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Soliton-like solutions and periodic form solutions for two variable-coefficient evolution equations using symbolic computation

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Summary. Some variable-coefficient generalizations of some nonlinear evolution equations (NLEEs) bear more realistic physical importance. By means of a generalized Riccati equation expansion (GREE) method and a symbolic computation system – Maple – we investigate the variable-coefficient Fisher-type equation and the nearly concentric KdV equation. As a result, rich families of exact analytic solutions for these two equations, including the non-travelling wave's and coefficient functions' soliton-like solutions, singular soliton-like solutions, and periodic form solutions, are obtained.

1 Introduction

Some variable-coefficient (VC) generalizations of some nonlinear evolution equations (NLEEs) bear more realistic physical importance. In recent years, a great deal of research work has been invested into the VC-NLEEs. For example, the VC-Fisher-type equation [1], the nearly concentric KdV equation (ncKdV) [2], [3], the VC-KP equation [4], the VC-KdV equation and VC-Burgers equation [5]–[8], the coupled VC-KdV equation [9], the VC-reaction-duffing equation [10], etc. The studies conducted in [1]–[10] revealed that the VC-NLEEs are able to model various physical situations more realistically than their constant-coefficient counterparts. However, solving variable-coefficients NLEEs is much more difficult than solving the constant coefficient NLEEs because of the existence of their coefficient function.

Recently, in [11] we presented a generalized Riccati equation expansion (GREE) method for constructing soliton-like solutions and periodic form solutions for some NLEEs, based on the various known tanh method and its various extended forms [12]–[32]. Because of introducing a more general ansatz than the ansatz in the above methods and with the help of a symbolic computation system – Maple – we can not only recover the exact solutions from the above mentioned methods but can also obtain new forms of solutions for given NLEEs.

In this paper, based on the computerized symbolic computation, we extend the power of the GREE method to two variable-coefficient NLEEs: the variable-coefficient Fisher-type equation [1] and the nearly concentric KdV equation [2], [3]. As a result, rich new families of exact solutions for these two equations, including the non-travelling wave's and coefficient functions' soliton-like solutions, singular soliton-like solutions, and periodic form solutions, are obtained. The solutions for the nearly concentric KdV equation obtained in [2] can be recovered by a set of solutions obtained in this paper.

The rest of the paper is organized as follows: in Sect. 2, we briefly describe the GREE method. In Sect. 3, we investigate the soliton-like solutions and periodic form solutions for the variable-coefficient Fisher-type equation and the nearly concentric KdV equation. Conclusions will be presented in the final Sect. 4.

2 Generalized Riccati equation expansion method

In the following, we describe the generalized Riccati equation expansion method for a given nonlinear evolution equation with a physical field $u(x, y, t)$ in three variables x, y, t,

$$H(u, u_t, u_x, u_y, u_{xx}, u_{xt}, u_{xy}, u_{yt}, \ldots,) = 0. \tag{1}$$

Step 1. We assume that Eq. (1) has solutions in the form

$$u(x, y, t) = a_0 + \sum_{i=1}^{m} [a_i \phi^i(\xi) + b_i \phi^{i-1}(\xi)\sqrt{R + \phi^2(\xi)} + g_i \phi^{-i}(\xi)], \tag{2}$$

where m is an integer to be determined by balancing the highest order derivative terms with the nonlinear terms in Eq. (1), R is a real constant and $a_0 = a_0(x, y, t)$, $a_i = a_i(x, y, t)$, $b_i = b_i(x, y, t)$, $g_i = g_i(x, y, t)$ $(i = 1, \ldots, m)$, $\xi = \xi(x, y, t)$ are all differentiable functions, and the new variable $\phi(\xi)$ satisfies

$$\frac{d\phi(\xi)}{d\xi} = R + \phi^2(\xi). \tag{3}$$

Step 2. Substituting Eq. (2) along with Eq. (3) into Eq. (1), multiplying with the most common denominator in the obtained system, and setting the coefficients of $\phi^j(\xi)(\sqrt{R + \phi^2(\xi)})^n$ $(j = 0, 1, \ldots; n = 0, 1)$ to zero, we obtain a set of over-determined partial differential equations with regard to arbitrary functions a_0, a_i, b_i, g_i $(i = 1, \ldots, m)$ and ξ.

Step 3. Solving the set of over-determined partial differential equations by use of the PDEtools package of *Maple*, we end up with the explicit expressions for a_0, a_i, b_i, g_i $(i = 1, \ldots, m)$ and ξ or the constraints among them.

Step 4. It is well known that the general solutions of the Riccati equation (3) are

$$\phi(\xi) = \begin{cases} -\sqrt{-R}\tanh(\sqrt{-R}\xi), & R < 0, \\ -\sqrt{-R}\coth(\sqrt{-R}\xi), & R < 0, \\ \sqrt{R}\tan(\sqrt{R}\xi), & R > 0, \\ -\sqrt{R}\cot(\sqrt{R}\xi), & R > 0, \\ -\frac{1}{\xi}, & R = 0. \end{cases} \tag{4}$$

Thus according to Eqs. (2) and (4) and the conclusions in *Step 3*, some soliton-like solutions, periodic form solutions and rational solutions of Eq. (1) can be obtained.

