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Full symmetry groups, Painlevé integrability and exact solutions of the nonisospectral BKP equation

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ABSTRACT

Based on the generalized symmetry group method presented by Lou and Ma [Lou and Ma, Non-Lie symmetry groups of (2 + 1)-dimensional nonlinear systems obtained from a simple direct method, J. Phys. A: Math. Gen. 38 (2005) L129], firstly, both the Lie point groups and the full symmetry group of the nonisospectral BKP equation are obtained, at the same time, a relationship is constructed between the new solutions and the old ones of equation. Secondly, the nonisospectral BKP can be proved to be Painlevé integrability by combining the standard WTC approach with the Kruskal's simplification, some solutions are obtained by using the standard truncated Painlevé expansion. Finally, based on the relationship by the generalized symmetry group method and some solutions by using the standard truncated Painlevé expansion, some interesting solution are constructed.

1. Introduction

The nonisospectral soliton equations are important physical models, because some of them can describe the waves in a certain type of nonuniform media [1–3]. There are many methods for finding solutions of nonisospectral soliton equations, such as Darboux transformation [4], IST [1,2,5] and so on. In recent years, the study of symmetries [6–8], symmetry groups [9], symmetry reductions [10,11] and group invariant solutions of nonlinear partial differential equations (PDEs) has become one of the most exciting and extremely active areas of research [12–16]. Some powerful methods to obtain the similarity reductions of a given PDE have been developed by mathematicians and physicist, such as, the Lie approach [6,9] and the direct method presented by Clarkson and Kruskal (CK) [10]. Most recently, Lou et al. develop a new symmetry group method, named generalized symmetry group method, in a series of papers [17–21]. By the new symmetry group method, both the Lie point symmetry groups and the full symmetry group can be obtained for some PDEs [22]. Furthermore, the expressions of the exact finite transformations of the Lie groups are much simpler than those obtained via the standard approaches for some nonlinear PDEs.

Here we use the generalized symmetry group method and the standard WTC approach with the Kruskal's simplification [23–27] to investigate (2 + 1)-dimensional nonisospectral BKP equation [28]:

$$\begin{aligned} &9u_t + y(u_{xxxxx} + 15uu_{xxx} + 15u_xu_{xx} + 45u^2u_x - 5u_{xxy} - 15uu_y - 15u_x\partial^{-1}u_y - 5\partial^{-1}u_{yy}) \\ &\quad - 3xu_y - 3u_{xx} - 3u^2 - 3u_x\partial^{-1}u - 6\partial^{-1}u_y = 0, \end{aligned} \tag{1}$$

where $u = u(x, y, t)$. The nonisospectral BKP equation had been researched, for example, Deng obtained the soliton solutions for the nonisospectral BKP equation are derived through Hirota method and Pfaffian technique [29].

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This paper is arranged as follows: In Section 2, both the Lie point groups and the full symmetry group of the nonisospectral BKP equation are obtained, at the same time, a relationship is constructed between the new solutions and the old ones of equation. In Section 3, we give the proof of the Painlevé integrability by combining the standard WTC approach with the Kruskal's simplification [23–27] and to obtain some exact solutions by using Painlevé expansion. Based on the relationship by the generalized symmetry group method and some solutions by using the standard truncated Painlevé expansion, some interesting solution are constructed. In Section 4, we give the conclusion of the article.

2. The full symmetry group of a (2 + 1)-dimensional nonisospectral BKP equation

In order to obtain the full symmetry group of the the nonisospectral BKP equation, firstly, we let

$$u = v_x. \tag{2}$$

Substituting Eqs. (2) into (1), then Eq. (1) becomes:

$$\begin{aligned}&(15v_x v_{xxxx} + 15v_{xx}v_{xxx} + 45v_x^2 v_{xx} + v_{xxxxxx} - 15v_x v_{xy} - 15v_{xx}v_y - 5v_{yy} - 5v_{xxxy})y \\ &\quad + 9v_{xt} - 3xv_{xy} - 3v_{xxx} - 3v_x^2 - 3v_{xx}v - 6v_y = 0,\end{aligned} \tag{3}$$

where $v = v(x, y, t)$.

