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Elliptic equation rational expansion method and new exact travelling solutions for Whitham–Broer–Kaup equations

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Abstract

Based on a new general ansatz and a general subepuation, a new general algebraic method named elliptic equation rational expansion method is devised for constructing multiple travelling wave solutions in terms of rational special function for nonlinear evolution equations (NEEs). We apply the proposed method to solve Whitham–Broer–Kaup equation and explicitly construct a series of exact solutions which include rational form solitary wave solution, rational form triangular periodic wave solutions and rational wave solutions as special cases. In addition, the links among our proposed method with the method by Fan [Chaos, Solitons & Fractals 2004;20:609], are also clarified generally.

1. Introduction

With the development of soliton theory, there have been a great amount of activities aiming to find methods for exact solutions of NEEs, such as Bäcklund transformation, Darboux transformation, Cole-Hopf transformation, various tanh methods, various Jacobi elliptic function methods, variable separation approach, Painlevé method, homogeneous balance method, similarity reduction method and so on [1–17]. Among those, the tanh method provides a straightforward and effective algorithm to obtain particular travelling solutions for a large number of NEEs. Recently, much research work has been concentrated on the various extensions and applications of the tanh method[7–17]. Recently, in Refs. [18], Fan developed a new algebraic method with symbolic computation for obtaining the above-mentioned various travelling wave solutions in a unified way, but also easily provides us with new and more general travelling wave solutions in terms of special functions such as hyperbolic, rational, triangular, Weierstrass and Jacobi elliptic double periodic functions. In Refs. [19], we extended the Fan's method [18] to a generalized method. As a result,

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we can not only successfully recover the previously known travelling wave solutions found by Fan's method but also obtain some new formal solutions. More recently, we present the Jacobi elliptic function rational expansion method [20] and the Riccati equation rational expansion method [21], in which the ansatzes are firstly express as rational form.

The present work is motivated by the desire to present a new subequation method, named elliptic equation rational expansion method, by proposing a more general ansatz so that it can be used to obtained more types and general formal solutions which contain not only the results obtained by using the various methods [6–21] but also other types of solutions. The appeal and success of the method lies in the fact: one circumvents integration to get explicit solutions based on the fact that soliton solutions are essentially of a localized nature. Writing the soliton solutions of a NEEs as the polynomials of auxiliary variables of the elliptic equation [22–24], the NEEs can changed into a nonlinear system of algebraic equations. The system can be solved with the help of symbolic computation. For illustration, we apply the generalized method to solve the Whitham–Broer–Kaup equation and successfully construct new and more general solutions including rational form solitary wave solutions, rational form triangular periodic solutions and ration wave solutions for the Whitham–Broer–Kaup equation.

This paper is organized as follows. In Section 2, we summarize the elliptic equation rational expansion method. In Section 3, we apply the method to the Whitham–Broer–Kaup equation and bring out many solutions. Conclusions will be presented in finally.

2. Summary of the elliptic equation rational expansion method

In the following we would like to outline the main steps of our method:

Step 1. For a given NEE system with some physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0, \tag{2.1}$$

by using the wave transformation

$$u_i(x,y,t) = U_i(\xi), \quad \xi = k(x + ly + \lambda t), \tag{2.2}$$

where k, l and λ are constants to be determined later. Then the nonlinear partial differential Eq. (2.1) is reduced to a nonlinear ordinary differential equation(ODE):

$$G_i(U_i, U_i', U_i'', \ldots) = 0. \tag{2.3}$$

Step 2. We introduce a new ansätz in terms of finite rational formal expansion in the following forms:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}\phi^j(\xi) + b_{ij}\phi^{j-1}(\xi)\phi'(\xi)}{(\mu\phi(\xi)+1)^j} \tag{2.4}$$

and the new variable $\phi = \phi(\xi)$ satisfying the elliptic equation [22–24]

$$\phi'^2 = \left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 = h_0 + h_1\phi + h_2\phi^2 + h_3\phi^3 + h_4\phi^4, \tag{2.5}$$

where h_ρ, a_{i0}, a_{ij} and b_{ij} ($\rho = 0,1,\ldots,4$; $i = 1,2,\ldots$; $j = 1,2,\ldots,m_i$) are constants to be determined later.

Step 3. The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions to occur is that differ effects that act to change wave forms in many nonlinear equations, i.e. dispersion, dissipation and nonlinearity, either separately or various combination are able to balance out. We define the degree of $U_i(\xi)$ as $D[U_i(\xi)] = n_i$, which gives rise to the degrees of other expressions as

$$D[U_i^{(\alpha)}] = n_i + \alpha, \quad D[U_i^\beta (U_j^{(\alpha)})^s] = n_i\beta + (\alpha + n_j)s. \tag{2.6}$$

Therefore we can get the value of m_i in Eq. (2.4). If n_i is a nonnegative integer, then we first make the transformation $U_i = V_i^{n_i}$.

Step 4. Substitute Eq. (2.4) into Eq. (2.3) along with Eq. (2.5) and then set all coefficients of $\phi^p(\xi)\left(\sqrt{\sum_{\rho=0}^4 h_\rho\phi^\rho}\right)^q$ ($p = 1,2,\ldots,q = 0,1.$) of the resulting system's numerator to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, μ, a_{i0}, a_{ij} and b_{ij} ($i = 1,2,\ldots$; $j = 1,2,\ldots,m_i$).

Step 5. Solving the over-determined system of nonlinear algebraic equations by use of *Maple*, we would end up with the explicit expressions for k, μ, a_{i0}, a_{ij} and b_{ij} ($i = 1,2,\ldots$; $j = 1,2,\ldots,m_i$).

Step 6. By using the results obtained in the above step, we can derive a series of fundamental solutions in rational form such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions. Because we interested in solitary wave, Jacobi and Weierstrass doubly periodic solutions. On the other hand, tan and cot type solutions appear in pairs with tanh and coth type solutions respectively, polynomial, rational triangular periodic solutions are omitted in this paper. By considering the different values of h_0, h_1, h_2, h_3 and h_4, Eq. (2.5) has many kinds of solitary wave, Jacobi and Weierstrass doubly periodic solutions which are listed as follows.

Case A. If $h_3 = h_4 = 0$, Eq. (2.5) possesses following solutions:

$$\phi = \sqrt{h_0}\xi, \quad h_1 = h_2 = 0, \ \ h_0 > 0; \tag{2.7}$$

$$\phi = -\frac{h_0}{h_1} + \frac{1}{4}h_1\xi^2, \quad h_2 = 0, \ \ h_1 \neq 0; \tag{2.8}$$

$$\phi = -\frac{h_1}{2h_2} + \exp\left(\sqrt{h_2}\xi\right), \quad h_0 = \frac{h_1^2}{4h_2}, \ \ h_2 > 0; \tag{2.9}$$

$$\phi = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2}\sin\left(\sqrt{-h_2}\xi\right), \quad h_0 = 0, \ \ h_2 < 0; \tag{2.10}$$

$$\phi = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2}\sinh\left(\sqrt{h_2}\xi\right), \quad h_0 = 0, \ \ h_2 > 0. \tag{2.11}$$

Case B. If $h_1 = h_3 = 0$, Eq. (2.5) possesses following solutions:

$$\phi = \sqrt{-\frac{h_2}{h_4}}\mathrm{sech}\left(\sqrt{h_2}\xi\right), \quad h_0 = 0, \ \ h_2 > 0, \ \ h_4 < 0; \tag{2.12}$$

$$\phi = \sqrt{-\frac{h_2}{2h_4}}\tanh\left(\sqrt{-\frac{h_2}{2}}\xi\right), \quad h_0 = \frac{h_2^2}{4h_4}, \ \ h_2 < 0, \ \ h_4 > 0; \tag{2.13}$$

$$\phi = \sqrt{-\frac{h_2}{h_4}}\sec\left(\sqrt{-h_2}\xi\right), \quad h_0 = 0, \ \ h_2 < 0; \ \ h_4 > 0; \tag{2.14}$$

$$\phi = \sqrt{\frac{h_2}{2h_4}}\tan\left(\sqrt{\frac{h_2}{2}}\xi\right), \quad h_0 = \frac{h_2^2}{4h_4}, \ \ h_2 > 0, \ \ h_4 > 0; \tag{2.15}$$

$$\phi = -\frac{1}{\sqrt{h_4}\xi}, \quad h_0 = h_2 = 0, \ \ h_4 > 0; \tag{2.16}$$

$$\phi = \mathrm{sn}(\xi), \quad h_0 = 1, \ \ h_2 = -(m^2+1), \ \ h_4 = m^2; \tag{2.17}$$

$$\phi = \mathrm{cd}(\xi), \quad h_0 = 1, \ \ h_2 = -(m^2+1), \ \ h_4 = m^2; \tag{2.18}$$

$$\phi = \mathrm{cn}(\xi), \quad h_0 = 1 - m^2, \ \ h_2 = 2m^2 - 1, \ \ h_4 = -m^2; \tag{2.19}$$