Remark 1: The GREE method is more general than the tanh method [12]–[15], extended tanh-function method [16]–[18], modified extended tanh-function method [19], and the generalized hyperbolic-function method [22]–[23]. If we set the parameters in Eq. (2) to different values, the above methods can be recovered by the GREE method. The concrete cases are as follows:

(1) Setting $a_i = \text{constant}$ $(i = 0, \ldots, m)$, $b_i = g_i = 0$ $(i = 1, \ldots, m)$, and taking ξ as a linear function of x, y, t, the extended tanh-function method can be recovered [16]–[18];

(2) If $g_i \neq 0$ ($b_i \neq 0$) in the former case, the GREE method is the same as the method in [19] and ([25]–[27]), respectively;
(3) Setting $b_i = g_i = 0$, the GREE method becomes the method in [33];
(4) Setting $g_i = 0$ and $\phi(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi)$, i.e., the first solution of Eq.(4), the generalized hyperbolic-function method can be obtained [22], [23];
(5) We call the solution (2) solitary wave solutions if they contain the variable $\xi(x, y, t)$ that is a linear form of $\{x, y, t\}$ and $a_i = \text{const.}$, $b_i = \text{const.}$, $g_i = 0$. Otherwise, we call them soliton-like solutions, or singular soliton-like solutions, or periodic forms solutions (e.g., see [3], [4], [20]–[23] for detail).

Remark 2: In general, it is very difficult, sometimes impossible, to solve the set of over-determined partial differential equations in *Step 2*. As the calculation goes on, in order to drastically simplify the work or to make the work feasible, we often choose special function forms for a_0, a_i, b_i, g_i ($i = 1, \ldots, m$) and ξ, on a trial-and-error basis (as we do in Sect. 3).

3 Applications of the GREE method

Example 1. Consider the variable-coefficient Fisher-type equation [1]

$$u_t = u_{xx} + b(t)u - a(t)u^2, \tag{5}$$

where $a(t)$ and $b(t)$ are both differentiable functions, which models such spatial spread as of an advantageous gene in a population or of early farming. Equation (5) is an interesting case of the two-species Lotka-Volterra competing system [33]. In [1], Gao and Tian made use of computerized symbolic computation and an auto-Bäcklund transformation and obtained a couple of families of soliton-like solutions.

By balancing the highest-order contributions from both the linear and nonlinear terms in Eq. (5), we obtain $m = 2$. Then we assume that Eq. (5) has the following formal solutions:

$$u(x,t) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + b_1\sqrt{R + \phi^2(\xi)} + b_2\phi(\xi)\sqrt{R + \phi(\xi)^2} + \frac{g_1}{\phi(\xi)} + \frac{g_2}{\phi^2(\xi)}, \tag{6}$$

where $a_0 = a_0(t)$, $a_1 = a_1(t)$, $a_2 = a_2(t)$, $b_1 = b_1(t)$, $b_2 = b_2(t)$ $g_1 = g_1(t)$, $g_2 = g_2(t)$, $p = p(t)$, $q = q(t)$, $\xi = xp + q$ are all differentiable functions and $\phi(\xi)$ satisfies Eq. (3).

Substituting Eq. (6) along with Eq. (3) into Eq. (5), collecting coefficients of monomials of $\phi(\xi)$, $\sqrt{R + \phi(\xi)^2}$ and x (notice that a_0, a_1, a_2, b_1, b_2, g_1, g_2, p and q are independent from x), then setting each coefficient to zero, we can obtain the following set of over-determined ordinary differential equations with respect to the unknown functions $a, b, a_0, a_1, a_2, b_1, b_2, g_1, g_2, p$ and q:

$$\begin{cases}
-g_2(6p^2R^2 - g_2a) = 0, \\
2a(b_1g_1 + g_2b_2) = 0, \\
-2b_2(-aa_2 + 3p^2) = 0, \\
aa_2^2 + ab_2^2 - 6p^2a_2 = 0, cr2aa_0g_1 + g_{1t} - 2p^2g_1R - 2g_2q_t - bg_1 + 2aa_1g_2 = 0, \\
-bg_2 - g_1Rq_t + ag_1^2 + 2aa_0g_2 + g_{2t} - 8p^2g_2R = 0, \\
ab_1^2 + a_{2t} - 8p^2a_2R + ab_2^2R - ba_2 + 2aa_0a_2 + aa_1^2 + a_1q_t = 0, \\
-ba_1 + 2aa_2g_1 + 2ab_1b_2R + a_{1t} + 2a_2Rq_t - 2p^2a_1R + 2aa_0a_1 = 0, \\
-5p^2b_2R + 2aa_0b_2 + 2aa_1b_1 + b_{2t} - bb_2 + b_1q_t = 0, \\
-2g_2p_t = 0, \\
2a_2Rp_t = 0, \\
2b_2p_t = 0, \\
aa_0^2 + a_1Rq_t + 2aa_2g_2 + a_{0t} - 2p^2g_2 - g_1q_t - ba_0 - 2p^2a_2R^2 + 2aa_1g_1 + ab_1^2R = 0,
\end{cases}$$

$$\begin{cases} b_2p_tR = 0, \\ b_2q_tR - bb_1 - p^2b_1R + 2aa_0b_1 + 2ab_2g_1 + b_{1t} = 0, \\ p_t(a_1R - g_1) = 0, \\ -2p^2b_1 + 2aa_2b_1 + 2b_2q_t + 2aa_1b_2 = 0, \\ b_1p_t = 0, \\ 2ab_1b_2 + 2aa_1a_2 + 2a_2q_t - 2p^2a_1 = 0, \\ 2ab_1g_2 = 0, \\ 2a_2p_t = 0, \\ -2g_2Rp_t = 0, \\ a_1p_t = 0, \\ -g_1Rp_t = 0, \\ 2ag_1g_2 - 2g_2Rq_t - 2p^2g_1R^2 = 0. \end{cases} \tag{7}$$

