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A NEW GENERAL ALGEBRAIC METHOD WITH SYMBOLIC COMPUTATION TO CONSTRUCT NEW TRAVELING SOLUTION FOR THE $(1+1)$-DIMENSIONAL DISPERSIVE LONG WAVE EQUATION

YONG CHEN
Nonlinear Science Center and Department of Mathematics
Ningbo University, Ningbo 315211, China
Department of Physics, Shanghai Jiao Tong University
Shanghai 200030, China
M. M. Key Laboratory, Chinese Academy of Sciences
Beijing 100080, China
chenyong@dlut.edu.cn

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A new algebraic method, named Riccati equation rational expansion (RERE) method, is devised for constructing multiple traveling wave solutions for nonlinear evolution equations (NEEs). With the aid of symbolic computation, we choose $(1+1)$-dimensional dispersive long wave equation (DLWE) to illustrate our method. As a result, we obtain many types of solutions including rational form solitary wave solutions, triangular periodic wave solutions and rational wave solutions.

Keywords: Riccati equation rational expansion method; triangular periodic wave solutions; rational form solitary wave solutions; symbolic computation.

1. Introduction

The tanh method provides a straightforward and effective algorithm to obtain particular traveling solutions for a large number of NEEs. Generally speaking, the various extensions and improvement of tanh method can be classified into two classes: One is called the direct method, which represents the solutions of given NEEs as the sum of a polynomial in exponential solutions.[1–3] The other is called the subequation method, which consists of looking for the solutions of given NEEs as a polynomial in a variable which satisfies an equation or equations (named subequation).[4–10]

The present work is motivated by the desire to present a new subequation method, named Riccati equation rational expansion (RERE) method, by proposing a more general ansätz so that it can be used to obtained more types and general formal solutions which contain not only the results obtained by using the known various tanh function methods[1,4–10] but also other types of solutions. For illustration, we apply the generalized method to solve $(1+1)$-dimensional dispersive long wave equation (DLWE) and successfully construct new and more general solutions including rational form solitary wave solutions, triangular periodic wave solutions and rational wave solutions for the $(1+1)$-DLWE.

This paper is organized as follows. In Sec. 2, we summarize the RERE method. In Sec. 3, we apply the RERE method to $(1+1)$-DLWE and obtain many new form solutions. Conclusions will be presented in Sec. 4.

2. The Riccati Equation Rational Expansion Method

In the following, we would like to outline the main steps of our method:

Step 1. For a given NEEs system with some physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0\,, \tag{1}$$

by using the wave transformation

$$u_i(x,y,t) = U_i(\xi)\,, \qquad \xi = k(x + ly + \lambda t)\,, \tag{2}$$

where k, l and λ are constants to be determined later, the nonlinear partial differential equation (1) is reduced to a nonlinear ordinary differential equation (ODE):

$$G_i(U_i, U_i', U_i'', \ldots) = 0\,. \tag{3}$$

Step 2. We introduce a new ansätz in terms of finite rational formal expansion in the following forms:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}\phi^j(\xi) + b_{ij}\phi^{j-1}(\xi)\sqrt{R+\phi^2(\xi)}}{(\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1)^j}\,, \tag{4}$$

and the new variable $\phi = \phi(\xi)$ satisfying

$$\phi' - (R+\phi^2) = \frac{d\phi}{d\xi} - (R+\phi^2) = 0\,, \tag{5}$$

where R, a_{i0}, a_{ij} and b_{ij} $(i = 1, 2, \ldots, j = 1, 2, \ldots, m_i)$ are constants to be determined later.

Step 3. By balancing the highest-order derivative term and the nonlinear term in Eq. (3), we can find the balance constant m_i (m_i is usually a positive integer). If m_i is a fraction or a negative integer, we first make the transformation,

$$U_i(\xi) = \phi_i^{m_i}(\xi)\,, \tag{6}$$

then substitute Eq. (6) into Eq. (3) and return to determine balance constant m_i again.