Let

$$v = \alpha + \beta V(\xi, \eta, \tau), \tag{4}$$

where $\alpha,\ \beta,\ \xi,\ \eta$ and τ are functions of $\{x, y, t\}$. Restricting $V(\xi, \eta, \tau) \equiv V$, and satisfies the same form as the nonisospectral BKP equations (3) but with new independent variables, i.e.,

$$\begin{aligned}&(15V_\xi V_{\xi\xi\xi\xi} + 15V_{\xi\xi}V_{\xi\xi\xi} + 45V_\xi^2 V_{\xi\xi} + V_{\xi\xi\xi\xi\xi\xi} - 15V_\xi V_{\xi\eta} - 15V_{\xi\xi}V_\eta - 5V_{\eta\eta} - 5V_{\xi\xi\xi\eta})\eta \\ &\quad + 9V_{\xi\tau} - 3\xi V_{\xi\eta} - 3V_{\xi\xi\xi} - 3V_\xi^2 - 3V_{\xi\xi}V - 6V_\eta = 0.\end{aligned} \tag{5}$$

Substituting Eqs. (4) into (3), then eliminate $V_{\xi\xi\xi\xi\xi\xi}$ by using Eq. (5), from that, the remained determining equations of the functions $\xi,\ \eta,\ \tau,\ \alpha,\ \beta$ can be got by vanishing the coefficients of V and its derivatives, then we find out the general solution of the determining equations by tedious calculations. The result reads

$$\tau = \tau_0, \quad \eta = (\delta^3 \tau_{0t}^{\frac{3}{5}} y^{\frac{2}{5}} + \eta_0)^{\frac{5}{2}}, \quad \xi = \frac{\delta \tau_{0t}^{\frac{1}{5}}}{y^{\frac{1}{5}}} \left(\delta^3 \tau_{0t}^{\frac{3}{5}} y^{\frac{2}{5}} + \eta_0\right)^{\frac{1}{2}} x + \xi_1(y, t), \tag{6}$$

$$\beta = \frac{\delta}{y^{\frac{1}{5}}} \left(\delta^3 \tau_{0t} y^{\frac{2}{5}} + \eta_0 \tau_{0t}^{\frac{2}{5}}\right)^{\frac{1}{2}}, \tag{7}$$

$$\begin{aligned}\alpha = &-\frac{1}{30}\frac{x^2\eta_0}{\left(\delta^3\tau_{0t}^{\frac{3}{5}}y^{\frac{2}{5}} + \eta_0\right)y} + \frac{x}{\tau_{0t}^{\frac{6}{5}}\left(\delta^3\tau_{0t}^{\frac{3}{5}}y^{\frac{2}{5}} + \eta_0\right)y^{\frac{2}{5}}}\left(\frac{3}{8}\delta^3 y^{\frac{4}{5}}\tau_{0tt}\tau_{0t}^{\frac{3}{5}} + \frac{3}{10}y^{\frac{2}{5}}\tau_{0tt}\eta_0 + \frac{3}{4}y^{\frac{2}{5}}\eta_{0t}\tau_{0t}^{\frac{3}{5}} + \frac{1}{2}\delta^2\eta_{0t}\eta_0 + \frac{1}{15}\delta^2\xi_0\tau_{0t}^{\frac{8}{5}}\right) \\ &+ \frac{1}{y^{\frac{1}{5}}}\left\{\int^y \left[-\frac{1}{3}\frac{y^{\frac{3}{5}}\xi_{1y}^2(y,t)\delta^3}{\tau_{0t}^{\frac{2}{5}}\left(\delta^3\tau_{0t}^{\frac{3}{5}}y^{\frac{2}{5}} + \eta_0\right)} + \frac{3}{5}\frac{\delta^4\xi_{1t}(y,t)}{\left(\delta^3\tau_{0t}^{\frac{3}{5}}y^{\frac{2}{5}} + \eta_0\right)^{\frac{1}{2}}y^{\frac{3}{5}}}\right]dy + \gamma_0\right\},\end{aligned} \tag{8}$$

where $\xi_0 \equiv \xi_0(t),\ \eta_0 \equiv \eta_0(t),\ \tau_0 \equiv \tau_0(t),\ \ \gamma_0 \equiv \gamma_0(t)$ are arbitrary functions of time t and

$$\xi_1(y, t) = \frac{1}{8\tau_t}(9\delta\tau_{0tt}y^{\frac{4}{5}} + 30y^{\frac{2}{5}}\eta_{0t}\delta^3 + 8\xi_0\tau_{0t})\left(\delta^3\tau_{0t}^{\frac{3}{5}}y^{\frac{2}{5}} + \eta_0\right)^{\frac{1}{2}},$$