Using the powerful PDEtools package of Maple, solving the set (7), we obtain the following results (*Note:* (1). The cases where $p = 0$ or p, q are all arbitrary constants, are omitted. (2). In the rest of the paper, C_i denotes an arbitrary constant and the expression $a_2 = a_2$ denotes a_2 as an arbitrary function of t, etc.):

Case 1

$$\begin{aligned} b_1 =& b_2 = g_1 = g_2 = 0, \quad p = C_1, \quad a_1 = \pm\frac{2\sqrt{R}e^{4RC_1^2t}a_2}{\theta_1}, \\ q =& \mp\int\frac{10C_1^2\sqrt{R}e^{4RC_1^2t}}{\theta_1}dt + C_4, \quad a = \frac{6C_1^2}{a_2}, \quad a_2 = a_2, \\ a_0 =& \frac{a_2R\left\{Re^{-2RC_1^2t}C_2\theta_1 + e^{6RC_1^2t}\theta_1 + C_6[e^{2RC_1^2t}RC_2 - e^{10RC_1^2t} + 4e^{-6RC_1^2t}C_2^2R^2]\right\}}{(C_6\theta_1e^{-6RC_1^2t}C_2R - C_6\theta_1e^{2RC_1^2t} + 3e^{-2RC_1^2t}C_2R - e^{6RC_1^2t})\theta_1}, \\ b =& 16\Big[\Big[((\frac{1}{16}a_{2t} - \frac{3}{2}Ra_2C_1^2)e^{8RC_1^2t} - \frac{1}{4}C_2R(-20Ra_2C_1^2 + a_{2t}))e^{6RC_1^2t} \\ &+ \frac{3}{4}C_2R((2Ra_2C_1^2 - \frac{1}{4}a_{2t})e^{8RC_1^2t} + C_2R(a_{2t} - 4Ra_2C_1^2))e^{-2RC_1^2t}\Big]\theta_1 \\ &+ C_6(-\frac{1}{4}(e^{-6RC_1^2t}RC_2 - e^{2RC_1^2t})((-\frac{1}{4}a_{2t} + 3Ra_2C_1^2)e^{8RC_1^2t} \\ &+ C_2R(-8Ra_2C_1^2 + a_{2t}))e^{(8RC_1^2t)} + R(C_2^2R(-\frac{1}{4}a_{2t}e^{8RC_1^2t} \\ &+ C_2R(4Ra_2C_1^2 + a_{2t}))e^{-6RC_1^2t} - ((\frac{15}{4}Ra_2C_1^2 - \frac{1}{4}a_{2t})e^{8RC_1^2t} \\ &+ C_2R(-11Ra_2C_1^2 + a_{2t}))C_2e^{2RC_1^2t} - 3(-\frac{1}{4}e^{8RC_1^2t} + C_2R)e^{10RC_1^2t}a_2C_1^2))\Big] \\ &/\Big[\theta_1a_2(C_6\theta_1e^{-6RC_1^2t}C_2R - C_6\theta_1e^{2RC_1^2t} + 3e^{-2RC_1^2t}C_2R - e^{6RC_1^2t})\theta_1^2\Big], \end{aligned} \tag{8}$$

where $\theta_1 = \sqrt{-e^{8RC_1^2t} + 4C_2R}$.

Case 2

$$g_1 = g_2 = 0, \quad a = \frac{3C_1^2}{a_2}, \quad a_1 = \mp b_1 = \pm\frac{\sqrt{R}e^{(RC_1^2t)}a_2}{\theta_2}, \quad p = C_1,$$

$$b_2 = \mp a_2, \quad a_2 = a_2, \quad q = \mp\int\frac{5C_1^2\sqrt{R}e^{(RC_1^2t)}}{\theta_2}dt + C_7,$$

$$b = \frac{1}{\theta_2}\Big[(-3C_3R((-2Ra_2C_1^2 + a_{2t})e^{2RC_1^2t} + C_3R(Ra_2C_1^2 - a_{2t}))e^{-2RC_1^2t} + (-24Ra_2C_1^2$$

$$+ 4a_{2t})e^{2RC_1^2t} + 20C_3R(Ra_2C_1^2 - \frac{1}{5}a_{2t}))\theta_2 + C_{11}(5((-\frac{6}{5}Ra_2C_1^2 + \frac{1}{5}a_{2t})(e^{(RC_1^2t)})^2$$

$$+ C_3R(Ra_2C_1^2 - \frac{1}{5}a_{2t}))(e^{(-3RC_1^2t)}RC_3 - 4e^{(-RC_1^2t)})e^{2RC_1^2t} + C_3R(((-12Ra_2C_1^2$$

$$+ 4a_{2t})e^{2RC_1^2t} + 8C_3R(Ra_2C_1^2 - \frac{1}{2}a_{2t}))e^{(-RC_1^2t)} + e^{(-3RC_1^2t)}C_3R(-a_{2t}e^{2RC_1^2t}$$

$$+ C_3R(Ra_2C_1^2 + a_{2t}))))\Big] / \Big[[C_{11}(e^{(-3RC_1^2t)}RC_3 - 4e^{(-RC_1^2t)})\theta_2 + 3e^{-2RC_1^2t}C_3R - 4]$$

$$\times a_2(-e^{2RC_1^2t} + C_3R)\Big],$$

$$a_0 = \frac{a_2R^2C_32e^{-2RC_1^2t}\theta_2 - 2C_{11}e^{(-RC_1^2t)} + C_{11}e^{(-3RC_1^2t)}C_3R}{(C_{11}\theta_2e^{(-3RC_1^2t)}C_3R - 4C_{11}\theta_2e^{(-RC_1^2t)} + 3e^{(-2RC_1^2t)}C_3R - 4)\theta_2}, \tag{9}$$

where $\theta_2 = \sqrt{-e^{2RC_1^2t} + C_3R}$.