Step 4. Substitute Eq. (4) into Eq. (3) along with Eq. (5) and then set all coefficients of $\phi^i(\xi)(\sqrt{R+\phi^2(\xi)})^j$ of the resulting system's numerator $(i = 1, 2, \ldots,$

$j = 0$, 1) to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, μ_1, μ_2, a_{i0}, a_{ij} and b_{ij} $(i = 1, 2, \ldots, j = 1, 2, \ldots, m_i)$.

Step 5. Solving the over-determined system of nonlinear algebraic equations by use of *Maple*, we would end up with the explicit expressions for k, μ_1, μ_2, a_{i0}, a_{ij} and b_{ij} $(i = 1, 2, \ldots, j = 1, 2, \ldots, m_i)$.

Step 6. It is well-known that the general solutions of Eq. (5) are
(1) when $R < 0$,

$$\phi(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi)\,, \qquad \phi(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi)\,, \tag{7}$$

(2) when $R = 0$,

$$\phi(\xi) = -\frac{1}{\xi}\,, \tag{8}$$

(3) when $R > 0$,

$$\phi(\xi) = \sqrt{R}\tan(\sqrt{R}\xi)\,, \qquad \phi(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi)\,. \tag{9}$$

Thus according to Eqs. (2), (4), (7)–(9) and the conclusions in Step 5, we can obtain the following rational formal traveling-wave solutions of Eq. (1).

(1) when $R < 0$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(-\sqrt{-R}\tanh(\sqrt{-R}\xi))^j \pm b_{ij}(-\sqrt{-R} \times \tanh(\sqrt{-R}\xi))^{j-1}\sqrt{-R}i\,\mathrm{sech}(\sqrt{-R}\xi)}{(1 - \mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm \mu_2\sqrt{-R}i\,\mathrm{sech}(\sqrt{-R}\xi))^j}\,, \tag{10a}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(-\sqrt{-R}\coth(\sqrt{-R}\xi))^j \pm b_{ij}(-\sqrt{-R} \times \coth(\sqrt{-R}\xi))^{j-1}\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)}{(1 - \mu_1\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi))^j}\,, \tag{10b}$$

(2) when $R = 0$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{(-1)^j(a_{ij} \mp b_{ij})}{(\xi - \mu_1 \pm \mu_2)^j}\,, \tag{11}$$

(3) when $R > 0$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(\sqrt{R}\tan(\sqrt{R}\xi))^j \pm b_{ij}(\sqrt{R}\tan(\sqrt{R}\xi))^{j-1}\sqrt{R}\sec(\sqrt{R}\xi)}{(1 + \mu_1\sqrt{R}\tan(\sqrt{R}\xi) \pm \mu_2\sqrt{R}\sec(\sqrt{R}\xi))^j}\,, \tag{12a}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(-\sqrt{R}\cot(\sqrt{R}\xi))^j \pm b_{ij}(-\sqrt{R}\cot(\sqrt{R}\xi))^{j-1}\sqrt{R}\csc(\sqrt{R}\xi)}{(1 - \mu_1\sqrt{R}\cot(\sqrt{R}\xi) \pm \mu_2\sqrt{R}\csc(\sqrt{R}\xi))^j}\,, \tag{12b}$$

where $\xi = k(x + ly + \lambda t)$ and $i = \sqrt{-1}$.

Remark. The more general the ansätz is, the more general and more formal the solutions of the NEEs will be. The ansätz proposed here is more general than the ansätz in the tanh function method,[1] extended tanh function method,[4] improved extended tanh function method,[5–7] projective Riccati equations method[8] and general projective Riccati equations method.[9,10] If we set the parameters in Eq. (4) to different values, the above methods can be recovered by the RERE method. The concrete case is as follows:

(1) Setting $\mu_1 = \mu_2 = b_1 = 0$, we just recover the solutions obtained by extended tanh function method[4];
(2) Setting $\mu_1 = \mu_2 = 0$, we just recover the solutions obtained by the improved extended tanh function method[5,6];
(3) Setting $\mu_1 = 0$ and $\mu_2 \neq 0$, we just recover the solutions obtained by the projective Riccati method.[9,10]
(4) The other solutions obtained here, to our knowledge, are all new formal exact solutions of NEEs.

It is clear to see that this method can make further extension which we will present in our following paper.

