac{C_1(C_2\alpha)^{1/3}(\exp(C_2)-1)(\exp(C_2(\eta+3)+C_0)-\exp(C_2\eta+C_0))}{\gamma^{1/3}(\exp(3C_2)-1)^{1/3}(1+C_1\exp(C_2(\eta+3)+C_0))(1+C_1\exp(C_2\eta+C_0))}\,,$$

from which we derive the solution of Eq. (1) via Eq. (6)

$$u_n=\frac{(C_2\alpha)^{4/3}\exp(C_2)(1+C_1\exp(C_2(n-(\alpha/\gamma)t+1)+C_0))(1+C_1\exp(C_2(n-(\alpha/\gamma)t-1)+C_0))}{\gamma^{4/3}(\exp(3C_2)-1)^{1/3}(\exp(C_2)-1)(1+C_1\exp(C_2(n-(\alpha/\gamma)t)+C_0))^2}\,,$$
$$v_n=-\frac{C_1C_2\alpha(\exp(C_2(n-(\alpha/\gamma)t+1)+C_0)-\exp(C_2(n-(\alpha/\gamma)t)+C_0))}{\gamma(1+C_1\exp(C_2(n-(\alpha/\gamma)t+1)+C_0))(1+C_1\exp(C_2(n-(\alpha/\gamma)t)+C_0))}\,,$$
$$w_n=\frac{C_1(C_2\alpha)^{2/3}(\exp(3C_2)-1)^{1/3}(\exp(C_2)-1)\exp(C_2(n-(\alpha/\gamma)t+1)+C_0)}{\gamma^{2/3}\exp(C_2)(1+C_1\exp(C_2(n-(\alpha/\gamma)t+2)+C_0))(1+C_1\exp(C_2(n-(\alpha/\gamma)t)+C_0))}\,,$$
$$q_n=-\frac{C_1(C_2\alpha)^{1/3}(\exp(C_2)-1)(\exp(C_2(n-(\alpha/\gamma)t+3)+C_0)-\exp(C_2(n-(\alpha/\gamma)t)+C_0))}{\gamma^{1/3}(\exp(3C_2)-1)^{1/3}(1+C_1\exp(C_2(n-(\alpha/\gamma)t+3)+C_0))(1+C_1\exp(C_2(n-(\alpha/\gamma)t)+C_0))}\,,\tag{8}$$

where C_i $(i=0,1,2)$ are arbitrary constants. Equation (8) is the solution (26) in Ref. [11] when we take $C_1=1$, $C_2=p$,

$$\gamma=-\frac{p\alpha\exp(3p/4)}{(\exp(3p)-1)^{1/4}(\exp(p)-1)^{3/4}}$$

and $C_0 = rz + sy + \eta^0$. Figure 1 depicts the one-soliton structure of the above u_n, v_n, w_n, and q_n. While from Fig. 2 we find that u_n, v_n, w_n, and q_n have the singularity when we take $C_1 < 0$.

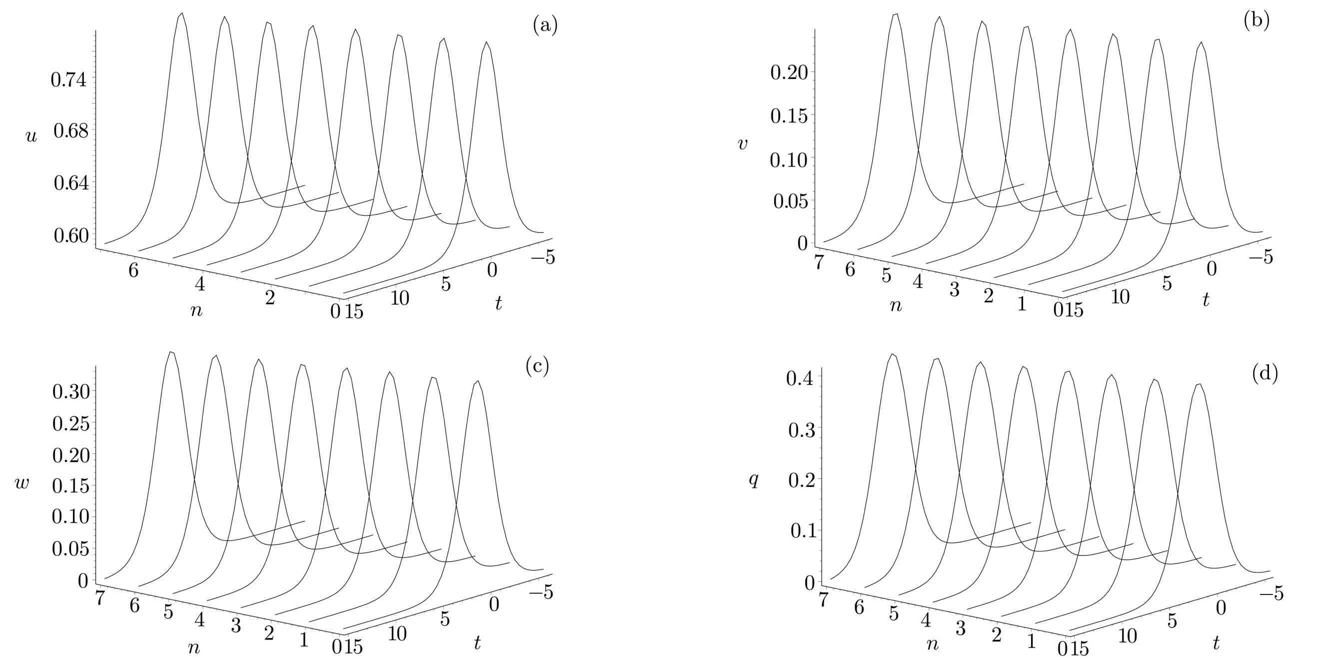

Fig. 1 The structure of function given by Eq. (8) with $\alpha = \gamma = C_1 = C_2 = 1$ and $C_0 = 0$: (a) u_n; (b) v_n; (c) w_n; (d) q_n.