$$\phi = \mathrm{dn}(\xi), \quad h_0 = m^2 - 1, \ \ h_2 = 2 - m^2, \ \ h_4 = -1; \tag{2.20}$$

$$\phi = \mathrm{ns}(\xi), \quad h_0 = m^2, \ \ h_2 = -(m^2+1), \ \ h_4 = 1; \tag{2.21}$$

$$\phi = \mathrm{dc}(\xi), \quad h_0 = m^2, \ \ h_2 = -(m^2+1), \ \ h_4 = 1; \tag{2.22}$$

$$\phi = \mathrm{nc}(\xi), \quad h_0 = -m^2, \ \ h_2 = 2m^2 - 1, \ \ h_4 = 1 - m^2; \tag{2.23}$$

$$\phi = \mathrm{nd}(\xi), \quad h_0 = -1, \ \ h_2 = 2 - m^2, \ \ h_4 = 1 - m^2; \tag{2.24}$$

$$\phi = \mathrm{cs}(\xi), \quad h_0 = 1 - m^2, \ \ h_2 = 2 - m^2, \ \ h_4 = 1; \tag{2.25}$$

$$\phi = \mathrm{sc}(\xi), \quad h_0 = 1, \ h_2 = 2 - m^2, \ h_4 = 1 - m^2; \tag{2.26}$$

$$\phi = \mathrm{sd}(\xi), \quad h_0 = 1, \ h_2 = 2m^2 - 1, \ h_4 = m^2(m^2 - 1); \tag{2.27}$$

$$\phi = \mathrm{ds}(\xi), \quad h_0 = m^2(m^2 - 1), \ h_2 = 2m^2 - 1, \ h_4 = 1; \tag{2.28}$$

$$\phi = \mathrm{ns}(\xi) \pm \mathrm{cs}(\xi), \quad h_0 = \frac{1}{4}, \ h_2 = \frac{1 - 2m^2}{2}, \ h_4 = \frac{1}{4}; \tag{2.29}$$

$$\phi = \mathrm{nc}(\xi) \pm \mathrm{sc}(\xi), \quad h_0 = \frac{1 - m^2}{4}, \ h_2 = \frac{1 + m^2}{2}, \ h_4 = \frac{1 - m^2}{4}; \tag{2.30}$$

$$\phi = \mathrm{ns}(\xi) \pm \mathrm{ds}(\xi), \quad h_0 = \frac{m^2}{4}, \ h_2 = \frac{m^2 - 2}{2}, \ h_4 = \frac{1}{4}; \tag{2.31}$$

$$\phi = \mathrm{sn}(\xi) \pm i\mathrm{cn}(\xi), \quad h_0 = \frac{m^2}{4}, \ h_2 = \frac{m^2 - 2}{2}, \ h_4 = \frac{m^2}{4}, \tag{2.32}$$

where m is a modulus. The Jacobi elliptic functions are doubly periodic and possess properties of triangular functions:

$$\mathrm{sn}^2(\xi) + \mathrm{cn}^2(\xi) = 1, \quad \mathrm{dn}^2(\xi) = 1 - m^2\mathrm{sn}^2\xi,$$

$$\mathrm{sn}'(\xi) = \mathrm{cn}(\xi)\mathrm{dn}(\xi), \quad \mathrm{cn}'(\xi) = -\mathrm{sn}(\xi)\mathrm{dn}(\xi), \quad \mathrm{dn}'(\xi) = -m^2\mathrm{sn}(\xi)\mathrm{cn}(\xi).$$

When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.

$$\mathrm{sn}(\xi) \to \tanh(\xi), \quad \mathrm{cn}(\xi) \to \mathrm{sech}(\xi).$$

When $m \to 0$, the Jacobi functions degenerate to the triangular functions, i.e.

$$\mathrm{sn}(\xi) \to \sin(\xi), \quad \mathrm{cn}(\xi) \to \cos(\xi).$$

The more detailed notations for the Weierstrass and Jacobi elliptic functions can be found in Refs. [23,24].

Case C. If $h_4 = 0$, Eq. (2.5) possesses following solutions:

$$\phi = -\frac{h_2}{h_3}\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right), \quad h_0 = h_1 = 0, \ h_2 > 0; \tag{2.33}$$

$$\phi = -\frac{h_2}{h_3}\sec^2\left(\frac{\sqrt{-h_2}}{2}\xi\right), \quad h_0 = h_1 = 0, \ h_2 < 0; \tag{2.34}$$

$$\phi = \frac{4}{h_3\xi^2}, \quad h_0 = h_1 = h_2 = 0; \tag{2.35}$$

$$\phi = \wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right), \quad h_2 = 0, \ h_3 > 0, \tag{2.36}$$

where $g_2 = -4\frac{h_1}{h_3}$ and $g_3 = -4\frac{h_0}{h_3}$ are called invariants of Weierstrass elliptic function.

Case D. If $h_0 = h_1 = 0$, Eq. (2.5) possesses following solutions:

$$\phi = -\frac{h_2\sec^2\left(\frac{\sqrt{-h_2}}{2}\xi\right)}{2\sqrt{-h_2h_4}\tan\left(\frac{\sqrt{-h_2}}{2}\xi\right) + h_3}, \quad h_2 < 0; \tag{2.37}$$

$$\phi = \frac{h_2\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3}, \quad h_2 > 0, \tag{2.38}$$

Thus according to Eqs. (2.2), (2.4), (2.7)–(2.38) and the conclusions in *Step 5*, we can obtain some rational formal travelling-wave solutions of Eq. (2.1).

Remark. The more general the ansatz is, the more general and more formal the solutions of the NEEs will be. If we set the parameters in Eq. (2.5) to different values, the above methods can be recovered by the RERE method. The concrete case is as follows: Setting $\mu = b_1 = 0$, we just recover the solutions obtained by the generalized method [18].

3. Exact solutions of the Whitham–Broer–Kaup equation

Let us consider the Whitham–Broer–Kaup (WBK) equation, i.e.,

$$\begin{cases} u_t + uu_x + v_x + \beta u_{xx} = 0, \\ v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = 0, \end{cases} \tag{3.1}$$

where α, $\beta \neq 0$ are all constants. Under Boussinesq approximation, Whitham [25], Broer [26] and Kaup [27] obtained nonlinear WBK equation. It is not difficult to see that when parameters α and β take different constants, system (3.1) includes many important mathematical and physical equations, such as when $\alpha = 0$, $\beta \neq 0$ system (3.1) becomes classical long wave equation that describe shallow water with dispersive [28], and when $\alpha = 1$, $\beta = 0$ system becomes variant Boussinesq equation [1]. Many mathematicians and physicists have devoted considerable effort to the study on WBK equation and make new developments in the regards (see, e.g., [29–32] for detail).

By considering the wave transformations $u(x,t) = U(\xi)$, $v(x,t) = V(\xi)$ and $\xi = k(x + \lambda t)$, we change the Eq. (3.1) to the form

$$\begin{cases} \lambda U' + UU' + V' + k\beta U'' = 0, \\ \lambda V' + (UV)' + \alpha k^2 U''' - \beta k V'' = 0. \end{cases} \tag{3.2}$$

According to the proposed method, we expand the solution of Eq. (3.2) in the form

$$\begin{cases} U(\xi) = a_0 + \sum\limits_{j=1}^{m_u} \dfrac{a_j\phi^j(\xi) + b_j\phi^{j-1}(\xi)\phi'(\xi)}{(\mu\phi(\xi)+1)^j}, \\ V(\xi) = A_0 + \sum\limits_{j=1}^{m_v} \dfrac{A_j\phi^j(\xi) + B_j\phi^{j-1}(\xi)\phi'(\xi)}{(\mu\phi(\xi)+1)^j}, \end{cases}$$

where $\phi(\xi)$ satisfies Eq. (2.5). Balancing the term U''' with term $(UV)'$ and the term U'' with term UU' in Eq. (3.2) gives $m_u = 1$ and $m_v = 2$. So we have

$$\begin{cases} U(\xi) = a_0 + \dfrac{a_1\phi(\xi) + b_1\phi'}{\mu\phi(\xi)+1}, \\ V(\xi) = A_0 + \dfrac{A_1\phi(\xi) + B_1\phi'}{\mu\phi(\xi)+1} + \dfrac{A_2\phi^2(\xi) + B_2\phi(\xi)\phi'}{(\mu\phi(\xi)+1)^2}, \end{cases} \tag{3.3}$$

where $\phi(\xi)$ satisfies Eq. (2.5).

With the aid of *Maple*, substituting (3.3) along with (2.5) into (3.1), and according to *Step 4* in Section 2 yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1 A_2, B_2, μ and k.