Case 3

$$b_1 = b_2 = 0, \quad q = \pm\int \frac{20C_1^2e^{16RC_1^2t}R}{\theta_3}dt + C_4, \quad a = \frac{6C_1^2R^2}{g_2},$$

$$a_2 = \frac{g_2}{R^2}, \quad g_1 = -a_1R = \pm\frac{4e^{16RC_1^2t}g_2}{\theta_3}, \quad p = C_1, \quad g_2 = g_2,$$

$$a_0 = -2g_2\Big[-3Re^{36RC_1^2t}\theta_3 + 4e^{4RC_1^2t}C_2\theta_3 - 28C_6e^{20RC_1^2t}RC_2 + 3C_6R^2e^{52RC_1^2t}$$

$$- 64C_6e^{-12RC_1^2t}C_2^2\Big] \Big/ \Big[R(4C_6\theta_3e^{-12RC_1^2t}C_2 - C_6\theta_3e^{20RC_1^2t}R - e^{36RC_1^2t}R + 12e^{4RC_1^2t}C_2)\theta_3\Big],$$

$$b = 24\Big[(-4R(R(g_2C_1^2R - \frac{1}{96}g_{2t})e^{32RC_1^2t} - \frac{40}{3}C_2(-\frac{1}{8}0g_{2t} + g_2C_1^2R))e^{36RC_1^2t}$$

$$+ 16C_2((g_2C_1^2R - \frac{1}{32}g_{2t})Re^{32RC_1^2t} - 8(-\frac{1}{16}g_{2t} + g_2C_1^2R)C_2)e^{4RC_1^2t})\theta_3$$

$$+ (R(-4e^{-12RC_1^2t}C_2 + e^{20RC_1^2t}R)(R(g_2C_1^2R - \frac{1}{2}4g_{2t})e^{32RC_1^2t} - \frac{16}{3}C_2(g_2C_1^2R$$

$$- \frac{1}{8}g_{2t}))e^{(32RC_1^2t)} - \frac{8}{3}C_2^2(g_{2t}Re^{32RC_1^2t} - 256C_2(g_2C_1^2R + \frac{1}{16}g_{2t}))e^{-12RC_1^2t}$$

$$+ 3R(-\frac{44}{3}C_2(R(g_2C_1^2R - \frac{1}{66}g_{2t})e^{32RC_1^2t} - \frac{400}{33}C_2(g_2C_1^2R - \frac{1}{5}0g_{2t}))e^{20RC_1^2t}$$

$$+ g_2C_1^2R^2e^{52RC_1^2t}(R(e^{(RC_1^2t)})^32 - 16C_2)))C_6\Big] / \Big[\theta_3(Re^{32RC_1^2t} - 16C_2)$$

$$\times g_2(C_6(-4e^{-12RC_1^2t}C_2 + e^{20RC_1^2t}R)\theta_3 + e^{36RC_1^2t}R - 12e^{4RC_1^2t}C_2)\Big], \tag{10}$$

where $\theta_3 = \sqrt{-Re^{32RC_1^2t} + 16C_2}$.

From Eqs. (4) and (6) and Cases 1–3, we obtain three new families of exact analytic solutions for Eq. (5):

Family 1

$$\begin{cases} u_{11} = a_0 - a_1\sqrt{-R}\tanh(\sqrt{-R}\xi) - a_2 R\tanh^2(\sqrt{-R}\xi), & R < 0, \\ u_{12} = a_0 - a_1\sqrt{-R}\coth(\sqrt{-R}\xi) - a_2 R\coth^2(\sqrt{-R}\xi), & R < 0, \\ u_{13} = a_0 + a_1\sqrt{R}\tan(\sqrt{R}\xi) + a_2 R\tan^2(\sqrt{R}\xi), & R < 0, \\ u_{14} = a_0 - a_1\sqrt{R}\cot(\sqrt{R}\xi) + a_2 R\cot^2(\sqrt{R}\xi), & R < 0, R > 0, \end{cases} \tag{11}$$

where $\xi = xp + q$ and a_0, a_2, p, q satisfy the constraints in Eq. (8).