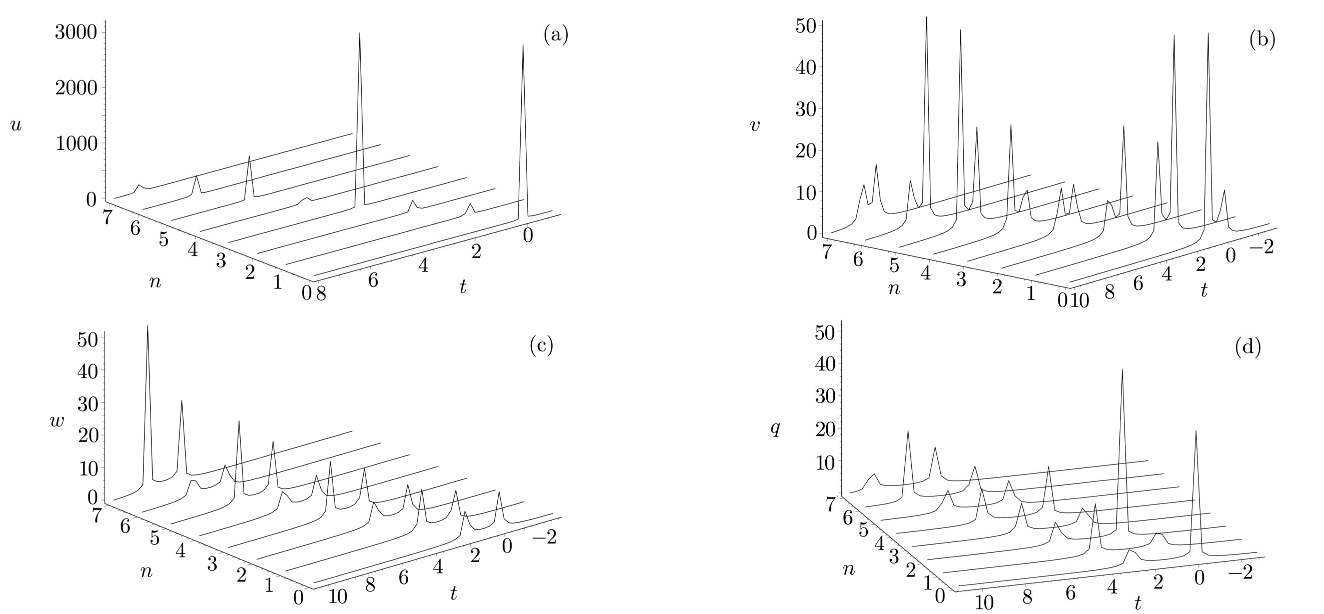

Fig. 2 The structure of function given by Eq. (8) with $C_1 = -1, \alpha = \gamma = C_2 = 1$ and $C_0 = 0$: (a) u_n; (b) v_n; (c) w_n; (d) q_n.

4 Summary and Discussion

In summary, we investigate the symmetry of the Blaszak–Marciniak four-field lattice equation by means of the classic method. The symmetry group of the Blaszak–Marciniak four-field lattice equation is obtained. Using the symmetry group, the similarity variable, similarity transformations, and the reduced equations are given. Solving the reduced equations, from the similarity transformations, we get the solution of the Blaszak–Marciniak four-field lattice equation, which have not only the one-soliton structure but also the singularity. Finding other solutions of the reduced equations, one can obtain other new solutions of the Blaszak–Marciniak four-field lattice equation.

References

[1] P.J. Olver, *Applications of Lie Group to Differential Equations*, Springer-Verlag, New York (1993).

[2] G.W. Bluman and S.C. Anco, *Symmetry and Integration Methods for Differential Equations*, Springer-Verlag, New York (2002).

[3] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York (1982).

[4] P.A. Clarkson and M. Kruskal, J. Math. Phys. **30** (1989) 2201.

[5] P.A. Clarkson, Chaos, Solitons & Fractals **5** (1995) 2261.

[6] Lou Sen-Yue and Ma Hong-Cai, J. Phys. A: Math. Gen. **38** (2005) L129; Ma Hong-Cai and Lou Sen-Yue, Z. Naturforsch **60a** (2005) 229.

[7] D. Levi and P. Winternitz, J. Math. Phys. **34** (1993) 3713.

[8] D. Levi and M.A. Rodriguez, J. Phys. A: Math. Gen. **32** (2005) 8303.

[9] R. Sahadevan and S. Khousalya, J. Math. Phys. **42** (2001) 3854.

[10] M. Blaszak and K. Marciniak, J. Math. Phys. **35** (1994) 4661.

[11] Ma Wen-Xiu, Hu Xing-Biao, Zhu Si-Ming, and Wu Yong-Tang, J. Math. Phys. **40** (1999) 6071.

[12] Hu Xing-Biao and H.W. Tam, Rept. Math. Phys. **46** (2000) 99.

[13] Zhu Zuo-Nong, Wu Xiao-Nan, Xue Wei-Min, and Zhu Zuo-Ming, Phys. Lett. A **296** (2002) 280.

[14] Zhu Zuo-Nong, Zhu Zuo-Ming, Wu Xiao-Nan, and Xue Wei-Min, J. Phys. Soc. Jpn. **71** (2002) 1864.