CZECHOSLOVAK JOURNAL OF PHYSICS

Volume 55 · 2005 Number 1 · pp. 1–104

The stochastic soliton-like solutions of stochastic mKdV equations

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Received 30 August 2004;
final version 27 October 2004

In this paper, the generally projective Riccati equations method is improved by means of a generalized transformation. The improved method can be applied to find not only some exact travelling wave solutions but also some soliton-like solutions with the aid of symbolic computation system — *Maple*. We choose Wick-type stochastic mKdV equations to illustrate the method. As a result, some stochastic soliton-like solutions are obtained.

PACS: 05.45.Yv, 02.30.Jr
Key words: extended projective Riccati equation method, stochastic mKdV equations, stochastic soliton-like solutions

1 Introduction

Finding exact solutions, in particular, soliton solutions, of nonlinear partial differential equations (PDEs) arising from many science fields is of prime significance. Numerous powerful methods have been presented, such as the inverse scattering method [1], Backlund transformation [2], Hirota method [3], variable separation method [4], homogenous balance method [5], various tanh methods [6–12], generalized hyperbolic-function method [13], generalized Riccati equation expansion method [14] and generally projective Riccati equations method [15–17]. Recent years, due to the availability of symbolic computation system like *Maple* or *Mathematica* which allows us to perform the complex computation on computer, the direct seeking for exact solutions for nonlinear PDEs has become more and more attractive.

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In this paper, we would like to extend the generally projective Riccati equations method, which was presented by Conte and Musette [15] and recently improved by Yan [16], Chen and Li [17], to find stochastic soliton-like solutions of a Wick-type stochastic mKdV equation as the following form [18]:

$$U_t + f(t)U^2\diamondsuit U_x - g(t)U_{xxx} = W(t)\diamondsuit R^{\diamondsuit}(t, U, U_x, U_{xxx}), \tag{1}$$

where "$\diamondsuit$" is the Wick production on the Hida distribution space $S(R^d)^*$, namely, $S(R^d)^*$ is the white noise functional space which is defined in Refs. [18,19]. Equation (1) is the perturbation of the mKdV equation with variable coefficients

$$u_t + f(t)u^2u_x - g(t)u_{xx} = 0 \tag{2}$$

by random force $W(t)\diamondsuit R^{\diamondsuit}(R, U, U_x, U_{xxx})$, where $f(t)$ and $g(t)$ are functions of t, $W(t)$, is Gaussian white noise, i.e., $W(t) = \dot{B}(t)$, where $B(t)$ is a Brownian motion, $R(t, u, u_x, u_{xxx}) = \alpha u^2 u_x - \beta u_{xxx}$ is a functional of u, u_x and u_{xxx} for some constants α, β and $R^{\diamondsuit}$ is the Wick version of the functional R.

Now, stochastic KdV equations have been studied by many authors, e.g., Wadati et al. [20], de Bouard and Debussche [21], and so on. In [19], Holden, Øsendal, Ubøe and Zhang gave white noise functional approach to research stochastic partial differential equations in Wick versions. We will use their theory, method and the extended projective Riccati equations method to give some new and more general exact solutions of Wick-type Stochastic mKdV equation (1).

2 The extended projective Riccati equations method

Based on the various tanh methods [6–12], generalized hyperbolic-function method [13], generalized Riccati equation expansion method [14] and generally projective Riccati equations method [15–17], now we establish the extended projective Riccati equations method.