Firstly, we can naturally present a more general ansätz, which reads,

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{\begin{array}{c} a_{ij}\phi^j(\xi) + b_{ij}\phi^{j-1}(\xi)\sqrt{R+\phi^2(\xi)} \\ + c_{ij}(\sqrt{R+\phi^2(\xi)}/\phi^j(\xi)) + d_{ij}\phi^{-j}(\xi) \end{array}}{\begin{array}{c} (\mu_{j1}\phi(\xi) + \mu_{j2}\sqrt{R+\phi^2(\xi)} \\ + \mu_{j3}(\sqrt{R+\phi^2(\xi)}/\phi(\xi)) + \mu_{j4}\phi^{-1}(\xi) + 1)^j \end{array}}\,, \tag{13}$$

where a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}, μ_{j1}, μ_{j2}, μ_{j3}, μ_{j4} $(i = 1, 2, \ldots, j = 1, 2, \ldots, m_i)$ and ξ are differentiable function to be determined later. We must point that μ_{ij} is no need to equal to μ_{kj}, when $i \neq k$ (this is different with other various existing tanh methods, which all require $\mu_{ij} = \mu_{kj}$, when $i \neq k$), because Eq. (13) is also satisfying solving the recurrent relation or derivative relation for the terms of polynomial for computation closed. Therefore, for some nonlinear equations, more types of solutions would be expected.

Secondly, we can present a more general ansätz, which reads,

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{\begin{array}{c} a_{ij}\phi^j(\xi) + b_{ij}\phi^{j-1}(\xi)\sqrt{\sum_{l=0}^{r} h_l\phi^l(\xi)} \\ + c_{ij}(\sqrt{\sum_{l=0}^{r} h_l\phi^l(\xi)}/\phi^j(\xi)) + d_{ij}\phi^{-j}(\xi) \end{array}}{\begin{array}{c} (\mu_{j1}\phi(\xi) + \mu_{j2}\sqrt{\sum_{l=0}^{r} h_l\phi^l(\xi)} \\ + \mu_{j3}(\sqrt{\sum_{l=0}^{r} h_l\phi^l(\xi)}/\phi(\xi)) + \mu_{j4}\phi^{-1}(\xi) + 1)^j \end{array}}\,, \tag{14}$$

where a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}, μ_{j1}, μ_{j2}, μ_{j3}, μ_{j4}, h_l $(i = 1, 2, \ldots, j = 1, 2, \ldots, m_i$, $l = 1, 2, \ldots, r)$ and ξ are differentiable function to be determined later. Here $\phi(\xi)$ satisfies a more general subequation, e.g.,

$$\phi'(\xi) = \frac{d\phi(\xi)}{d\xi} = \sqrt{\sum_{l=0}^{r} h_l\phi^l(\xi)}\,. \tag{15}$$

We can easily see that they are different ansätz because of the different choices of the subequations.

3. Exact Solutions of the (1 + 1)-Dimensional Dispersive Long Wave Equation

Let us consider the $(1+1)$-dimensional dispersive long wave equation (DLWE), i.e.,

$$\begin{cases} v_t + vv_x + w_x = 0\,, \\ w_t + (wv)_x + \dfrac{1}{3}v_{xxx} = 0\,, \end{cases} \tag{16}$$

where $w-1$ is the elevation of the water wave, v is the surface velocity of water along x-direction. The equation system (16) can be traced back to the works of Broer,[11] Kaup,[12] Jaulent–Miodek,[13] Martinez,[14] Kupershmidt,[15] etc. A good understanding of all solutions of Eq. (16) is very helpful for coastal and civil engineers to apply the nonlinear water wave model in a harbor and coastal design. Therefore, finding more types of exact solutions of Eq. (16) is of fundamental interest in fluid dynamics. There are several papers devoted to this equation.[16–19]

By considering the wave transformations $v(x,t) = V(\xi)$, $w(x,t) = W(\xi)$ and $\xi = k(x+\lambda t)$, we change Eq. (16) to the form

$$\begin{cases} \lambda V' + VV' + W' = 0\,, \\ \lambda W' + (WV)' + \dfrac{1}{3}k^2 V'' = 0\,. \end{cases} \tag{17}$$