Family 2

$$\begin{cases} u_{21} = a_0 - a_1\sqrt{-R}[\tanh(-\sqrt{-R}\xi) + \mathrm{i}\,\mathrm{sech}(-\sqrt{-R}\xi)] \\ \qquad -R[a_2\tanh^2(\sqrt{-R}\xi) + \mathrm{i}b_2\mathrm{sech}(\sqrt{-R}\xi)\tanh(\sqrt{-R}\xi)], & R < 0, \\ u_{22} = a_0 - a_1\sqrt{-R}[\coth(-\sqrt{-R}\xi) + \mathrm{csch}(-\sqrt{-R}\xi)] \\ \qquad -R(a_2\coth^2(\sqrt{-R}\xi) + b_2\mathrm{csch}(\sqrt{-R}\xi)\coth(\sqrt{-R}\xi)), & R < 0, \\ u_{23} = a_0 + a_1\sqrt{R}\tan(\sqrt{R}\xi) + b_1\sqrt{R}\sec(\sqrt{R}\xi) + \\ \qquad R(a_2\tan^2(\sqrt{R}\xi) + b_2\sec(\sqrt{R}\xi)\tan(\sqrt{R}\xi)), & R > 0, \\ u_{24} = a_0 - a_1\sqrt{R}\tan(\sqrt{R}\xi) + b_1\sqrt{R}\sec(\sqrt{R}\xi) + \\ \qquad R(a_2\cot^2(\sqrt{R}\xi) + b_2\cot(\sqrt{R}\xi)\csc(\sqrt{R}\xi)), & R > 0, \end{cases} \tag{12}$$

where $\xi = xp + q$ and p, q, a_0, a_2, b_2 satisfy the constraints in Eq. (9).

Family 3

$$\begin{cases} u_{31} = a_0 - a_1\sqrt{-R}[\tanh(\sqrt{-R}\xi) + \coth(\sqrt{-R}\xi)] - \\ \qquad a_2 R[\tanh^2(\sqrt{-R}\xi) + \coth^2(\sqrt{-R}\xi)], & R < 0, \\ u_{31} = a_0 + a_1\sqrt{R}[\tan(\sqrt{R}\xi) + \cot(\sqrt{R}\xi)] - \\ \qquad a_2 R[\tan^2(\sqrt{R}\xi) + \cot^2(\sqrt{R}\xi)], & R > 0, \end{cases} \tag{13}$$

where $\xi = xp + q$ and a_0, a_1, p, q satisfy the constraints in Eq. (10).

Example 2. Consider the nearly concentric KdV equation, or the ncKdV [2], [3], [34]

$$(2H_t + \frac{H}{t} + 3HH_x + \frac{H_{xxx}}{3})_x + \frac{H_{yy}}{t^2} = 0, \tag{14}$$

which arises in cylindrical shallow-water waves subject to certain kinds of small transversal disturbances [3], [34]. The relation between Eq. (14) and the Kadomstev-Petviashvili equation has been discussed in [34]. For Eq. (14), a class of the solitonic solutions has been found via the generalized tanh method [2]. In [3], Gao and Tian have also found an auto-Bäcklund transformation and some exact solitonic solutions.

The balancing procedure yields $m = 2$. Therefore, we may choose

$$H = (A_0 + A_1 x + A_2 x^2) + a_1\phi(\xi) + a_2\phi^2(\xi) + [b_1 + b_2\phi(\xi)]\sqrt{R + \phi^2(\xi)} + \frac{g_1}{\phi(\xi)} + \frac{g_2}{\phi^2(\xi)}, \tag{15}$$

where $A_0 = A_0(y,t)$, $A_1 = A_1(y,t)$, $A_2 = A_2(y,t)$, $a_1 = a_1(y,t)$, $a_2 = a_2(y,t)$, $b_1 = b_1(y,t)$, $b_2 = b_2(y,t)$, $g_1 = g_1(y,t)$, $g_2 = g_2(y,t)$, $\xi = x[p_0(t) + yp_1(t)] + q(y,t)$ are all differentiable functions and $\phi(\xi)$ satisfies Eq. (4).

Substituting Eq. (15) along with Eq. (3) into Eq. (14), multiplying $\phi(\xi)^6\sqrt{R+\phi(\xi)^2}$ in the obtained system, collecting coefficients of monomials of $\phi(\xi)$, $\sqrt{R+\phi(\xi)^2}$ and x with the aid of Maple, then setting all coefficients to zero, we can deduce a set of over-determined partial differential equations with respect to the unknown functions $A_0, A_1, A_2, a_1, a_2, b_1, b_2, g_1, g_2, p_0$, p_1 and q. Because the set includes 55 partial differential equations, we do not list them in the paper for simplification.

Solving the set of partial differential equations, we obtain the following results:

Case I

$$
\begin{aligned}
a_1 &= b_1 == b_2 = g_1 = g_2 = 0,\\
a_2 &= -\frac{8}{3}p_0p_1y - \frac{4}{3}p_1^2y^2 - \frac{4}{3}p_0^2, \quad A_2 = -\frac{p_1^2}{3t^2(p_1^2y^2+2p_0p_1y+p_0^2)},\\
q &= -\frac{1}{6}ty^3(2p_1't+p_1) + \frac{1}{2}(-tp_0-2t^2p_0')y^2 + F_1y + F_2,\\
A_1 &= -\frac{(-2p_1ytp_0 - 2t^2p_0'p_1y + 2p_1F_1 - y^2tp_1^2 + 2t^2p_0'p_0 + 2t^2p_1'yp_0)}{3t^2(p_1^2y^2+2p_0p_1y+p_0^2)},\\
A_0 &= -\frac{1}{36t^2(p_1^2y^2+2p_0p_1y+p_0^2)}\Big[32Ry^4t^2p_1^4 + 128p_1yRp_0^3t^2 + 192p_1^2y^2Rp_0^2t^2\\
&\quad + 128Ry^3t^2p_0p_1^3 + 12t^4y^4{p_1'}^2 - p_1^2t^2y^4 + 24t^2p_0F_2' + 48y^2t^4{p_0'}^2 + 32Rp_0^4t^2\\
&\quad - 8p_1t^3y^4p_1' - 36p_1t^3y^3p_0' - 4p_1t^2y^3p_0 + 24p_1t^2y^2F_1' + 24p_1t^2yF_2' + 48t^4y^3p_1'p_0'\\
&\quad - 12y^2t^3p_0p_0' + 4t^3y^3p_1'p_0 - 24t^2y^2p_1'F_1 - 48yt^2p_0'F_1 - 12ty^2p_1F_1 - 24ytp_0F_1\\
&\quad + 24t^2p_0F_1'y + 12F_1^2 - 8p_1t^4y^4p_1'' - 24p_1t^4y^3p_0'' - 24t^4p_0y^2p_0'' - 8t^4p_0y^3p_1''\Big],
\end{aligned}
\tag{16}
$$