Given a nonlinear PDE with, say, two variables $\{x, t\}$,

$$p(u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots) = 0. \tag{3}$$

Step 1. We assume that (3) has the following solutions:

$$u(x,t) = a_0 + \sum_{i=1}^{m} \sigma^{i-1}(\xi)[a_i\sigma(\xi) + b_i\tau(\xi)], \tag{4}$$

where $a_0 = a_0(x,t)$, $a_i = a_i(x,t)$, $b_i = b_i(x,t)$, $(i = 1, \ldots, m)$, $\xi = \xi(x,t)$ are all unknown functions of $\{x, t\}$, $\sigma(\xi)$ and $\tau(\xi)$ satisfy the projective Riccati equations

$$\sigma'(\xi) = \epsilon\sigma(\xi)\tau(\xi), \quad \tau'(\xi) = R + \epsilon\tau^2(\xi) - \mu\sigma(\xi), \quad \epsilon = \pm 1, \tag{5}$$

$$\tau^2(\xi) = -\epsilon[R - 2\mu\sigma(\xi) + \frac{\mu^2 - 1}{R}\sigma^2(\xi)], \quad R \neq 0, \tag{6}$$

where R, μ are constants and $' = \mathrm{d}/\mathrm{d}\xi$.

The parameter m can be found by balancing the highest-order derivative term and the nonlinear terms in (5) (m is usually a positive integer). If m is a fraction or a negative integer, we first make the transformation $u(x,t)=\varphi^m(x,t)$.

Step 2. Substituting (4)–(6) into (3), we can obtain a set of algebraic polynomials for $\tau^i(\xi)\sigma^j(\xi)(i=0,1; j=0,1,\ldots)$. Setting the coefficients of these terms $\tau^i(\xi)\sigma^j(\xi)$ to zero, we get a system of over-determined PDEs with respect to unknown functions $\{a_0,\, a_i,\, b_i(i=1,\ldots,m),\, \xi\}$.

Step 3. Solving the above system by use of symbolic computation system *Maple*, we would end up with the explicit expressions for μ, a_0, a_i, b_i $(i=1,\ldots,m)$ and ξ or the constraints among them.

In order that the equations derived in *Step* 2 can be solved easily, we may choose special forms of a_i, b_i and ξ on a trial-and-error basis. (As we do in Section 3.)

We know that Eqs. (5) and (6) have the following solutions:

Case 1. When $\epsilon=-1$,

$$\begin{cases} \sigma_1(\xi)=\dfrac{R\,\mathrm{sech}(\sqrt{R}\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\xi)+1}, & \tau_1(\xi)=\dfrac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\xi)+1},\\ \sigma_2(\xi)=\dfrac{R\,\mathrm{csch}(\sqrt{R}\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\xi)+1}, & \tau_2(\xi)=\dfrac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\xi)+1}. \end{cases} \tag{7}$$

Case 2. When $\epsilon=1$,

$$\begin{cases} \sigma_3(\xi)=\dfrac{R\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1}, & \tau_3(\xi)=\dfrac{\sqrt{R}\tan(\sqrt{R}\xi)}{\mu\ \sec(\sqrt{R}\xi)+1},\\ \sigma_4(\xi)=\dfrac{R\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1}, & \tau_4(\xi)=-\dfrac{\sqrt{R}\cot(\sqrt{R}\xi)}{\mu\ \csc(\sqrt{R}\xi)+1}. \end{cases} \tag{8}$$

Step 4. Thus according to (4), (7), (8) and the conclusions in *Step 3*, we can obtain many families of exact solutions for equation (3).

3 The stochastic soliton-like solutions of stochastic mKdV equations

In this section, we will give exact solutions of (1). Taking the Hermite transform of (1), we get the equation

$$\widetilde{U_t}(t,x,z)+f(t)\widetilde{U}^2(t,x,z)\widetilde{U_x}(t,x,z)-g(t)\widetilde{U}_{xxx}(x,t,z)=$$

$$\widetilde{W}(t,z)[\alpha\widetilde{U}^2(t,x,z)\widetilde{U}_x(t,x,z)-\beta\widetilde{U}_{xxx}(t,x,z)]. \tag{9}$$

Set $\mathit{\Gamma}(t,z)=(f(t)-\alpha\widetilde{W}(t,z))$ and $\mathit{\Omega}(t,z)=(g(t)-\beta\widetilde{W}(t,z))$. Equation (9) gives

$$\widetilde{U}_t(t,x,z)+\mathit{\Gamma}(t,z)\widetilde{U}^2(t,x,z)\widetilde{U}_x(t,x,z)-\mathit{\Omega}(t,z)\widetilde{U}_{xxx}(t,x,z)=0, \tag{10}$$

where the Hermite transformation of $W(t)$ is defined by $\widetilde{W}(t,z)=\sum_{k=1}^{\infty}\eta_k(t)z_k$ when $z=(z_1,z_2,\cdots,)\in(C^N)_c$ is parameter.