According to the proposed method, we expand the solution of Eq. (17) in the form

$$\begin{cases} V(\xi) = a_0 + \displaystyle\sum_{j=1}^{m_v} \frac{a_j\phi^j(\xi) + b_j\phi^{j-1}(\xi)\sqrt{R+\phi^2(\xi)}}{(\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1)^j}\,, \\ W(\xi) = A_0 + \displaystyle\sum_{j=1}^{m_w} \frac{A_j\phi^j(\xi) + B_j\phi^{j-1}(\xi)\sqrt{R+\phi^2(\xi)}}{(\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1)^j}\,, \end{cases}$$

where $\phi(\xi)$ satisfies Eq. (5). Balancing the term V''' with term $(WV)'$ and the term W' with term VV' in Eq. (17) gives $m_v = 1$ and $m_w = 2$. So we have

$$\begin{cases} V(\xi) = a_0 + \dfrac{a_1\phi(\xi) + b_1\sqrt{R+\phi^2(\xi)}}{\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1}\,, \\ W(\xi) = A_0 + \dfrac{A_1\phi(\xi) + B_1\sqrt{R+\phi^2(\xi)}}{\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1} + \dfrac{A_2\phi^2(\xi) + B_2\phi(\xi)\sqrt{R+\phi^2(\xi)}}{(\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1)^2}\,, \end{cases} \tag{18}$$

where $\phi(\xi)$ satisfies Eq. (5).

With the aid of *Maple*, substituting Eq. (18) along with Eq. (5) into Eq. (17), yields a set of algebraic equations for $\phi^i(\xi)(\sqrt{R+\phi^2(\xi)})^j$ $(i = 0, 1, \ldots, j = 0, 1)$.

Setting the coefficients of these terms $\phi^i(\xi)(\sqrt{R+\phi^2(\xi)})^j$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1 A_2, B_2, μ_1, μ_2 and k.

By use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method,[20] solving the over-determined algebraic equations, we get the following results.

Here we just consider the condition $R < 0$.

Case 1.

$$\begin{gathered} k = k\,, \qquad \mu_1 = 0\,, \qquad \mu_2 = \mu_2\,, \qquad a_0 = -\lambda\,, \qquad a_1 = \pm\frac{1}{3}\sqrt{3}k\,, \\ b_1 = \pm\frac{1}{3}\sqrt{3-3\mu_2^2R}k\,, \qquad A_0 = -\frac{1}{3}k^2R\,, \qquad A_1 = 0\,, \\ B_1 = \frac{1}{3}\mu_2Rk^2\,, \qquad A_2 = -\frac{1}{3}k^2\,, \qquad B_2 = \pm\frac{1}{9}\sqrt{3}k^2\sqrt{3-3\mu_2^2R}\,. \end{gathered} \tag{19}$$

Case 2.

$$\begin{gathered} B_1 = \frac{2}{3}\mu_2Rk^2\,, \qquad a_0 = \frac{\pm k\mu_2R - \lambda\sqrt{3-3\mu_2^2R}}{\sqrt{3-3\mu_2^2R}}\,, \qquad \mu_1 = a_1 = A_1 = B_2 = 0\,, \\ \mu_2 = \mu_2\,, \qquad k = k\,, \qquad b_1 = \pm\frac{2}{3}\sqrt{3-3\mu_2^2R}k\,, \qquad A_0 = -\frac{k^2R(-1+2\mu_2^2R)}{3(-1+\mu_2^2R)}\,, \\ A_2 = -\frac{2}{3}k^2\,. \end{gathered} \tag{20}$$

Case 3.

$$\begin{gathered} k = k\,, \qquad \mu_1 = \mu_1\,, \qquad \mu_2 = b_1 = B_1 = B_2 = 0\,, \qquad a_0 = \pm\frac{2}{3}\sqrt{3}kR\mu_1 - \lambda\,, \\ a_1 = \pm\frac{2}{3}\sqrt{3}(\mu_1^2R+1)k\,, \qquad A_0 = -\frac{2}{3}(\mu_1^2R+1)k^2R\,, \\ A_1 = \frac{4}{3}\mu_1Rk^2(\mu_1^2R+1)\,, \qquad A_2 = -\frac{2}{3}k^2(\mu_1^2R+1)^2\,. \end{gathered} \tag{21}$$