By use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method[33], solving the over-determined algebraic equations, we get the following results.

$$k = \pm 2\frac{\sqrt{-(\alpha+\beta^2)(-h_0\mu^4 + h_1\mu^3 - h_2\mu^2 + h_3\mu - h_4)}(a_0+\lambda)}{(\alpha+\beta^2)(-4h_0\mu^3 + 3h_1\mu^2 - 2h_2\mu + h_3)}, \quad a_0 = a_0, \quad b_1 = b_1,$$

$$A_0 = \frac{(a_0+\lambda)^2(-6\mu^2h_3h_1 + 16\mu^3h_0h_3 + 4\mu^3h_2h_1 + h_3^2 - 12\mu^4h_0h_2 + 12\mu^5h_0h_1)}{(-4h_0\mu^3 + 3h_1\mu^2 - 2h_2\mu + h_3)^2} - \frac{(a_0+\lambda)^2(8\mu^6h_0^2 + 12\mu h_4h_1 - 4h_4h_2 - 3\mu^4h_1^2 - 24\mu^2h_0h_4)}{(-4h_0\mu^3 + 3h_1\mu^2 - 2h_2\mu + h_3)^2},$$

$$B_1 = \pm 8\frac{\beta(a_0+\lambda)^2(h_0\mu^4 - h_1\mu^3 + h_2\mu^2 - h_3\mu + h_4)\sqrt{(\alpha+\beta^2)(h_0\mu^4 - h_1\mu^3 + h_2\mu^2 - h_3\mu + h_4)}}{(4h_0\mu^3 - 3h_1\mu^2 + 2h_2\mu - h_3)^2(\alpha+\beta^2)},$$

$$B_2 = \pm 8\frac{\beta\mu(a_0+\lambda)^2(h_0\mu^4-h_1\mu^3+h_2\mu^2-h_3\mu+h_4)\sqrt{(\alpha+\beta^2)(h_0\mu^4-h_1\mu^3+h_2\mu^2-h_3\mu+h_4)}}{(4h_0\mu^3-3h_1\mu^2+2h_2\mu-h_3)^2(\alpha+\beta^2)},$$

$$a_1 = -4\frac{(a_0+\lambda)(-h_0\mu^4+h_1\mu^3-h_2\mu^2+h_3\mu-h_4)}{-4h_0\mu^3+3h_1\mu^2-2h_2\mu+h_3},$$

$$A_1 = 4\frac{(a_0+\lambda)^2(-h_0\mu^4+h_1\mu^3-h_2\mu^2+h_3\mu-h_4)}{-4h_0\mu^3+3h_1\mu^2-2h_2\mu+h_3},$$
$$A_2 = 8\frac{(a_0+\lambda)^2(-h_0\mu^4+h_1\mu^3-h_2\mu^2+h_3\mu-h_4)^2}{(4h_0\mu^3-3h_1\mu^2+2h_2\mu-h_3)^2}. \tag{3.4}$$

Then according to Eqs. (3.4), we obtain following solutions of the WBK equation.

Note: Since tan- and cot-type solution appear in pairs with tanh- and coth-type solutions, respectively, we omit them in this paper. In addition, some rational solutions are also omitted.

Family 1. When $h_3 = h_4 = 0$ and $h_0 = -\frac{h_1^2}{4h_2}$, we obtain the following solutions for the WBK equation, as follows:

$$u_{11} = a_0 - \frac{(a_0+\lambda)(-\mu^4h_1^2+4h_1\mu^3h_2-4h_2^2\mu^2)(-h_1+2h_2\exp(\sqrt{h_2}\xi))}{(-h_1^2\mu^3+3h_1h_2\mu^2-2h_2^2\mu)(\mu(-h_1+2h_2\exp(\sqrt{h_2}\xi))+2h_2)}, \tag{3.5.1}$$

$$\begin{aligned} v_{11} = A_0 &+ \frac{(a_0+\lambda)^2(-\mu^4h_1^2+4h_1h_2\mu^3-4h_2^2\mu^2)(-h_1+2h_2\exp(\sqrt{h_2}\xi))}{(-h_1^2\mu^3+3h_1h_2\mu^2-2h_2^2\mu)(\mu(-h_1+2h_2\exp(\sqrt{h_2}\xi))+2h_2)} \\ &- \frac{(a_0+\lambda)^2(-\mu^4h_1^2+4h_1h_2\mu^3-4h_2^2\mu^2)^2(-h_1+2h_2\exp(\sqrt{h_2}\xi))^2}{2(-h_1^2\mu^3+3h_1h_2\mu^2-2h_2^2\mu)^2(\mu(-h_1+2h_2\exp(\sqrt{h_2}\xi))+2h_2)^2} \\ &\mp \frac{2(a_0+\lambda)^2(\mu^4h_1^2-4h_1h_2\mu^3+4h_2^2\mu^2)^{\frac{3}{2}}\beta h_2\sqrt{\alpha+\beta^2}\exp(\sqrt{h_2}\xi)}{(h_1^2\mu^3-3h_1h_4\mu^2+2h_2h_4)^2(\alpha+\beta^2)(\mu(-h_1+2h_2\exp(\sqrt{h_2}\xi))+2h_2)} \\ &\pm 2\frac{(a_0+\lambda)^2\sqrt{\mu^4h_1^2-4h_1h_2\mu^3+4h_2^2\mu^2}\mu\beta\sqrt{\alpha+\beta^2}(-h_1+2h_2\exp(\sqrt{h_2}\xi))h_2^2\exp(\sqrt{h_2}\xi)}{(h_1^2\mu^3-3h_1h_4\mu^2+2h_2h_4)^2(\alpha+\beta^2)(\mu(-h_1+2h_2\exp(\sqrt{h_2}\xi))+2h_2)^2}, \end{aligned} \tag{3.5.2}$$

where $\xi = k(x+\lambda t)$, k and A_0 are determined by (3.4), μ, $h_2 > 0$, h_1, a_0 and λ are arbitrary constants.

Family 2. When $h_0 = h_3 = h_4 = 0$, we obtain the following solutions for the WBK equation, as follows:

$$u_{21} = a_0 - 4\frac{(h_2\mu^2-h_1\mu^3)(a_0+\lambda)(-h_1+h_1\sinh(\sqrt{h_2}\xi))}{(-3h_1\mu^2+2h_2\mu)(\mu(-h_1+h_1\sinh(\sqrt{h_2}\xi))+2h_2)}, \tag{3.6.1}$$

$$\begin{aligned} v_{21} = &-\frac{(a_0+\lambda)^2(-4\mu^3h_2h_1+3\mu^4h_1^2)}{(-3h_1\mu^2+2h_2\mu)^2} + 4\frac{(h_2\mu^2-h_1\mu^3)(a_0+\lambda)^2(-h_1+h_1\sinh(\sqrt{h_2}\xi))}{(-3h_1\mu^2+2h_2\mu)(\mu(-h_1+h_1\sinh(\sqrt{h_2}\xi))+2)} \\ &- 8\frac{(h_2\mu^2-h_1\mu^3)^2(a_0+\lambda)^2(-h_1+h_1\sinh(\sqrt{h_2}\xi))^2}{(-3h_1\mu^2+2h_2\mu)^2(\mu(-h_1+h_1\sinh(\sqrt{h_2}\xi))+2h_2)^2} \\ &\mp 8\frac{(h_2\mu^2-h_1\mu^3)(a_0+\lambda)^2\beta\sqrt{(\alpha+\beta^2)(h_2\mu^2-h_1\mu^3)}h_1h_2^{\frac{3}{2}}\cosh(\sqrt{h_2}\xi)}{(-3h_1\mu^2+2h_2\mu)^2(\alpha+\beta^2)(\mu(-h_1+h_1\sinh(\sqrt{h_2}\xi))+2h_2)} \\ &\pm 8\frac{(h_2\mu^2-h_1\mu^3)(a_0+\lambda)^2\mu\beta\sqrt{(\alpha+\beta^2)(h_2\mu^2-h_1\mu^3)}(-h_1+h_1\sinh(\sqrt{h_2}\xi))h_1h_2^{\frac{5}{2}}\cosh(\sqrt{h_2}\xi)}{(-3h_1\mu^2+2h_2\mu)^2(\alpha+\beta^2)(\mu(-h_1+h_1\sinh(\sqrt{h_2}\xi))+2h_2)^2}, \end{aligned} \tag{3.6.2}$$

where $\xi = k(x+\lambda t)$, k and A_0 are determined by (3.4), $h_2 > 0$, h_1, μ, a_0 and λ are arbitrary constants.