Firstly, we investigate Eq. (10). According to the extended method, we suppose that the solution of (10) has the form

$$\widetilde{U}(t,x,z) = a_0 + a_1\tau(\xi) + b_1\sigma(\xi), \tag{11}$$

where $a_0 = a_0(t,z)$, $a_1 = a_1(t,z)$, $b_1 = b_1(t,z)$, $\xi = x\lambda(t,z) + \eta(t,z)$, $\sigma(\xi)$ and $\tau(\xi)$ satisfy Eqs. (5) and (6).

Substituting (5), (6) and (11) into (10), collecting coefficients of monomials of $\tau(\xi)$, $\sigma(\xi)$ and x of the resulting system's numerator (Notice that $\Gamma(t,z)$, $\Omega(t,z)$, $\lambda(t,z)$, $\eta(t,z)$, $a_0(t,z)$, $a_1(t,z)$, and $b_1(t,z)$ are independent of x), then setting each coefficients to zero, we obtain the following over-determined PDEs system with respect to differentiable functions $\{\Gamma(t,z)$, $\Omega(t,z)$, $\lambda(t,z)$, $\eta(t,z)$, $a_0(t,z)$, $a_1(t,z)$, $b_1(t,z)\}$ under $\epsilon = -1$.

Note 1. In the following system, $\Gamma = \Gamma(t,z)$, $\Omega = \Omega(t,z)$, $\lambda = \lambda(t,z)$, $\eta = \eta(t,z)$, $a_0 = a_0(t,z)$, $a_1 = a_1(t,z)$, $b_1 = b_1(t,z)$, $a_{0t} = \partial a_0/\partial t$ and so on.

$$a_{0t}R^2 = 0, \tag{12}$$

$$-Rb_1\lambda(R\Gamma b_1^2 - 6\lambda^2\mu^2\Omega + 6\Omega\lambda^2 + 3\Gamma a_1^2\mu^2 - 3\Gamma a_1^2) = 0, \tag{13}$$

$$R^2 a_{1t} = 0, \tag{14}$$

$$R\lambda a_1(-3a_1^2\Gamma\mu + 3a_1^2\Gamma\mu^3 + 12\lambda^2\Omega\mu - 12\lambda^2\Omega\mu^3 + 4\Gamma a_0 b_1 - 4\Gamma a_0 b_1\mu^2 + 5R\Gamma b_1^2\mu) = 0, \tag{15}$$

$$-\lambda a_1(\mu-1)(\mu+1)(\Gamma a_1^2\mu^2 - 6\lambda^2\mu^2\Omega - \Gamma a_1^2 + 6\Omega\lambda^2 + 3R\Gamma b_1^2) = 0, \tag{16}$$

$$R\lambda(-7R\Omega b_1\mu\lambda^2 + 4R\Gamma a_1^2 b_1\mu - 2\Gamma a_0 a_1^2\mu^2 - 2R\Gamma a_0 b_1^2 + 2\Gamma a_0 a_1^2) = 0, \tag{17}$$

$$-a_1 R\lambda_t(\mu-1)(\mu+1) = 0, \tag{18}$$

$$\begin{aligned} -Ra_1(&4\Omega\lambda^3 R + \Gamma\lambda a_0^2\mu^2 - a_1^2\Gamma\lambda R - 6R\Gamma\lambda a_0 b_1\mu + 2R^2\Gamma\lambda b_1^2 \\ &- 7R\Omega\lambda^3\mu^2 - \eta_t + \eta_t\mu^2 + 3a_1^2 R\lambda\mu^2\Gamma - \Gamma\lambda a_0^2) = 0, \end{aligned} \tag{19}$$