where $F_1 = F_1(t), F_2 = F_2(t), p_0 = p_0(t), p_1 = p_1(t)$ are all arbitrary functions and $F_1' = F_1'(t)$, etc.

Case II

$$
\begin{aligned}
a_1 &= b_1 = b_2 = g_1 = 0, \quad g_2 = a_2R^2 = -\frac{4}{3}R^2p_1^2y^2 - \frac{8}{3}R^2p_0p_1y - \frac{4}{3}R^2p_0^2,\\
q &= -\frac{1}{6}ty^3(2p_1't+p1) + \frac{1}{2}(-tp_0-2t^2p_0')y^2 + F_3y + F_4,\\
A_2 &= -\frac{p_1^2}{3t^2(p_1^2y^2+2p_0p_1y+p_0^2)},\\
A_1 &= -\frac{1}{3}\frac{(-2p_1ytp_0 - 2t^2p_0'p_1y + 2p_1F_3 - y^2tp_1^2 + 2t^2p_0'p_0 + 2t^2p_1'yp_0)}{t^2(p_1^2y^2+2p_0p_1y+p_0^2)},\\
A_0 &= -\frac{1}{36t^2(p_1^2y^2+2p_0p_1y+p_0^2)}(32Ry^4t^2p_1^4 + 128p_1yRp_0^3t^2 + 192p_1^2y^2Rp_0^2t^2\\
&\quad + 128Ry^3t^2p_0p_1^3 + 12t^4y^4{p_1'}^2 - p_1^2t^2y^4 + 48y^2t^4{p_0'}^2 + 32Rp_0^4t^2 - 8p_1t^3y^4p_1'\\
&\quad - 36p_1t^3y^3p_0' - 4p_1t^2y^3p_0 + 48t^4y^3p_1'p_0' - 12y^2t^3p_0p_0' + 4t^3y^3p_1'p_0 + 12F_3^2\\
&\quad + 24t^2p_1yF_4' + 24t^2p_1y^2F_3' + 24t^2p_0F_4' - 24t^2y^2p_1'F_3 + 24t^2p_0F_3'y - 12ty^2p_1F_3\\
&\quad - 24ytp_0F_3 - 48yt^2p_0'F_3 - 8p_1t^4y^4p_1'' - 24p_1t^4y^3p_0'' - 24t^4p_0y^2p_0'' - 8t^4p_0y^3p_1''),
\end{aligned}
\tag{17}
$$

where $F_3 = F_3(t), F_4 = F_4(t), p_0 = p_0(t), p_1 = p_1(t)$ are all arbitrary functions and $F_3' = F_3'(t)$, etc.

Case III

$$a_1 = b_1 = g_1 = g_2 = 0, \quad a_2 = -\frac{4}{3}p_0p_1y - \frac{2}{3}p_1^2y^2 - \frac{2}{3}p_0^2,$$

$$b_2 = \pm\left(\frac{4}{3}p_0p_1y + \frac{2}{3}p_1^2y^2 + \frac{2}{3}p_0^2\right), \quad A_2 = -\frac{p_1^2}{3t^2(p_1^2y^2 + 2p_0p_1y + p_0^2)},$$

$$q = -\frac{1}{6}ty^3(2p_1't + p_1) + \frac{1}{2}(-tp_0 - 2t^2p_0')y^2 + F_1y + F_2,$$

$$A_1 = -\frac{(-2p_1ytp_0 - 2t^2p_0'p_1y + 2p_1F_1 - y^2tp_1^2 + 2t^2p_0'p_0 + 2t^2p_1'yp_0)}{3t^2(p_1^2y^2 + 2p_0p_1y + p_0^2)},$$

$$\begin{aligned} A_0 = &-\frac{1}{36t^2(p_1^2y^2 + 2p_0p_1y + p_0^2)}(20Ry^4t^2p_1^4 + 80p_1yRp_0^3t^2 + 120p_1^2y^2Rp_0^2t^2 \\ &+ 80Ry^3t^2p_0p_1^3 + 12t^4y^4{p_1'}^2 - p_1^2t^2y^4 + 24t^2p_0F_2' + 48y^2t^4{p_0'}^2 + 20Rp_0^4t^2 \\ &- 8p_1t^3y^4p_1' - 36p_1t^3y^3p_0' - 4p_1t^2y^3p_0 + 24p_1t^2y^2F_1' + 24p_1t^2yF_2' + 48t^4y^3p_1'p_0' \\ &- 12y^2t^3p_0p_0' + 4t^3y^3p_1'p_0 - 24t^2y^2p_1'F_1 - 48yt^2p_0'F_1 - 12ty^2p_1F_1 - 24ytp_0F_1 \\ &+ 24t^2p_0F_1'y + 12F_1^2 - 8p_1t^4y^4p_1'' - 24p_1t^4y^3p_0'' - 24t^4p_0y^2p_0'' - 8t^4p_0y^3p_1''), \end{aligned} \tag{18}$$

where $F_1 = F_1(t), F_2 = F_2(t), p_0 = p_0(t), p_1 = p_1(t)$ are all arbitrary functions and $F_1' = F_1'(t)$, etc.