while the constants δ possess discrete values determined by

$$\delta = 1, \quad \frac{-1 + \sqrt{5} + \mathrm{i}\sqrt{10 + 2\sqrt{5}}}{4}, \quad -\frac{1 + \sqrt{5} - \mathrm{i}\sqrt{10 - 2\sqrt{5}}}{4}, \quad -\frac{1 + \sqrt{5} + \mathrm{i}\sqrt{10 - 2\sqrt{5}}}{4}, \quad \frac{-1 + \sqrt{5} - \mathrm{i}\sqrt{10 + 2\sqrt{5}}}{4}. \tag{9}$$

In summary, the following theorem holds:

Theorem. If $V \equiv V(x, y, t)$ is a solution of the Eq. (3), then so is

$$v = \alpha + \beta V(\xi, \eta, \tau), \tag{10}$$

where $\alpha, \beta, \xi, \eta, \tau$ are given by Eqs. (6)–(8), discrete value of the δ are given by Eq. (9).

The relationship is constructed between the new solutions and the old ones of Eq. (3). Thus from the theorem of Eq. (3) with Eq. (2), we also can obtain the relationship between the new solutions and the old ones of Eq. (1).

From the symmetry group theorem, for the nonisospectral BKP equation, the symmetry group is divided into five sectors which correspond to

$$\delta = 1,$$
$$\delta = \frac{-1+\sqrt{5}+\mathrm{i}\sqrt{10+2\sqrt{5}}}{4},$$
$$\delta = -\frac{1+\sqrt{5}-\mathrm{i}\sqrt{10-2\sqrt{5}}}{4},$$
$$\delta = -\frac{1+\sqrt{5}+\mathrm{i}\sqrt{10-2\sqrt{5}}}{4},$$
$$\delta = \frac{-1+\sqrt{5}-\mathrm{i}\sqrt{10+2\sqrt{5}}}{4},$$

of theorem respectively. That is to say, the full symmetry group, $\mathcal{G}_{CBKP}$, expressed by theorem for the complex the nonisospectral BKP equation is the product of the usual Lie point symmetry group $\mathcal{S}$ $(\delta = 1)$ and the discrete group $\mathcal{D}_5$

$$\mathcal{G}_{CBKP} = \mathcal{D}_5 \otimes \mathcal{S}, \tag{11}$$
$$\mathcal{D}_5 \equiv \{I,\ R_1,\ R_2,\ R_3,\ R_4\}, \tag{12}$$

where I is the identity transformation, and

$$R_1 : v(x,y,t) \to \frac{-1+\sqrt{5}-\mathrm{i}\sqrt{10+2\sqrt{5}}}{4}\, v\left(\frac{-1+\sqrt{5}+\mathrm{i}\sqrt{10+2\sqrt{5}}}{4}x,\ y,\ t\right),$$
$$R_2 : v(x,y,t) \to \frac{-1+\sqrt{5}+\mathrm{i}\sqrt{10+2\sqrt{5}}}{4}\, v\left(\frac{-1+\sqrt{5}-\mathrm{i}\sqrt{10+2\sqrt{5}}}{4}x,\ y,\ t\right),$$
$$R_3 : v(x,y,t) \to -\frac{1+\sqrt{5}-\mathrm{i}\sqrt{10-2\sqrt{5}}}{4}\, v\left(-\frac{1+\sqrt{5}+\mathrm{i}\sqrt{10-2\sqrt{5}}}{4}x,\ y,\ t\right),$$
$$R_4 : v(x,y,t) \to -\frac{1+\sqrt{5}+\mathrm{i}\sqrt{10-2\sqrt{5}}}{4}\, v\left(-\frac{1+\sqrt{5}-\mathrm{i}\sqrt{10-2\sqrt{5}}}{4}x,\ y,\ t\right).$$