$$-R^2 b_1\lambda_t = 0, \tag{20}$$

$$-R^2(R\Gamma\lambda a_1^2 b_1 + \Gamma\lambda a_0^2 b_1 - R\Omega\lambda^3 b_1 - 2\Gamma\lambda a_0 a_1^2\mu + b_1\eta_t) = 0, \tag{21}$$

$$R^2(\Gamma\lambda a_0^2 a_1\mu + b_{1t} + a_1\mu\eta_t - R\Omega\lambda^3 a_1\mu - 2R\Gamma\lambda a_0 a_1 b_1 + R\Gamma\lambda a_1^3\mu) = 0, \tag{22}$$

$$a_1\mu\lambda_t R^2 = 0. \tag{23}$$

With the aid of *Maple*, solving the above PDEs' system, we have the following solutions:

Case 1.

$$\begin{gathered} a_0 = b_1 = \mu = 0, \quad \lambda = F_2(z), \quad a_1 = F_1(z), \\ \Omega = \frac{\Gamma F_1(z)^2}{6F_2(z)^2}, \quad \eta = -\frac{1}{3}F_1(z)^2 F_2(z) R\int \Gamma \mathrm{d}t + F_3(z), \end{gathered} \tag{24}$$

where $\Gamma = \Gamma(t,z)$ is an arbitrary function of $\{t,z\}$, $F_1(z)$, $F_2(z)$ and $F_3(z)$ are all arbitrary functions of z.

Case 2.

$$\begin{aligned} a_0 = \mu = 0, \quad b_1 = \pm\frac{F_1(z)}{\sqrt{-R}}, \quad \lambda = F_2(z), \quad \Omega = \frac{2\Gamma F_1(z)^2}{3F_2(z)^2}, \\ a_1 = F_1(z), \quad \eta = -\frac{1}{3}F_1(z)^2F_2(z)R\int\Gamma\mathrm{d}t + F_3(z)\,, \end{aligned} \tag{25}$$

where $\Gamma = \Gamma(t,z)$ is an arbitrary function of $\{t,z\}$, $F_1(z)$, $F_2(z)$ and $F_3(z)$ are all arbitrary functions of z.

Case 3.

$$\begin{aligned} a_0 = b_1 = 0, \quad \lambda = F_2(z), \quad \Omega = \frac{2\Gamma F_1(z)^2}{3F_2(z)^2}, \mu = \pm 1, \\ a_1 = F_1(z), \quad \eta = -\frac{1}{3}F_1(z)^2F_2(z)R\int\Gamma\mathrm{d}t + F_3(z). \end{aligned} \tag{26}$$

where $\Gamma = \Gamma(t,z)$ is an arbitrary function of $\{t,z\}$, $F_1(z)$, $F_2(z)$ and $F_3(z)$ are all arbitrary functions of z.

Case 4.

$$\begin{aligned} & a_1 = 0, \quad a_0 = F_1(z), \quad \Omega = \frac{24}{49}\frac{\Gamma F_1(z)^2(\mu^2-1)}{\mu^2 R F_2(z)^2}, \\ & b_1 = -\frac{12}{7}\frac{F_1(z)(\mu^2-1)}{\mu R}, \quad \lambda = F_2(z), \\ & \eta = -\frac{1}{49}\frac{(25F_2(z)F_1(z)^2\int\Gamma dt\mu^2 + 24F_2(z)F_1(z)^2\int\Gamma dt - 49F_3(z)\mu^2)}{\mu^2}, \end{aligned} \tag{27}$$

where $\Gamma = \Gamma(t,z)$ is an arbitrary function of $\{t,z\}$, μ is an arbitrary constant, $F_1(z)$, $F_2(z)$ and $F_3(z)$ are all arbitrary functions of z.