From Eqs. (4) and (15) and Cases I–III, we can obtain the following three families of solutions for the ncKdV equation (here the cases of $R = 0$ are omitted):

Family I

$$\begin{cases} u_{11} = a_0 - a_2R\tanh^2(\sqrt{-R}\xi), & R < 0, \\ u_{12} = a_0 - a_2R\coth^2(\sqrt{-R}\xi), & R < 0, \\ u_{13} = a_0 + a_2R\tan^2(\sqrt{R}\xi), & R > 0, \\ u_{14} = a_0 + a_2R\cot^2(\sqrt{R}\xi), & R > 0, \end{cases} \tag{19}$$

where $\xi = x(p_0 + yp_1) + q$, $a_0 = A_0 + A_1x + A_2x^2$ and A_0, A_1, A_2, a_2, q are determined by Eq. (16) .

Family II

$$\begin{cases} u_{21} = a_0 - a_2R[\tanh^2(\sqrt{-R}\xi) + \coth^2(\sqrt{-R}\xi)], & R < 0, \\ u_{22} = a_0 - a_2R[\tan^2(\sqrt{R}\xi) + \cot^2(\sqrt{R}\xi)], & R > 0, \end{cases} \tag{20}$$

where $\xi = x(p_0 + yp_1) + q$, $a_0 = A_0 + A_1x + A_2x^2$ and A_0, A_1, A_2, a_2, q are determined by Eq. (17).

Family III

$$\begin{cases} u_{31} = a_0 - R[a_2\tanh^2(\sqrt{-R}\xi) + \mathrm{i}b_2\mathrm{sech}(\sqrt{-R}\xi)\tanh(\sqrt{-R}\xi)], & R < 0, \\ u_{32} = a_0 - R(a_2\coth^2(\sqrt{-R}\xi) + b_2\mathrm{csch}(\sqrt{-R}\xi)\coth(\sqrt{-R}\xi)), & R < 0, \\ u_{33} = a_0 + R(a_2\tan^2(\sqrt{R}\xi) + b_2\sec(\sqrt{R}\xi)\tan(\sqrt{R}\xi)), & R > 0, \\ u_{34} = a_0 + R(a_2\cot^2(\sqrt{R}\xi) + b_2\cot(\sqrt{R}\xi)\csc(\sqrt{R}\xi)), & R > 0, \end{cases} \tag{21}$$

where $\xi = x(p_0 + yp_1) + q$, $a_0 = A_0 + A_1 x + A_2 x^2$ and $A_0, A_1, A_2, a_2, b_2, q$ are determined by Eq. (18).

Remark 3

(1) It is necessary to point out that we only seek for some special solutions of the Fisher-type equation (5) and the ncKdV equation (14). We take ncKdV equations as an example to illustrate the process here. Firstly, we assume that the solution of Eq. (14) is as follows:

$$H = \sum_{i=0}^{n} A_i x^i + \sum_{i=1}^{m} [a_i \phi^i(\xi) + b_i \phi^{i-1}(\xi)\sqrt{R + \phi^2(\xi)} + g_i \phi^{-i}(\xi)], \quad \xi = xp(y,t) + q(y,t), \tag{22}$$

where $A_i = A_i(y,t)$, $a_i = a_i(y,t)$, $b_i = b_i(y,t)$, $g_i = g_i(y,t)$ and $\phi(\xi)$ satisfies Eq. (3).

Then substituting Eq. (22) along with Eq. (3) into Eq. (14), we obtain a selection $m = n = 2$. Therefore Eq. (22) is changed into

$$H = \sum_{i=0}^{2} A_i x^i + \sum_{i=1}^{2} [a_i \phi^i(\xi) + b_i \phi^{i-1}(\xi)\sqrt{R + \phi^2(\xi)} + g_i \phi^{-i}(\xi)]. \tag{23}$$

Substituting Eqs. (23) and (3) into Eq. (14), according to *Step 3* in Sect. 2, we can derive an over-determined PDEs' system which can not be solved by *Maple*. But we find that the term $p_{yy} = d^2p(y,t)/dy^2$ occurs many times in the over-determined system of PDEs. In order to further simplify computation, we set $p(y,t) = p_0(t) + yp_1(t)$, i.e., $p_{yy} = 0$. So Eq. (23) is changed into Eq. (15). Proceeding as before, we are lucky to obtain the results in the paper with *Maple*. Otherwise, we will have to take some more simple forms of A_i, a_i, b_i, g_i, p and q.

(2) It is easy to see that, when setting $R = -1$, $x = \xi$, $y = \psi$, $t = \tau$, $p_0 = \frac{\Omega_0(\tau)}{2}$, $p_1 = \frac{\Omega_1(\tau)}{2}$, $F_1 = \frac{\Gamma_1(\tau)}{2}$, $F_2 = \frac{\Gamma_2(\tau)}{2}$ in the solution u_{11} in Family I, i.e., u_{11} in Eq. (19), the solutions obtained in [3] can be recovered. But to our knowledge, the other solutions obtained here have not been found before.

(3) The results obtained do not rely on any boundary conditions in the horizontal directions, and thus might be more suitable for some physical phenomena. At the same time we find it hard to find out certain physically meaningful solutions for a mathematical boundary condition to be based on. This remains an open question for us.