From theorem, by restricting $(f \equiv f(t),\ g \equiv g(t),\ h \equiv h(t)),\ p \equiv p(t))$

$$\tau = t + \epsilon f, \quad \xi_0 = \epsilon g, \quad \eta_0 = \epsilon h, \quad \gamma_0 = \epsilon p,$$

then Eq. (10) can be written as

$$v = V + \epsilon\sigma(V),$$
$$\sigma(V) = V_t g + \left(\frac{5}{2}y^{\frac{3}{5}}h + \frac{3}{2}yg_t\right)V_y + \left(\frac{9}{8}yg_{tt} + y^{\frac{1}{5}}f + \frac{1}{2}xg_t + \frac{1}{2}\frac{xh}{y^{\frac{2}{5}}} + \frac{15}{4}y^{\frac{3}{5}}h_t\right)V_x + \frac{9}{16}yg_{ttt} + \frac{1}{2}Vg_t + \frac{1}{2}\frac{Vh}{y^{\frac{2}{5}}} + \frac{3}{8}xg_{tt} + \frac{1}{15}\frac{xf}{y^{\frac{4}{5}}}$$
$$- \frac{1}{30}\frac{x^2h}{y^{\frac{7}{5}}} + \frac{3}{4}\frac{xh_t}{y^{\frac{2}{5}}} + \frac{3}{2}y^{\frac{1}{5}}f_t + \frac{45}{16}y^{\frac{3}{5}}h_{tt} + \frac{p}{y^{\frac{1}{5}}},$$

The equivalent vector expression of the above symmetry is

$$\Gamma = \left\{y^{\frac{1}{5}}f\frac{\partial}{\partial x} - \left(\frac{1}{15}\frac{xf}{y^{\frac{4}{5}}} + \frac{3}{2}y^{\frac{1}{5}}f_t\right)\frac{\partial}{\partial V}\right\} + \left\{\left(\frac{9}{8}yg_{tt} + \frac{1}{2}xg_t\right)\frac{\partial}{\partial x} + \frac{3}{2}yg_t\frac{\partial}{\partial y} + g\frac{\partial}{\partial t} - \left(\frac{9}{16}yg_{ttt} + \frac{1}{2}Vg_t + \frac{3}{8}xg_{tt}\right)\right\}\frac{\partial}{\partial V}\bigg\}$$
$$+ \left\{\left(\frac{1}{2}\frac{xh}{y^{\frac{2}{5}}} + \frac{15}{4}y^{\frac{3}{5}}h_t\right)\frac{\partial}{\partial x} + \frac{5}{2}y^{\frac{3}{5}}h\frac{\partial}{\partial y} - \left(\frac{1}{2}\frac{Vh}{y^{\frac{2}{5}}} - \frac{1}{30}\frac{x^2h}{y^{\frac{7}{5}}} + \frac{3}{4}\frac{xh_t}{y^{\frac{2}{5}}} + \frac{45}{16}y^{\frac{3}{5}}h_{tt}\right)\frac{\partial}{\partial V}\right\} - \frac{p}{y^{\frac{1}{5}}}\frac{\partial}{\partial V}$$
$$\equiv \sigma_1(f) + \sigma_2(g) + \sigma_3(h) + \sigma_4(p),$$

which is exactly the same as that we obtained by the standard Lie approach.

The commutation relations for the Kac–Moody–Virasoro algebra among $\sigma_1(f),\ \sigma_2(g), \sigma_3(h)$ and $\sigma_4(p)$ are as follows:

$$[\sigma_1(f_1), \sigma_1(f_2)] = 0, \quad [\sigma_1(f), \sigma_4(p)] = 0, \quad [\sigma_3(h), \sigma_4(p)] = 0, \quad [\sigma_4(p_1), \sigma_4(p_2)] = 0,$$
$$[\sigma_1(f), \sigma_2(g)] = \sigma_1\left(gf_t - \frac{1}{5}g_tf\right), \quad [\sigma_1(f), \sigma_3(h)] = \sigma_4\left(\frac{1}{2}fh_t - \frac{3}{2}f_th\right),$$
$$[\sigma_2(g_1), \sigma_2(g_2)] = \sigma_2(g_1g_{2t} - g_2g_{1t}), \quad [\sigma_2(g), \sigma_3(h)] = \sigma_3\left(gh_t - \frac{3}{5}g_th\right),$$
$$[\sigma_2(g), \sigma_4(p)] = \sigma_4\left(gp_t + \frac{1}{5}g_tp\right), \quad [\sigma_3(h_1), \sigma_3(h_2)] = \frac{15}{4}\sigma_1(h_1h_{2t} - h_2h_{1t}).$$

We can use the relationship form this theorem to obtain new solutions from old solutions. So in the next section, we prove the nonisospectral BKP equation is Painlevé integrability and obtain a solution by using the standard truncated Painlevé expansion.