Case 4.

$$\begin{gathered} k = k\,, \qquad \mu_1 = \mu_1\,, \qquad a_0 = -\lambda\,, \qquad \mu_2 = a_1 = B_1 = B_2 = 0\,, \\ b_1 = \pm\frac{2}{3}\sqrt{3\mu_1^2R+3k}\,, \qquad A_0 = -\frac{1}{3}Rk^2(2\mu_1^2R+1)\,, \\ A_1 = \frac{4}{3}R(\mu_1^2R+1)k^2\mu_1\,, \qquad A_2 = -\frac{2}{3}k^2(\mu_1^2R+1)^2\,. \end{gathered} \tag{22}$$

Case 5.

$$B_1 = \pm\frac{1}{9}\sqrt{3}k^2\sqrt{3\mu_1^2R+3}\mu_1 R\,, \qquad \mu_2 = 0\,, \qquad a_0 = \pm\frac{1}{3}\sqrt{3}kR\mu_1 - \lambda\,,$$
$$a_1 = \pm\frac{1}{3}\sqrt{3}(\mu_1^2R+1)k\,, \qquad A_0 = -\frac{1}{3}k^2R(\mu_1^2R+1)\,, \qquad b_1 = \pm\frac{1}{3}\sqrt{3\mu_1^2R+3k}\,,$$
$$A_1 = \frac{2}{3}k^2R\mu_1(\mu_1^2R+1)\,, \qquad k = k\,, \qquad \mu_1 = \mu_1\,, \qquad A_2 = -\frac{1}{3}k^2(\mu_1^2R+1)^2\,,$$
$$B_2 = \pm\frac{1}{9}\sqrt{3}k^2\sqrt{3\mu_1^2R+3}(\mu_1^2R+1)\,. \tag{23}$$

Case 6.

$$\mu_1 = \pm 1\,, \qquad \mu_2 = \pm 1\,, \qquad a_0 = \pm\frac{1}{3}\sqrt{3}kR - \lambda\,, \qquad b_1 = \pm\frac{2}{3}\sqrt{3}k\,,$$
$$a_1 = 0\,, \qquad k = k\,, \qquad A_0 = \frac{k^2R(1+R^2)}{3(-1+R)}\,, \qquad A_1 = -\frac{4Rk^2}{3(-1+R)}\,, \tag{24}$$
$$B_1 = -2\frac{k^2R(1+R)}{3(-1+R)}\,, \qquad A_2 = 2\frac{k^2(1+R)}{3(-1+R)}\,, \qquad B_2 = \frac{4Rk^2}{3(-1+R)}\,.$$

Case 7.

$$k = k\,, \qquad \mu_1 = \pm 1\,, \qquad \mu_2 = \pm 1\,, \qquad a_0 = \pm\frac{1}{3}\sqrt{3}kR - \lambda\,,$$
$$a_1 = \pm\frac{1}{3}(1+R)k\sqrt{3}\,, \qquad A_0 = \frac{k^2R(1+R)}{3(-1+R)}\,, \qquad b_1 = \pm\frac{1}{3}(1+R)k\sqrt{3}\,,$$
$$A_1 = -\frac{2k^2R(1+R)}{3(-1+R)}\,, \qquad B_1 = -\frac{2k^2R(1+R)}{3(-1+R)}\,, \qquad A_2 = \frac{k^2(1+R)^2}{3(-1+R)}\,,$$
$$B_2 = \frac{k^2(1+R)^2}{3(-1+R)}\,. \tag{25}$$

Case 8.

$$\mu_1 = \pm 1\,, \qquad \mu_2 = \pm 1\,, \qquad a_0 = \pm\frac{1}{3}\sqrt{3}kR - \lambda\,, \qquad a_1 = \pm\frac{1}{3}(1+R)k\sqrt{3}\,,$$
$$k = k\,, \qquad b_1 = \pm\frac{1}{3}\sqrt{3}(-1+R)k\,, \qquad A_0 = -\frac{1}{3}k^2R\,, \qquad A_1 = \frac{2}{3}k^2R\,, \tag{26}$$
$$B_1 = 0\,, \qquad A_2 = -\frac{1}{3}k^2(1+R)\,, \qquad B_2 = -\frac{1}{3}k^2(-1+R)\,.$$

From Eqs. (17), (18) and Cases 1–8, we obtain the following solutions for Eq. (16).