(4) In order to understand the significance of these solutions expressed by Eqs. (19)–(21), we choose the first solution u_{11} of Eq. (19) to draw six figures to show its properties with some selected parameters indicated in the figures' captions. The values chosen for the figures are purely for the purpose of drawing the figures and for a qualitative analysis. In reality, the detailed application requires a careful choice of the parameters and functions in the solutions.

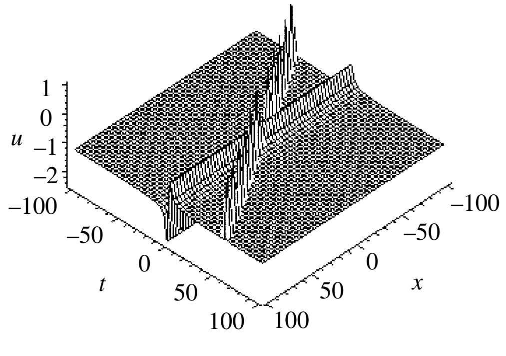

Fig. 1. Plot of u_{11} with $R = -1$, $p_0 = \frac{3}{2}$, $p_1 = 0$, $F_1(t) = 2$, $F_2(t) = \frac{t}{2}$, $y = 2$

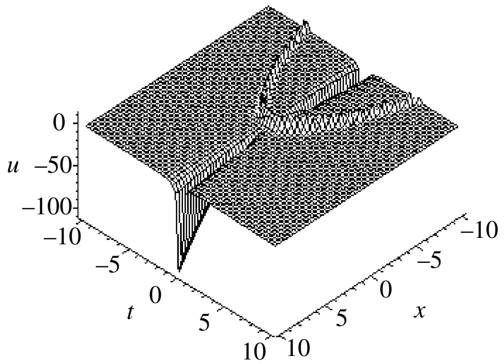

Fig. 2. The same as Fig.1 except that $p_1 = 1$, $F_1(t) = 2t + \frac{1}{2}t^2$

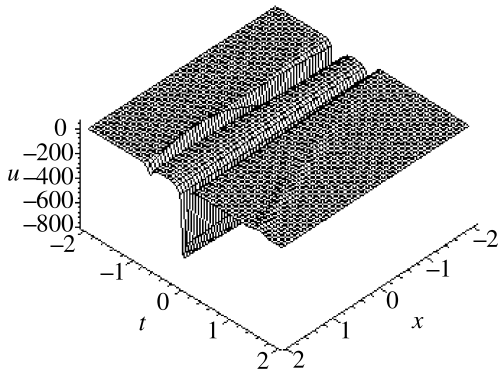

Fig. 3. The same as Fig.1 except that $p_1 = t$, $F_1(t) = \frac{\sin(t)}{2}$

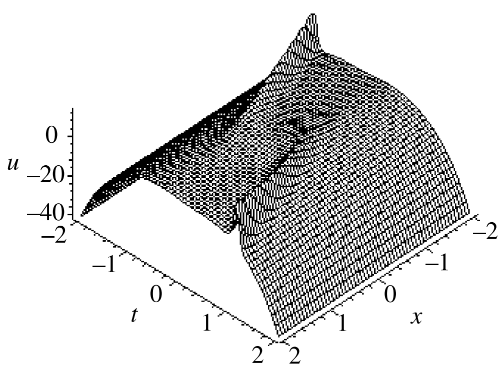

Fig. 4. The same as Fig.1 except that $p_1 = t^2$, $F_1(t) = 2t$

4 Conclusions

In summary, rich exact analytic solutions including soliton-like solutions and periodic form solutions of two variable-coefficient NLEEs, the variable-coefficient Fisher-type equation and the vcKdV equation, are obtained by using a generalized Riccati equation expansion method and symbolic computation. It is shown that this method can be applied to not only some constant-coefficients NLEEs [11] but also to some variable-coefficients NLEEs. This method can also be extended to a couple of other NLEEs.

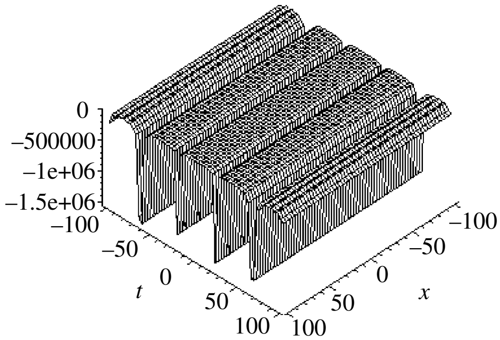

Fig. 5. Plots of u_{11} with $R=-1$, $p_0=\sin(t)$, $p_1=0$, $F_1(t)=2t$, $F_2(t)=\frac{t}{2}+t^3$, $y=2$

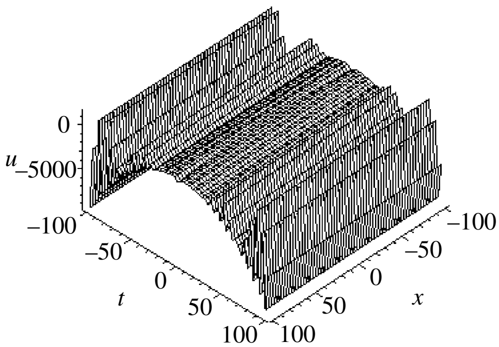

Fig. 6. The same as Fig. 5 except that $p_1=2$

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