Family 3. When $h_0 = h_1 = h_3 = 0$, we obtain the following solutions for the WBK equation, as follows:

$$u_{31} = a_0 - \frac{2(h_2\mu^2 + h_4)(a_0 + \lambda)\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)}{h_2\mu\left(\mu\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi) + 1\right)}, \tag{3.7.1}$$

$$\begin{aligned} v_{31} = &-\frac{(a_0+\lambda)^2 h_4}{h_2\mu^2} + 2\frac{(h_2\mu^2 + h_4)(a_0+\lambda)^2\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)}{h_2\mu\left(\mu\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)+1\right)} + 2\frac{(h_2\mu^2+h_4)^2(a_0+\lambda)^2(\operatorname{sech}(\sqrt{h_2}\xi))^2}{h_2\mu^2 h_4\left(\mu\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)+1\right)^2} \\ &\mp 2\frac{(h_2\mu^2+h_4)(a_0+\lambda)^2\beta\sqrt{(\alpha+\beta^2)(h_2\mu^2+h_4)}\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)\tanh(\sqrt{h_2}\xi)}{h_2^{\frac{3}{2}}\mu^2(\alpha+\beta^2)\left(\mu\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)+1\right)} \\ &\pm 2\frac{(h_2\mu^2+h_4)\sqrt{h_2}(a_0+\lambda)^2\beta\sqrt{(\alpha+\beta^2)(h_2\mu^2+h_4)}\operatorname{sech}^2(\sqrt{h_2}\xi)\tanh(\sqrt{h_2}\xi)}{\mu(\alpha+\beta^2)h_4\left(\mu\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)+1\right)^2}, \end{aligned} \tag{3.7.2}$$

where $\xi = k(x + \lambda t)$, k is determined by (3.4), $h_2 > 0$, $h_4 < 0$, μ, a_0 and λ are arbitrary constants.

Family 4. When $h_1 = h_3 = 0$ and $h_0 = -\frac{h_2^2}{4h_4}$, we obtain the following solutions for the WBK equation, as follows:

$$u_{41} = a_0 - \frac{(4h_2h_4\mu^2 + 4h_4^2 + h_2^2\mu^4)(a_0+\lambda)\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)}{(h_2^2\mu^3 + 2h_2h_4\mu)\left(\mu\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)+2\right)}, \tag{3.8.1}$$

$$\begin{aligned} v_{41} = A_0 &+ \frac{(4h_2h_4\mu^2 + 4h_4^2 + h_2^2\mu^4)(a_0+\lambda)^2\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)}{(h_2^2\mu^3 + 2h_2h_4\mu)\left(\mu\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)+2\right)} \\ &+ \frac{(a_0+\lambda)^2(4h_2h_4\mu^2 + 4h_4^2 + h_2^2\mu^4)^2 h_2\tanh^2\left(\frac{\sqrt{-2h_2}}{2}\xi\right)}{(h_2^2\mu^3 + 2h_2h_4^2\mu)^2 h_4\left(\mu\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)+2\right)^2} \\ &\pm \frac{(4h_2h_4\mu^2 + 4h_4^2 + h_2^2\mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}h_2\operatorname{sech}^2\left(\frac{\sqrt{-2h_2}}{2}\xi\right)}{(h_2^2\mu^3 + 2h_2h_4\mu)^2(\alpha+\beta^2)\left(\mu\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)+2\right)} \\ &\pm \frac{(4h_2h_4\mu^2 + 4h_4^2 + h_2^2\mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}h_2\mu\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)\operatorname{sech}^2\left(\frac{\sqrt{-2h_2}}{2}\xi\right)}{(h_2^2\mu^3 + 2h_2h_4\mu)^2(\alpha+\beta^2)\left(\mu\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)+2\right)^2}, \end{aligned} \tag{3.8.2}$$

where $\xi = k(x + \lambda t)$, k, a_0, A_0 are determined by (3.4), $h_2 < 0$, $h_4 > 0$, μ, a_0 and λ are arbitrary constants.

Family 5. When $h_0 = h_1 = h_4 = 0$, we obtain the following solutions for the WBK equation, as follows:

$$u_{51} = a_0 + 4\frac{(h_2\mu^2 - h_3\mu)(a_0+\lambda)h_2\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)\left(-h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)+h_3\right)}, \tag{3.9.1}$$

$$v_{51} = A_0 + 4\frac{(h_2\mu^2 - h_3\mu)(a_0+\lambda)^2 h_2 \operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)\left(h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)} - 8\frac{(h_2\mu^2 - h_3\mu)^2(a_0+\lambda)^2 h_2^2 \operatorname{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)^2\left(h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)^2}$$
$$\mp 8\frac{(h_2\mu^2 - h_3\mu)(a_0+\lambda)^2\beta\sqrt{(\alpha+\beta^2)(h_2\mu^2 - h_3\mu)}\,h_2^{\frac{3}{2}}\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)^2(\alpha+\beta^2)\left(-h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + h_3\right)}$$
$$\pm 8\frac{(h_2\mu^2 - h_3\mu)(a_0+\lambda)^2\mu\beta\sqrt{(\alpha+\beta^2)(h_2\mu^2 - h_3\mu)}\,h_2^{\frac{5}{2}}\operatorname{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)^2(\alpha+\beta^2)\left(-h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + h_3\right)^2}, \tag{3.9.2}$$

where $\xi = k(x + \lambda t)$, k is determined by (3.4), $h_2 > 0$, h_3, μ, a_0 and λ are arbitrary constants.

Family 6. When $h_0 = h_1 = 0$, we obtain the following solutions for the WBK equation, as follows:

$$u_{61} = a_0 - 4\frac{(h_2\mu^2 + h_4 - h_3\mu)(a_0+\lambda)h_2\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)\left(h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)}, \tag{3.10.1}$$

$$v_{61} = -\frac{(a_0+\lambda)^2(4h_4h_2 - h_3^2)}{(2h_2\mu - h_3)^2} + \frac{4(h_2\mu^2 + h_4 - h_3\mu)(a_0+\lambda)^2 h_2\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)\left(h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)}$$
$$- 8\frac{(h_2\mu^2 + h_4 - h_3\mu)^2(a_0+\lambda)^2 h_2^2\operatorname{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)^2\left(h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)^2}$$
$$\mp\frac{8(h_2\mu^2 + h_4 - h_3\mu)^{\frac{3}{2}}h_2^{\frac{3}{2}}(a_0+\lambda)^2\beta\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)^2\sqrt{\alpha+\beta^2}\left(h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)}$$
$$\pm\frac{8(h_2\mu^2 + h_4 - h_3\mu)^{\frac{3}{2}}h_2^2h_4^{\frac{1}{2}}(a_0+\lambda)^2\beta\operatorname{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)^2\left(h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)\left(2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)}$$
$$\pm\frac{8(h_2\mu^2 + h_4 - h_3\mu)^{\frac{3}{2}}\mu h_2^{\frac{5}{2}}(a_0+\lambda)^2\beta\operatorname{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)^2\sqrt{\alpha+\beta^2}\left(h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)^2}$$
$$\pm\frac{8(h_2\mu^2 + h_4 - h_3\mu)^{\frac{3}{2}}h_2^3h_4^{\frac{1}{2}}(a_0+\lambda)^2\beta\operatorname{sech}^6\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2h_2\mu - h_3)^2\left(h_2\mu\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)^2\left(2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)}, \tag{3.10.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.4), $h_2 > 0$, h_3, h_4, μ, a_0 and λ are arbitrary constants.

Family 7. When $h_2 = h_4 = 0$, we obtain the following solutions for the WBK equation, as follows:

$$u_{71} = a_0 - 4\frac{(-h_0\mu^4 + h_1\mu^3 + h_3\mu)(a_0+\lambda)\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)}{(-4h_0\mu^3 + 3h_1\mu^2 + h_3)\left(\mu\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)+1\right)}, \tag{3.11.1}$$

$$\begin{aligned} v_{71} = {} & A_0 + 4\frac{(-h_0\mu^4 + h_1\mu^3 + h_3\mu)(a_0+\lambda)^2\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)}{(-4h_0\mu^3 + 3h_1\mu^2 + h_3)\left(\mu\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)+1\right)} - 8\frac{(-h_0\mu^4 + h_1\mu^3 + h_3\mu)^2(a_0+\lambda)^2\wp^2\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)}{(-4h_0\mu^3 + 3h_1\mu^2 + h_3)^2\left(\mu\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)+1\right)^2} \\ & \pm 8\frac{(a_0+\lambda)^2(h_3\mu + h_1\mu^3 - h_0\mu^4)^{\frac{3}{2}}\beta\sqrt{\alpha+\beta^2}\sqrt{h_0 + h_1\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right) + h_3\wp^3\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)}}{(4h_0\mu^3 - 3h_1\mu^2 - h_3)^2(\alpha+\beta^2)\left(\mu\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)+1\right)} \\ & + 8\frac{(a_0+\lambda)^2(h_0\mu^4 - h_1\mu^3 - h_3\mu)^{\frac{3}{2}}\mu\beta\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)\sqrt{h_0 + h_1\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right) + h_3\wp^3\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)}}{(-4h_0\mu^3 + 3h_1\mu^2 + h_3)^2\sqrt{\alpha+\beta^2}\left(\mu\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)+1\right)^2}, \end{aligned} \tag{3.11.2}$$

where $\xi = k(x + \lambda t)$, $g_2 = -4\frac{h_1}{h_3}$, $g_3 = -4\frac{h_0}{h_3}$, k and A_0 are determined by (3.4), $h_3 > 0$, h_0, h_1, μ, a_0 and λ are arbitrary constants.