3. Painlevé integrability of the (2 + 1)-dimensional the nonisospectral BKP equation and some solution from the truncated Painlevé expansion

Painlevé analysis is one of the most powerful method to prove the integrability of a model developed by WTC (Weiss–Tabor–Canvela) [23–27]. If one needs only to prove the Painlevé property of a model, one may use the Kruskal's simplification for WTC method. Furthermore, the Painlevé analysis can also be used to find some exact solutions no matter whether the model is integrable or not.

At first, the Painlevé expansion may have the form:

$$v = \sum_{j=0}^{\infty} v_j f^{j-\alpha}, \tag{13}$$

where the arbitrary function $f \equiv f(x,y,t)$ may have different forms in different approaches, $v_j \equiv v_j(x,y,t)$ $(j = 0, 1, 2 \ldots \infty)$. Using any one possible form, the final conclusion will be exactly the same. In order to give out a complete treatment, it is convenient by using the Kruskal's simplification, i.e.,

$$f = x + \psi(y,t), \tag{14}$$

with $\psi(y,t) \equiv \psi$ being an arbitrary function of y and t.

By substituting $v = v_0 f^{\alpha}$ into Eq. (3), comparing the leading order terms for $f \to 0$, we get two possible branch:

$$\alpha = 1, \tag{15}$$

$$v_0 = 2f_x, \tag{16}$$

or

$$v_0 = 4f_x. \tag{17}$$

For the first branch, by equating the coefficients of f^{j-1}, the polynomial equation in j is derived as:

$$j^6 - 21j^5 + 145j^4 - 375j^3 + 214j^2 + 396j - 360 = 0. \tag{18}$$

Using Eq. (18), the resonances are found to be

$$j = -1,\ 1,\ 2,\ 3,\ 6,\ 10.$$

For $j = 1, 2$ and 3, consequently v_1, v_2 and v_3 are arbitrary functions. For $j = 4$ and 5, we obtain

$$v_4 = \frac{1}{90y}\left(3x\psi_y - 9\psi_t + 5y\psi_y^2\right), \tag{19}$$

$$v_5 = -\frac{1}{360y}(5y\psi_{yy} + 12\psi_y). \tag{20}$$

For $j = 6$, v_6 is arbitrary function. For $j = 7, 8$ and 9, we get

$$v_7 = -\frac{1}{8640y^2}\left(3x\psi_y - 9\psi_t + 5y\psi_y^2 + 5y^2\psi_y\psi_{yy}\right), \tag{21}$$

$$v_8 = -\frac{1}{907200y^2}\Big(720y\psi_y^3x - 2160y\psi_y^2\psi_t + 600y^2\psi_y^4 - 180y\psi_{yy} - 315\psi_y - 1296x\psi_t\psi_y + 1944\psi_t^2 + 216x^2\psi_y^2 - 100y^2\psi_{yyy}\Big), \tag{22}$$

$$v_9 = -\frac{1}{604800y^3}\Big(75y^3\psi_y^2\psi_{yy} - 108yx\psi yt - 360y^2\psi_y\psi_{yt} + 162y\psi_{tt} - 309x\psi_y^2 y + 927\psi_y\psi_t y - 445\psi_y^3 y^2 - 18x^2\psi_y + 54x\psi_t + 18yx^2\psi_{yy} + 180y^2\psi_{yy}\psi_t + 60y^2x\psi_y\psi_{yy}\Big). \tag{23}$$

Then for $j = 10$, v_{10} is arbitrary function. Up to now, we prove that all the resonance conditions are satisfied for the first branch. It is concluded that the Eq. (3) passes the P-test in the first branch.

For the second branch, we can get the polynomial equation in j is

$$j^6 - 21j^5 + 115j^4 - 15j^3 - 836j^2 + 36j + 720 = 0, \tag{24}$$

then we get the resonances

$$j = -1,\ -2,\ 1,\ 5,\ 6,\ 12,$$

we can also prove that all the resonance conditions are satisfied for the second branch.

In all, it is concluded that the Eq. (3) passes the P-test and hence it is expected to be integrable whether in the first branch or second branch.