Case 5.

$$\begin{aligned} a_0 = a_1 = \mu = 0, \quad \lambda = F_1(z), \quad b_1 = F_2(z), \\ \Omega = -\frac{1}{6}\frac{R\Gamma F_2(z)^2}{F_1(z)^2}, \quad \eta = -\frac{1}{6}R^2F_1(z)F_2(z)^2\int\Gamma dt + F_3(z), \end{aligned} \tag{28}$$

where $\Gamma = \Gamma(t,z)$ is an arbitrary function of $\{t,z\}$, $F_1(z)$, $F_2(z)$ and $F_3(z)$ are all arbitrary functions of z.

Thus from (7), (8), (11) and Cases 1-5, we obtain the following exact solutions of Eq. (10). For simplicity, we only list the sech-tanh-type solutions in the paper.

Family 1.

$$\widetilde{U}_1(t,x,z) = F_1(z)\sqrt{R}\tanh\left\{\sqrt{R}\left[F_2(z)x - \frac{1}{3}F_1(z)^2F_2(z)R\int\Gamma dt + F_3(z)\right]\right\}, \tag{29}$$

where $\Omega = \Gamma F_1(z)^2/6F_2(z)^2$.

Family 2.

$$\widetilde{U}_2(t,x,z) = F_1(z)\sqrt{R}\left[\mathrm{sech}\left[\sqrt{R}(F_2(z)x+\eta)\right] + \mathrm{i}\tanh[\sqrt{R}[F_2(z)x+\eta]\right]\Bigg], \tag{30}$$

where $\Omega = 2\Gamma F_1(z)^2/3F_2(z)^2$, $\eta = -\frac{1}{3}F_1(z)^2F_2(z)R\int \Gamma dt + F_3(z)$.

Family 3.

$$\widetilde{U}_3(t,x,z) = \frac{F_1(z)\sqrt{R}\tanh\left[\sqrt{R}(F_2(z)x+\eta))\right]}{\pm\mathrm{sech}[\sqrt{R}(F_2(z)x+\eta)]+1}, \tag{31}$$

where $\Omega = 2\Gamma F_1(z)^2/3F_2(z)^2$, $\eta = -\frac{1}{3}F_1(z)^2F_2(z)R\int \Gamma dt + F_3(z)$.

Family 4.

$$\widetilde{U}_4(t,x,z) = F_1(z) - \frac{12}{7}\frac{F_1(z)(\mu^2-1)}{\mu}\frac{\mathrm{sech}\left[\sqrt{R}(F_2(z)x+\eta))\right]}{\mu\,\mathrm{sech}\left[\sqrt{R}(F_2(z)x+\eta)\right]+1}, \tag{32}$$

where $\Omega = \frac{24}{49}\Gamma F_1(z)^2(\mu^2-1)/\mu^2RF_2(z)^2$ and

$$\eta = -\frac{1}{49}\frac{(25F_2(z)F_1(z)^2\int \Gamma dt\mu^2 + 24F_2(z)F_1(z)^2\int \Gamma dt - 49F_3(z)\mu^2)}{\mu^2}.$$

Family 5.

$$\widetilde{U}_5(t,x,z) = F_2(z)R\,\mathrm{sech}\left[\sqrt{R}\left(F_1(z)x - \frac{1}{6}R^2F_1(z)F_2(z)^2\int \Gamma dt + F_3(z)\right)\right], \tag{33}$$

where $\Omega = -\frac{1}{6}R\Gamma F_2(z)^2/F_1(z)^2$.

By use of (29)–(33), the definition of $\widetilde{W}$, Theorem 2.1 in Ref. [18] and $\exp^{\diamond}\{\mathrm{B(t)}\} = \exp\{\mathrm{B(t)} - \frac{1}{2}\mathrm{t}^2\}$ (see Lemma 2.6.16 in [19]), we have the following five families of stochastic solitary solutions:

Note 2. In the rest of the paper, $\Omega(t) = g(t) - \beta\dot{B}(t)$ and $\Gamma(t) = f(t) - \alpha\dot{B}(t)$.