Family 8. When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = -(m^2 + 1)$ and $h_4 = m^2$, we obtain the following solutions for the WBK equation, as follows:

$$u_{81} = a_0 - 4\frac{(-(m^2+1)\mu^2 + m^2 + \mu^4)(a_0+\lambda)\mathrm{sn}(\xi)}{(4\mu^3 - 2(m^2+1)\mu)(\mu\,\mathrm{sn}(\xi)+1)}, \tag{3.12.1}$$

$$\begin{aligned} v_{81} = {} & A_0 + 4\frac{(-(m^2+1)\mu^2 + m^2 + \mu^4)(a_0+\lambda)^2\mathrm{sn}(\xi)}{(4\mu^3 - 2(m^2+1)\mu)(\mu\,\mathrm{sn}(\xi)+1)} - 8\frac{(-(m^2+1)\mu^2 + m^2 + \mu^4)^2(a_0+\lambda)^2\mathrm{sn}^2(\xi)}{(4\mu^3 - 2(m^2+1)\mu)^2(\mu\,\mathrm{sn}(\xi)+1)^2} \\ & \mp 8\frac{((-m^2-1)\mu^2 + m^2 + \mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}\mathrm{cn}(\xi)\mathrm{dn}(\xi)}{(4\mu^3 + 2(-m^2-1)\mu)^2(\alpha+\beta^2)(\mu\mathrm{sn}(\xi)+1)} \\ & \pm 8\frac{((-m^2-1)\mu^2 + m^2 + \mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\mu\beta\sqrt{\alpha+\beta^2}\mathrm{sn}(\xi)\mathrm{cn}(\xi)\mathrm{dn}(\xi)}{(4\mu^3 + 2(-m^2-1)\mu)^2(\alpha+\beta^2)(\mu\,\mathrm{sn}(\xi)+1)^2}, \end{aligned} \tag{3.12.2}$$

where $\xi = k(x + \lambda t)$, k and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 9. When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = -(m^2 + 1)$ and $h_4 = m^2$, we obtain the following solutions for the WBK equation, as follows:

$$u_{91} = a_0 - 4\frac{((-m^2-1)\mu^2 + m^2 + \mu^4)(a_0+\lambda)\mathrm{cd}(\xi)}{(4\mu^3 + 2(-m^2-1)\mu)(\mu\,\mathrm{cd}(\xi)+1)}, \tag{3.13.1}$$

$$\begin{aligned} v_{91} = {} & A_0 + 4\frac{((-m^2-1)\mu^2 + m^2 + \mu^4)(a_0+\lambda)^2\mathrm{cd}(\xi)}{(4\mu^3 + 2(-m^2-1)\mu)(\mu\,\mathrm{cd}(\xi)+1)} - 8\frac{(a_0+\lambda)^2((-m^2-1)\mu^2 + m^2 + \mu^4)^2\mathrm{cd}^2(\xi, m)}{(4\mu^3 + 2(-m^2-1)\mu)^2(\mu\,\mathrm{cd}(\xi)+1)^2} \\ & \mp 8\frac{((-m^2-1)\mu^2 + m^2 + \mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}\mathrm{sn}(\xi)(1-m^2)}{(4\mu^3 + 2(-m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{dn}^2(\xi)(\mu\,\mathrm{cd}(\xi)+1)} \\ & \pm 8\frac{((-m^2-1)\mu^2 + m^2 + \mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\mu\beta\sqrt{\alpha+\beta^2}\mathrm{cd}(\xi)\mathrm{sn}(\xi)(1-m^2)}{(4\mu^3 + 2(-m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{dn}^2(\xi)(\mu\,\mathrm{cd}(\xi)+1)^2}, \end{aligned} \tag{3.13.2}$$

where $\xi = k(x + \lambda t)$, k and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 10. When $h_1 = h_3 = 0$, $h_0 = 1 - m^2$, $h_2 = 2m^2 - 1$ and $h_4 = -m^2$, we obtain the following solutions for the WBK equation, as follows:

$$u_{101} = a_0 - 4\frac{(a_0+\lambda)(-(1-m^2)\mu^4 - (2m^2-1)\mu^2 + m^2)\mathrm{cn}(\xi)}{(-4(1-m^2)\mu^3 - 2(2m^2-1)\mu)(\mu\,\mathrm{cn}(\xi)+1)}, \tag{3.14.1}$$

$$\begin{aligned} v_{101} = {} & A_0 + 4\frac{(a_0+\lambda)^2(-(1-m^2)\mu^4 - (2m^2-1)\mu^2 + m^2)\mathrm{cn}(\xi)}{(-4(1-m^2)\mu^3 - 2(2m^2-1)\mu)(\mu\,\mathrm{cn}(\xi)+1)} - 8\frac{(a_0+\lambda)^2(-(1-m^2)\mu^4 - (2m^2-1)\mu^2 + m^2)^2\mathrm{cn}^2(\xi)}{(-4(1-m^2)\mu^3 - 2(2m^2-1)\mu)^2(\mu\,\mathrm{cn}(\xi)+1)^2} \\ & \mp 8\frac{(a_0+\lambda)^2((2m^2-1)\mu^2 - m^2 + (1-m^2)\mu^4)^{\frac{3}{2}}\beta\sqrt{\alpha+\beta^2}\,\mathrm{dn}(\xi)\mathrm{sn}(\xi)}{(4(1-m^2)\mu^3 + 2(2m^2-1)\mu)^2(\alpha+\beta^2)(\mu\,\mathrm{cn}(\xi)+1)} \\ & \pm 8\frac{(a_0+\lambda)^2((1-m^2)\mu^4 + (2m^2+1)\mu^2 + m^2)^{\frac{3}{2}}\mu\beta\sqrt{\alpha+\beta^2}\,\mathrm{cn}(\xi)\mathrm{dn}(\xi)\mathrm{sn}(\xi)}{(-4(1-m^2)\mu^3 - 2(2m^2-1)\mu)^2(\alpha+\beta^2)(\mu\,\mathrm{cn}(\xi)+1)^2}, \end{aligned} \tag{3.14.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 11. When $h_1 = h_3 = 0$, $h_0 = m^2 - 1$, $h_2 = 2 - m^2$ and $h_4 = -1$, we obtain the following solutions for the WBK equation, as follows:

$$u_{111} = a_0 - 4\frac{((2-m^2)\mu^2 - 1 + (m^2-1)\mu^4)(a_0+\lambda)\mathrm{dn}(\xi)}{(4(m^2-1)\mu^3 + 2(2-m^2)\mu)(\mu\,\mathrm{dn}(\xi)+1)}, \tag{3.15.1}$$

$$\begin{aligned} v_{111} = {} & A_0 + 4\frac{((2-m^2)\mu^2 - 1 + (m^2-1)\mu^4)(a_0+\lambda)^2\mathrm{dn}(\xi)}{(4(m^2-1)\mu^3 + 2(2-m^2)\mu)(\mu\,\mathrm{dn}(\xi)+1)} - 8\frac{(a_0+\lambda)^2((2-m^2)\mu^2 - 1 + (m^2-1)\mu^4)^2\mathrm{dn}^2(\xi)}{(4(m^2-1)\mu^3 + 2(2-m^2)\mu)^2(\mu\,\mathrm{dn}(\xi,m)+1)^2} \\ & \mp 8\frac{((2-m^2)\mu^2 - 1 + (m^2-1)\mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}\,m^2\mathrm{cn}(\xi)\mathrm{sn}(\xi)}{(4(m^2-1)\mu^3 + 2(2-m^2)\mu)^2(\alpha+\beta^2)(\mu\,\mathrm{dn}(\xi)+1)} \\ & \pm 8\frac{((2-m^2)\mu^2 - 1 + (m^2-1)\mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\mu\beta\sqrt{\alpha+\beta^2}\,\mathrm{dn}(\xi)m^2\mathrm{cn}(\xi)\mathrm{sn}(\xi)}{(4(m^2-1)\mu^3 + 2(2-m^2)\mu)^2(\alpha+\beta^2)(\mu\,\mathrm{dn}(\xi)+1)^2}, \end{aligned} \tag{3.15.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 12. When $h_1 = h_3 = 0$, $h_0 = m^2$, $h_2 = -(1 + m^2)$ and $h_4 = 1$, we obtain the following solutions for the WBK equation, as follows:

$$u_{121} = a_0 - 4\frac{((-m^2-1)\mu^2 + 1 + m^2\mu^4)(a_0+\lambda)\mathrm{ns}(\xi)}{(4m^2\mu^3 + 2(-m^2-1)\mu)(\mu\,\mathrm{ns}(\xi)+1)}, \tag{3.16.1}$$