Then we use Painlevé expansion to obtain some solutions of Eq. (3). Because in the first branch, when $j = 1$, v_1 is a resonance point, let

$$v = \frac{2f_x}{f} = 2[\ln(f)]_x, \tag{25}$$

$$f = a(t) + \exp(p(t)x + q(t)y + w(t)). \tag{26}$$

Substituting Eqs. (25) and (26) into Eq. (3), collecting the coefficient of $\exp(p(t)x + q(t)y + w(t)), x$ and y, then we can get

$$q(t) = 3p_t(t), \quad w(t) = \ln[a(t)] + \frac{1}{3}\int^t p^2(t)dt, \tag{27}$$

$a(t)$ are arbitrary function of t, $p(t)$ should be satisfied

$$45p_t^2(t) - 27p(t)p_{tt}(t) - p^6(t) + 15p^3(t)p_t(t) = 0. \tag{28}$$

So we get the solution of Eq. (3),

$$v = \frac{2p(t)e^{\left(p(t)x+3p_t(t)y+\frac{1}{3}\int^t p^2(t)dt\right)}}{1+e^{\left(p(t)x+3p_t(t)y+\frac{1}{3}\int^t p^2(t)dt\right)}}.$$

Then we also obtain the solution of Eq. (1),

$$u = p^2(t)\mathrm{sech}^2\left(xp(t) + 3p_t(t)y + \frac{1}{3}\int p^2(t)dt\right). \tag{29}$$

If we let

$$p(t) = \frac{k_1(t) + q_1(t)}{2}, \quad k_1(t) = \frac{3}{2\sqrt{9c_1 - 6t}}, \quad q_1(t) = \frac{3}{2\sqrt{9c_2 - 6t}},$$

then Eq. (28) become one-soliton solution of the nonisospectral BKP, which had been given in [29]. If we give out the appropriate form of $p(t)$, we can obtain more abundant solutions of Eq. (1).

By theorem, we also can obtain the new solution of Eq. (3),

$$v = \alpha + \beta\frac{2p(\tau)e^{\left(p(\tau)\xi+3p_\tau(\tau)\eta+\frac{1}{3}\int^\tau p^2(\tau)d\tau\right)}}{\left(1+e^{(p(\tau)\xi+3p_\tau(\tau)\eta+\frac{1}{3}\int^\tau p^2(\tau)d\tau)}\right)}, \tag{30}$$

where α, β, ξ, η and τ are determined by Eqs. (6)–(8), $p(\tau)$ is satisfied

$$45p_\tau^2(\tau) - 27p(\tau)p_{\tau\tau}(\tau) - p^6(\tau) + 15p^3(\tau)p_\tau(\tau) = 0.$$

In the second branch, if we let f have the same form as Eq. (25), we cannot obtain a solution of Eq. (3), which contains three variables x, y and t. So we let

$$f = xp(t) + q(y,t). \tag{31}$$

Substituting Eqs. (17) and (31) into Eq. (3), collecting the coefficient of x and y, then we get

$$q(y,t) = \frac{7}{3}y^{\frac{3}{7}}F_1(t) + \frac{7}{3}p_t(t)y + F_2(t), \tag{32}$$

where $p(t), F_1(t)$ and $F_2(t)$ is arbitrary function of t, then we can obtain the solution of Eq. (3)

$$v = \frac{12p(t)}{3xp(t) + 7y^{\frac{3}{7}}F_1(t) + 7p_t(t)y + 3F_2(t)},$$

then the solution of Eq. (1) is

$$u = \frac{-36p(t)^2}{\left(3xp(t) + 7y^{\frac{3}{7}}F_1(t) + 7p_t(t)y + 3F_2(t)\right)^2}.$$

We can also get a new solution of Eq. (3) by theorem.

4. Conclusions

In summary, making use of the generalized symmetry group method and symbolic computation, the fully symmetry transformation groups for the nonisospectral BKP equation are given. The full symmetry group of the nonisospectral BKP equation is a product of one discrete group ($\mathcal{D}_5$) and one infinite dimensional Kac–Moody–Virasoro type Lie group with four arbitrary functions. The relationship is constructed between the new solutions and the old ones of equation, which is a relationship of group invariant solution. If we have obtained a solution of the nonisospectral BKP equation by other method, we can use this relationship to obtain another solution. Then we prove the nonisospectral BKP can pass the Painlevé test and hence it is expected to be integrable. So we obtain a solution of the nonisospectral BKP by the standard truncated Painlevé expansion, then we get new general solution by the relationship, which is the known one-soliton solution [29]. It is necessary to point out that the general solution contain all its group invariant solutions, so it is enough to submit it to the relationship for one time only.

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