Family 1.

$$U_1(t,x) = F_1\sqrt{R}\tanh\left\{\sqrt{R}\left[F_2x - \frac{1}{3}F_1^2F_2R\left(\int f(t)dt - \alpha(B(t) - \frac{1}{2}t^2\right) + F_3\right]\right\}, \tag{34}$$

where $\Omega(t) = \Gamma(t)F_1^2/6F_2^2$; F_1, F_2 and F_3 are all arbitrary constants.

Family 2.

$$U_2(t,x) = F_1\sqrt{R}[\mathrm{sech}[\sqrt{R}(F_2x+\delta(t))] + \mathrm{i}\tanh[\sqrt{R}[F_2x+\delta(t)]]], \tag{35}$$

where $\Omega(t) = 2\Gamma(t)F_1^2/3F_2^2$, $\delta(t) = -\frac{1}{3}F_1^2F_2R[\int f(t)dt - \alpha(B(t) - \frac{1}{2}t^2)] + F_3$; F_1, F_2 and F_3 are all arbitrary constants.

Family 3.

$$U_3(t,x) = \frac{F_1\sqrt{R}\tanh\left[\sqrt{R}(F_2x+\delta(t))\right]}{\pm\mathrm{sech}\left[\sqrt{R}\,(F_2x+\delta(t))\right]+1} \tag{36}$$

where $\Omega(t) = 2\Gamma(t)F_1^2/3F_2^2$, $\delta(t) = -\frac{1}{3}F_1^2F_2R[\int f(t)dt - \alpha(B(t) - \frac{1}{2}t^2)] + F_3$; F_1, F_2 and F_3 are all arbitrary constants.

Family 4.

$$U_4(t,x) = F_1 - \frac{12}{7}\frac{F_1(\mu^2-1)}{\mu}\frac{\operatorname{sech}\left[\sqrt{R}(F_2x+\delta(t))\right]}{\mu\operatorname{sech}\left[\sqrt{R}(F_2x+\delta(t))\right]+1}\,, \tag{37}$$

where $\Omega(t) = \frac{24}{49}\Gamma(t)F_1^2(\mu^2-1)/\mu^2RF_2^2$ and

$$\delta(t) = -\frac{1}{49}\frac{(25F_2F_1^2\mu^2 + 24F_2F_1^2)[\int f(t)dt - \alpha(B(t) - \frac{1}{2}t^2)] - 49F_3\mu^2}{\mu^2};$$

F_1, F_2 and F_3 are all arbitrary constants.

Family 5.

$$U_5(t,x) = F_2R\operatorname{sech}\left\{\sqrt{R}\left[F_1x - \tfrac{1}{6}R^2F_1F_2^2\left[\int f(t)dt - \alpha\left(B(t) - \tfrac{1}{2}t^2\right)\right] + F_3\right]\right\}, \tag{38}$$

where $\Omega(t) = -\frac{1}{6}R\Gamma(t)F_2^2/F_1^2$; F_1, F_2 and F_3 are all arbitrary constants.

Owing to the arbitrariness of F_1, F_2, F_3, it is not difficult to verify that the solution (3.16) obtained in [18] can be reproduced by the solution (34) obtained by us. But, to our knowledge, the other solutions obtained were not reported before.

4 Summary and discussion

In summary, based on the extended projective Riccati equations method and symbolic computation, by means of the theory of stochastic partial differential equations in Wick version [17–21], we study stochastic mKdV equation (1). As a result, some new and more general formal solutions for Eq. (1) are obtained, which include the nontravelling wave and coefficient functions stochastic soliton-like solutions, singular stochastic soliton-like solutions, and stochastic triangular functions solutions.

The work was supported by China Postdoctoral Science Foundation, Nature Science Foundation of Zhejiang Province of China (Y604056), and the National Outstanding Youth Foundation of China (No. 19925522).

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