$$\begin{aligned} v_{121} = {} & A_0 + 4\frac{((-m^2-1)\mu^2 + 1 + m^2\mu^4)(a_0+\lambda)^2\mathrm{ns}(\xi)}{(4m^2\mu^3 + 2(-m^2-1)\mu)(\mu\,\mathrm{ns}(\xi)+1)} - 8\frac{((-m^2-1)\mu^2 + 1 + m^2\mu^4)^2(a_0+\lambda)^2\mathrm{ns}^2(\xi)}{(4m^2\mu^3 + 2(-m^2-1)\mu)^2(\mu\,\mathrm{ns}(\xi)+1)^2} \\ & \mp 8\frac{((-m^2-1)\mu^2 + 1 + m^2\mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}\,\mathrm{cn}(\xi)\mathrm{dn}(\xi)}{(4m^2\mu^3 + 2(-m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{sn}^2(\xi)(\mu\,\mathrm{ns}(\xi)+1)} \\ & \pm 8\frac{(a_0+\lambda)^2((-m^2-1)\mu^2 + 1 + m^2\mu^4)^{\frac{3}{2}}\mu\beta\sqrt{\alpha+\beta^2}\,\mathrm{ns}(\xi)\mathrm{cn}(\xi)\mathrm{dn}(\xi)}{(4m^2\mu^3 + 2(-m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{sn}^2(\xi)(\mu\,\mathrm{ns}(\xi)+1)^2}, \end{aligned} \tag{3.16.2}$$

where $\xi = k(x + \lambda t)$, k and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 13. When $h_1 = h_3 = 0$, $h_0 = m^2$, $h_2 = -(1 + m^2)$ and $h_4 = 1$, we obtain the following solutions for the WBK equation, as follows:

$$u_{131} = a_0 - 4\frac{((-m^2-1)\mu^2+1+m^2\mu^4)(a_0+\lambda)\mathrm{dc}(\xi)}{(4m^2\mu^3+2(-m^2-1)\mu)(\mu\,\mathrm{dc}(\xi)+1)}, \tag{3.17.1}$$

$$\begin{aligned} v_{131} = {} & A_0 + 4\frac{((-m^2-1)\mu^2+1+m^2\mu^4)(a_0+\lambda)^2\mathrm{dc}(\xi)}{(4m^2\mu^3+2(-m^2-1)\mu)(\mu\,\mathrm{dc}(\xi)+1)} - 8\frac{(a_0+\lambda)^2((-m^2-1)\mu^2+1+m^2\mu^4)^2\mathrm{dc}^2(\xi)}{(4m^2\mu^3+2(-m^2-1)\mu)^2(\mu\,\mathrm{dc}(\xi)+1)^2} \\ & \mp 8\frac{((-m^2-1)\mu^2+1+m^2\mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}\mathrm{sn}(\xi)(1-m^2)}{(4m^2\mu^3+2(-m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{cn}^2(\xi)(\mu\,\mathrm{dc}(\xi)+1)} \\ & \pm 8\frac{((-m^2-1)\mu^2+1+m^2\mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\mu\beta\sqrt{\alpha+\beta^2}\mathrm{dc}(\xi)\mathrm{sn}(\xi)(1-m^2)}{(4m^2\mu^3+2(-m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{cn}^2(\xi)(\mu\,\mathrm{dc}(\xi)+1)^2}, \end{aligned} \tag{3.17.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 14. When $h_1 = h_3 = 0$, $h_0 = -m^2$, $h_2 = 2m^2 - 1$ and $h_4 = 1 - m^2$, we obtain the following solutions for the WBK equation, as follows:

$$u_{141} = a_0 - 4\frac{((2m^2-1)\mu^2+1-m^2-m^2\mu^4)(a_0+\lambda)\mathrm{nc}(\xi)}{(-4m^2\mu^3+2(2m^2-1)\mu)(\mu\,\mathrm{nc}(\xi)+1)}, \tag{3.18.1}$$

$$\begin{aligned} v_{141} = {} & A_0 + 4\frac{((2m^2-1)\mu^2+1-m^2-m^2\mu^4)(a_0+\lambda)^2\mathrm{nc}(\xi)}{(-4m^2\mu^3+2(2m^2-1)\mu)(\mu\,\mathrm{nc}(\xi)+1)} - 8\frac{(a_0+\lambda)^2((2m^2-1)\mu^2+1-m^2-m^2\mu^4)^2\mathrm{nc}^2(\xi)}{(-4m^2\mu^3+2(2m^2-1)\mu)^2(\mu\,\mathrm{nc}(\xi)+1)^2} \\ & \mp 8\frac{((2m^2-1)\mu^2+1-m^2-m^2\mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}\mathrm{dn}(\xi)\mathrm{sn}(\xi)}{(4m^2\mu^3-2(2m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{cn}^2(\xi)(\mu\,\mathrm{nc}(\xi)+1)} \\ & \pm 8\frac{(a_0+\lambda)^2((2m^2-1)\mu^2+1-m^2-m^2\mu^4)^{\frac{3}{2}}\mu\beta\sqrt{\alpha+\beta^2}\mathrm{nc}(\xi)\mathrm{dn}(\xi)\mathrm{sn}(\xi)}{(-4m^2\mu^3+2(2m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{cn}^2(\xi)(\mu\,\mathrm{nc}(\xi)+1)^2}, \end{aligned} \tag{3.18.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 15. When $h_1 = h_3 = 0$, $h_0 = -1$, $h_2 = 2 - m^2$ and $h_4 = m^2 - 1$, we obtain the following solutions for the WBK equation, as follows:

$$u_{151} = a_0 - 4\frac{((2-m^2)\mu^2+m^2-1-\mu^4)(a_0+\lambda)\mathrm{nd}(\xi)}{(-4\mu^3+2(2-m^2)\mu)(\mu\,\mathrm{nd}(\xi)+1)}, \tag{3.19.1}$$

$$\begin{aligned} v_{151} = {} & A_0 + 4\frac{((2-m^2)\mu^2+m^2-1-\mu^4)(a_0+\lambda)^2\mathrm{nd}(\xi)}{(-4\mu^3+2(2-m^2)\mu)(\mu\,\mathrm{nd}(\xi)+1)} - 8\frac{(a_0+\lambda)^2((2-m^2)\mu^2+m^2-1-\mu^4)^2\mathrm{nd}^2(\xi)}{(-4\mu^3+2(2-m^2)\mu)^2(\mu\,\mathrm{nd}(\xi)+1)^2} \\ & \mp 8\frac{((2-m^2)\mu^2+m^2-1-\mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}m^2\mathrm{cn}(\xi)\mathrm{sn}(\xi)}{(-4\mu^3+2(2-m^2)\mu)^2(\alpha+\beta^2)\mathrm{dn}^2(\xi)(\mu\,\mathrm{nd}(\xi)+1)} \\ & \pm 8\frac{((2-m^2)\mu^2+m^2-1-\mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\mu\beta\sqrt{\alpha+\beta^2}\mathrm{nd}(\xi)m^2\mathrm{cn}(\xi)\mathrm{sn}(\xi)}{(-4\mu^3+2(2-m^2)\mu)^2(\alpha+\beta^2)\mathrm{dn}^2(\xi)(\mu\,\mathrm{nd}(\xi)+1)^2}, \end{aligned} \tag{3.19.2}$$

where $\xi = k(x + \lambda t)$, k and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 16. When $h_1 = h_3 = 0$, $h_0 = 1 - m^2$, $h_2 = 2 - m^2$ and $h_4 = 1$, we obtain the following solutions for the WBK equation, as follows:

$$u_{161} = a_0 - 4\frac{(-(1-m^2)\mu^4-1-(2-m^2)\mu^2)(a_0+\lambda)\mathrm{cs}(\xi)}{(-4(1-m^2)\mu^3-2(2-m^2)\mu)(\mu\,\mathrm{cs}(\xi)+1)}, \tag{3.20.1}$$

$$v_{161} = A_0 + 4\frac{(-(1-m^2)\mu^4 - 1 - (2-m^2)\mu^2)(a_0+\lambda)^2\mathrm{cs}(\xi)}{(-4(1-m^2)\mu^3 - 2(2-m^2)\mu)(\mu\,\mathrm{cs}(\xi)+1)} - 8\frac{(a_0+\lambda)^2(-(1-m^2)\mu^4 - 1 - (2-m^2)\mu^2)^2\mathrm{cs}^2(\xi)}{(-4(1-m^2)\mu^3 - 2(2-m^2)\mu)^2(\mu\,\mathrm{cs}(\xi)+1)^2}$$
$$\mp 8\frac{((1-m^2)\mu^4 + 1 + (2-m^2)\mu^2)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}\,\mathrm{dn}(\xi,m)}{(-4(1-m^2)\mu^3 - 2(2-m^2)\mu)^2(\alpha+\beta^2)\mathrm{sn}^2(\xi)(\mu\,\mathrm{cs}(\xi)+1)}$$
$$\pm 8\frac{(a_0+\lambda)^2(-(1-m^2)\mu^4 - 1 - (2-m^2)\mu^2)^{\frac{3}{2}}\mu\beta\sqrt{\alpha+\beta^2}\,\mathrm{cs}(\xi)\mathrm{dn}(\xi)}{(-4(1-m^2)\mu^3 - 2(2-m^2)\mu)^2(\alpha+\beta^2)\mathrm{sn}^2(\xi)(\mu\,\mathrm{cs}(\xi,m)+1)^2}, \tag{3.20.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 17. When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = 2 - m^2$ and $h_4 = 1 - m^2$, we obtain the following solutions for the WBK equation, as follows:

$$u_{171} = a_0 - 4\frac{((2-m^2)\mu^2 + 1 - m^2 + \mu^4)(a_0+\lambda)\mathrm{sc}(\xi)}{(4\mu^3 + 2(2-m^2)\mu)(\mu\,\mathrm{sc}\xi + 1)}, \tag{3.21.1}$$

$$v_{171} = A_0 + 4\frac{((2-m^2)\mu^2 + 1 - m^2 + \mu^4)(a_0+\lambda)^2\mathrm{sc}(\xi)}{(4\mu^3 + 2(2-m^2)\mu)(\mu\,\mathrm{sc}(\xi)+1)} - 8\frac{(a_0+\lambda)^2((2-m^2)\mu^2 + 1 - m^2 + \mu^4)^2\mathrm{sc}^2(\xi)}{(4\mu^3 + 2(2-m^2)\mu)^2(\mu\,\mathrm{sc}(\xi)+1)^2}$$
$$\mp 8\frac{((2-m^2)\mu^2 + 1 - m^2 + \mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}\,\mathrm{dn}(\xi)}{(4\mu^3 + 2(2-m^2)\mu)^2(\alpha+\beta^2)\mathrm{cn}^2(\xi)(\mu\,\mathrm{sc}(\xi)+1)}$$
$$\pm 8\frac{((2-m^2)\mu^2 + 1 - m^2 + \mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\mu\beta\sqrt{\alpha+\beta^2}\,\mathrm{sc}(\xi)\mathrm{dn}(\xi)}{(4\mu^3 + 2(2-m^2)\mu)^2(\alpha+\beta^2)\mathrm{cn}^2(\xi)(\mu\,\mathrm{sc}(\xi)+1)^2}, \tag{3.21.2}$$

where $\xi = k(x + \lambda t)$, k and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 18. When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = 2m^2 - 1$ and $h_4 = m^2(m^2 - 1)$, we obtain the following solutions for the WBK equation, as follows:

$$u_{181} = a_0 - 4\frac{((2m^2-1)\mu^2 + m^2(m^2-1) + \mu^4)(a_0+\lambda)\mathrm{sd}(\xi)}{(4\mu^3 + 2(2m^2-1)\mu)(\mu\,\mathrm{sd}(\xi)+1)}, \tag{3.22.1}$$

$$v_{181} = A_0 + 4\frac{((2m^2-1)\mu^2 + m^2(m^2-1) + \mu^4)(a_0+\lambda)^2\mathrm{sd}(\xi)}{(4\mu^3 + 2(2m^2-1)\mu)(\mu\,\mathrm{sd}(\xi)+1)} - 8\frac{((2m^2-1)\mu^2 + m^2(m^2-1) + \mu^4)^2(a_0+\lambda)^2\mathrm{sd}^2(\xi)}{(4\mu^3 + 2(2m^2-1)\mu)^2(\mu\,\mathrm{sd}(\xi)+1)^2}$$
$$\mp 8\frac{((2m^2-1)\mu^2 + m^2(m^2-1) + \mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\beta\sqrt{\alpha+\beta^2}\,\mathrm{cn}(\xi)}{(4\mu^3 + 2(2m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{dn}^2(\xi)(\mu\,\mathrm{sd}(\xi)+1)}$$
$$\pm 8\frac{((2m^2-1)\mu^2 + m^2(m^2-1) + \mu^4)^{\frac{3}{2}}(a_0+\lambda)^2\mu\beta\sqrt{\alpha+\beta^2}\,\mathrm{sd}(\xi)\mathrm{cn}(\xi)}{(4\mu^3 + 2(2m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{dn}^2(\xi)(\mu\,\mathrm{sd}(\xi)+1)^2}, \tag{3.22.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 19. When $h_1 = h_3 = 0$, $h_0 = m^2(m^2 - 1)$, $h_2 = 2m^2 - 1$ and $h_4 = 1$, we obtain the following solutions for the WBK equation, as follows:

$$u_{191} = a_0 - 4\frac{(a_0+\lambda)((2m^2-1)\mu^2 + 1 + m^2(m^2-1)\mu^4)\mathrm{ds}(\xi)}{(4m^2(m^2-1)\mu^3 + 2(2m^2-1)\mu)(\mu\,\mathrm{ds}(\xi)+1)}, \tag{3.23.1}$$

$$v_{191} = A_0 + 4\frac{(a_0+\lambda)^2((2m^2-1)\mu^2+1+m^2(m^2-1)\mu^4)\mathrm{ds}(\xi)}{(4m^2(m^2-1)\mu^3+2(2m^2-1)\mu)(\mu\,\mathrm{ds}(\xi)+1)} - 8\frac{(a_0+\lambda)^2((2m^2-1)\mu^2+1+m^2(m^2-1)\mu^4)^2\mathrm{ds}^2(\xi)}{(4m^2(m^2-1)\mu^3+2(2m^2-1)\mu)^2(\mu\,\mathrm{ds}(\xi)+1)^2}$$
$$\mp 8\frac{(a_0+\lambda)^2((2m^2-1)\mu^2+1+m^2(m^2-1)\mu^4)^{\frac{3}{2}}\beta\sqrt{\alpha+\beta^2}\mathrm{cn}(\xi)}{(4m^2(m^2-1)\mu^3+2(2m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{sn}^2(\xi)(\mu\,\mathrm{ds}(\xi)+1)}$$
$$\pm 8\frac{(a_0+\lambda)^2((2m^2-1)\mu^2+1+m^2(m^2-1)\mu^4)^{\frac{3}{2}}\mu\beta\sqrt{\alpha+\beta^2}\mathrm{ds}(\xi)\mathrm{cn}(\xi)}{(4m^2(m^2-1)\mu^3+2(2m^2-1)\mu)^2(\alpha+\beta^2)\mathrm{sn}^2(\xi)(\mu\,\mathrm{ds}(\xi,m)+1)^2}, \tag{3.23.2}$$

where $\xi = k(x+\lambda t)$, k, a_0 and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 20. When $h_1 = h_3 = 0$, $h_0 = \frac{1}{4}$, $h_2 = \frac{1-2m^2}{2}$ and $h_4 = \frac{1}{4}$, we obtain the following solutions for the WBK equation, as follows:

$$u_{201} = a_0 + \frac{(\mu^4+1+2(1-m^2)\mu^2)(a_0+\lambda)(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi))}{(-\mu^3-2(1-m^2)\mu)(\mu(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi,m))+1)}, \tag{3.24.1}$$

$$v_{201} = A_0 + \frac{(\mu^4+1+2(1-m^2)\mu^2)(a_0+\lambda)^2(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi))}{(\mu^3+(1-m^2)\mu)(\mu(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi))+1)} - \frac{(a_0+\lambda)^2(\mu^4+1+2(1-m^2)\mu^2)^2(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi))^2}{2(\mu^3+(1-m^2)\mu)^2(\mu(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi))+1)^2}$$
$$\mp\frac{(a_0+\lambda)^2(2(1-m^2)\mu^2+1+\mu^4)^{\frac{3}{2}}\beta\sqrt{\alpha+\beta^2}(\mathrm{cn}(\xi)\mathrm{dn}(\xi)\pm\mathrm{dn}(\xi))}{(\mu^3+2(1-m^2)\mu)^2(\alpha+\beta^2)(\mu(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi))+1)\mathrm{sn}^2(\xi)}$$
$$\pm\frac{(a_0+\lambda)^2(2(1-m^2)\mu^2+1+\mu^4)^{\frac{3}{2}}\beta\sqrt{\alpha+\beta^2}(\mathrm{cn}(\xi)\mathrm{dn}(\xi)\pm\mathrm{dn}(\xi))\mu(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi))}{(\mu^3+2(1-m^2)\mu)^2(\alpha+\beta^2)(\mu(\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi))+1)^2\mathrm{sn}^2(\xi)}, \tag{3.24.2}$$

where $\xi = k(x+\lambda t)$, k, a_0 and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 21. When $h_1 = h_3 = 0$, $h_0 = \frac{1-m^2}{4}$, $h_2 = \frac{1+m^2}{2}$ and $h_4 = \frac{1-m^2}{4}$, we obtain the following solutions for the WBK equation, as follows:

$$u_{211} = a_0 - \frac{((1-m^2)\mu^4+1+2(1+m^2)\mu^2-m^2)(a_0+\lambda)(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi,m))}{((1-m^2)\mu^3+(1+m^2)\mu)(\mu(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi))+1)}, \tag{3.25.1}$$

$$v_{211} = A_0 + \frac{((1-m^2)\mu^4+1+2(1+m^2)\mu^2+m^2)(a_0+\lambda)^2(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi))}{((1-m^2)\mu^3+(1+m^2)\mu)(\mu(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi))+1)}$$
$$-\frac{((1-m^2)\mu^4+1+2(1+m^2)\mu^2-m^2)^2(a_0+\lambda)^2(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi))^2}{2((1-m^2)\mu^3+(1+m^2)\mu)^2(\mu(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi))+1)^2}$$
$$\mp\frac{((1-m^2)\mu^4+1+2(1+m^2)\mu^2-m^2)^{\frac{3}{2}}(a_0+\lambda)^2\beta(\mathrm{dn}(\xi)\mathrm{sn}(\xi)\pm\mathrm{dn}(\xi))}{((1-m^2)\mu^3-(1+m^2)\mu)^2\sqrt{\alpha+\beta^2}(\mu(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi))+1)\mathrm{cn}^2(\xi)}$$
$$\pm\frac{((1-m^2)\mu^4+1+2(1+m^2)\mu^2-m^2)^{\frac{3}{2}}(a_0+\lambda)^2\beta(\mathrm{dn}(\xi)\mathrm{sn}(\xi)\pm\mathrm{dn}(\xi))\mu(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi))}{((1-m^2)\mu^3-(1+m^2)\mu)^2\sqrt{\alpha+\beta^2}(\mu(\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi))+1)^2\mathrm{cn}^2(\xi)}, \tag{3.25.2}$$

where $\xi = k(x+\lambda t)$, k, a_0 and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 22. When $h_1 = h_3 = 0$, $h_0 = \frac{m^4}{4}$, $h_2 = \frac{m^2-2}{2}$ and $h_4 = \frac{1}{4}$, we obtain the following solutions for the WBK equation, as follows:

$$u_{221} = a_0 - \frac{(a_0+\lambda)(2(m^2-2)\mu^2+1+m^4\mu^4)(\mathrm{ns}(\xi)\pm\mathrm{ds}(\xi))}{(m^4\mu^3+(m^2-2)\mu)(\mu(\mathrm{ns}(\xi)\pm\mathrm{ds}(\xi))+1)}, \tag{3.26.1}$$

$$v_{221} = A_0 + \frac{(a_0+\lambda)^2 2((m^2-2)\mu^2+1+m^4\mu^4)(\mathrm{ns}(\xi)\pm\mathrm{ds}(\xi))}{(m^4\mu^3+(m^2-2)\mu)(\mu(\mathrm{ns}(\xi)\pm\mathrm{ds}(\xi))+1)} - \frac{(a_0+\lambda)^2(2(m^2-2)\mu^2+1+m^4\mu^4)^2(\mathrm{ns}(\xi)\pm\mathrm{ds}(\xi))^2}{2(m^4\mu^3+(m^2-2)\mu)^2(\mu(\mathrm{ns}(\xi)\pm\mathrm{ds}(\xi))+1)^2}$$
$$\mp \frac{(a_0+\lambda)^2(2(m^2-2)\mu^2+1+m^4\mu^4)^{\frac{3}{2}}\beta\sqrt{\alpha+\beta^2}(\mathrm{cn}(\xi)\mathrm{dn}(\xi)\pm\mathrm{cn}(\xi))}{(m^4\mu^3+(m^2-2)\mu)^2(\alpha+\beta^2)(\mu(\mathrm{ns}(\xi)\pm\mathrm{ds}(\xi))+1)\mathrm{sn}^2(\xi)}$$
$$\pm \frac{(a_0+\lambda)^2(2(m^2-2)\mu^2+1+m^4\mu^4)^{\frac{3}{2}}\beta\sqrt{\alpha+\beta^2}(\mathrm{cn}(\xi)\mathrm{dn}(\xi)\pm\mathrm{cn}(\xi))\mu(\mathrm{ns}(\xi)\pm\mathrm{ds}(\xi))}{(m^4\mu^3+(m^2-2)\mu)^2(\alpha+\beta^2)(\mu(\mathrm{ns}(\xi)\pm\mathrm{ds}(\xi))+1)^2\mathrm{sn}^2(\xi)}, \tag{3.26.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

Family 23. When $h_1 = h_3 = 0$, $h_0 = \frac{m^2}{4}$, $h_2 = \frac{m^2-2}{2}$ and $h_4 = \frac{m^2}{4}$, we obtain the following solutions for the WBK equation, as follows:

$$u_{231} = a_0 - \frac{(2(m^2-2)\mu^2+m^2+m^2\mu^4)(a_0+\lambda)(\mathrm{sn}(\xi)\pm i\,\mathrm{cn}(\xi))}{(m^2\mu^3+(m^2-2)\mu)(\mu(\mathrm{sn}(\xi)\pm i\,\mathrm{cn}(\xi))+1)}, \tag{3.27.1}$$

$$v_{231} = A_0 + \frac{(2(m^2-2)\mu^2+m^2+m^2\mu^4)(a_0+\lambda)^2(\mathrm{sn}(\xi)\pm i\,\mathrm{cn}(\xi))}{(m^2\mu^3+(m^2-2)\mu)(\mu(\mathrm{sn}(\xi)\pm i\,\mathrm{cn}(\xi))+1)}$$
$$- \frac{(2(m^2-2)\mu^2+m^2+m^2\mu^4)^2(a_0+\lambda)^2(\mathrm{sn}(\xi)\pm i\,\mathrm{cn}(\xi))^2}{2(m^2\mu^3+(m^2-2)\mu)^2(\mu(\mathrm{sn}(\xi)\pm i\,\mathrm{cn}(\xi))+1)^2}$$
$$\mp \frac{(a_0+\lambda)^2(2(m^2-2)\mu^2+m^2+m^2\mu^4)^{\frac{3}{2}}\beta\sqrt{\alpha+\beta^2}(\mathrm{cn}(\xi)\mathrm{dn}(\xi)\mp i\,\mathrm{dn}(\xi)\mathrm{sn}(\xi))}{(m^2\mu^3+(m^2-2)\mu)^2(\alpha+\beta^2)(\mu(\mathrm{sn}(\xi)\pm i\,\mathrm{cn}(\xi))+1)}$$
$$\pm \frac{(a_0+\lambda)^2(2(m^2-2)\mu^2+m^2+m^2\mu^4)^{\frac{3}{2}}\beta\mu(\mathrm{cn}(\xi)\mathrm{dn}(\xi)\mp i\,\mathrm{dn}(\xi)\mathrm{sn}(\xi))(\mathrm{sn}(\xi)\pm i\,\mathrm{cn}(\xi))}{(m^2\mu^3+(m^2-2)\mu)^2\sqrt{\alpha+\beta^2}(\mu(\mathrm{sn}(\xi)\pm i\,\mathrm{cn}(\xi))+1)}, \tag{3.27.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.4), μ, a_0 and λ are arbitrary constants.

4. Summary and conclusions

In summary, we have proposed an unified algebraic method: general equation rational expansion method with symbolic computation, which greatly exceeds the applicability of the existing the tanh method, the extended tanh method, the general tanh method, the projective Riccati equation method and the general projective Riccati equation method in obtaining multiple travelling wave solutions of general nonlinear evolution equations. In fact, we naturally present a more general ansatz, which reads,

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}\phi^j(\xi) + b_{ij}\phi^{j-1}(\xi)\sqrt{\sum_{l=0}^{r} h_l\phi^l(\xi)} + c_{ij}\frac{\sqrt{\sum_{l=0}^{r} h_l\phi^l(\xi)}}{\phi^j(\xi)} + d_{ij}\phi^{-j}(\xi)}{\left(\mu_{j1}\phi(\xi) + \mu_{j2}\sqrt{\sum_{l=0}^{r} h_l\phi^l(\xi)} + \mu_{j3}\frac{\sqrt{\sum_{l=0}^{r} h_l\phi^l(\xi)}}{\phi(\xi)} + \mu_{j4}\phi^{-1}(\xi) + 1\right)^j}, \tag{4.1}$$

and the new variable $\phi = \phi(\xi)$ satisfying

$$(\phi'(\xi))^2 = \left(\frac{\mathrm{d}\phi(\xi)}{\mathrm{d}\xi}\right)^2 = \sum_{l=0}^{r} h_l\phi^l(\xi), \tag{4.2}$$

where a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}, μ_{j1}, μ_{j2}, μ_{j3}, μ_{j4}, h_l $(i = 1,2,\ldots;\ j = 1,2,\ldots,m_i;\ l = 1,2,\ldots,r)$ and ξ are differentiable function to be determined later. Therefore, for some nonlinear equations, more types of non-travelling solutions, such as soliton-like solutions, would be expected.

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