Family 1.

$$v_{11} = -\lambda \pm \frac{\sqrt{3}k\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{3-3\mu_2^2R}k\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)}{3(1 \pm i\mu_2\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi))}\,, \tag{27a}$$

$$w_{11} = -\frac{1}{3}k^2R \pm \frac{i\mu_2 Rk^2\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)}{3(1 \pm i\mu_2\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi))} + \frac{k^2R\tanh^2(\sqrt{-R}\xi) \pm ik^2\sqrt{1-\mu_2^2R}R\tanh(\sqrt{-R}\xi)\,\mathrm{sech}(\sqrt{-R}\xi)}{3(1 \pm i\mu_2\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi))^2}, \quad (27b)$$

$$v_{12} = -\lambda \pm \frac{\sqrt{3}k\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{3-3\mu_2^2R}k\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)}{3(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi))}, \quad (28a)$$

$$w_{12} = -\frac{1}{3}k^2R \pm \frac{\mu_2 Rk^2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)}{3(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi))} + \frac{k^2R\coth^2(\sqrt{-R}\xi) \pm k^2\sqrt{1-\mu_2^2R}R\coth(\sqrt{-R}\xi)\,\mathrm{csch}(\sqrt{-R}\xi)}{3(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi))^2}, \quad (28b)$$

where $\xi = k(x + \lambda t)$, $R < 0$, μ_2, k and λ are arbitrary constants.

Family 2.

$$v_{21} = \frac{\pm k\mu_2 R - \lambda\sqrt{3-3\mu_2^2R}}{\sqrt{3-3\mu_2^2R}} \pm \frac{2i\sqrt{3-3\mu_2^2R}k\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)}{3(1 \pm i\mu_2\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi))}, \quad (29a)$$

$$w_{21} = A_0 \pm \frac{2i\mu_2 Rk^2\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)}{3(1 \pm i\mu_2\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi))} + \frac{2Rk^2\tanh^2(\sqrt{-R}\xi)}{3(1 \pm i\mu_2\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi))^2}, \quad (29b)$$

$$v_{22} = \frac{\pm k\mu_2 R - \lambda\sqrt{3-3\mu_2^2R}}{\sqrt{3-3\mu_2^2R}} \pm \frac{2\sqrt{3-3\mu_2^2R}k\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)}{3(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi))}, \quad (30a)$$

$$w_{22} = A_0 \pm \frac{2\mu_2 Rk^2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)}{3(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi))} + \frac{2Rk^2\coth^2(\sqrt{-R}\xi)}{3(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi))^2}, \quad (30b)$$

where $\xi = k(x+\lambda t)$, A_0 is determined by Eq. (20), $R < 0$, μ_2, k and λ are arbitrary constants.

Family 3.

$$v_{31} = \pm\frac{2}{3}\sqrt{3}kR\mu_1 - \lambda \pm \frac{2\sqrt{3}(\mu_1^2R+1)k\sqrt{-R}\tanh(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)}, \quad (31a)$$

$$w_{31} = -\frac{2}{3}(\mu_1^2R+1)k^2R - \frac{4(k^2R\mu_1 + k^2\mu_1^3R^2)\sqrt{-R}\tanh(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)} + \frac{(2k^2\mu_1^4R^2 + 4k^2R\mu_1^2 + 2k^2)R\tanh^2(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)^2}, \quad (31b)$$

$$v_{32} = \pm\frac{2}{3}\sqrt{3}kR\mu_1 - \lambda \pm \frac{2\sqrt{3}(\mu_1^2R+1)k\sqrt{-R}\coth(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1)}\,, \tag{32a}$$

$$w_{32} = -\frac{2}{3}(\mu_1^2R+1)k^2R - \frac{4(k^2R\mu_1 + k^2\mu_1^3R^2)\sqrt{-R}\coth(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1)} + \frac{(2k^2\mu_1^4R^2 + 4k^2R\mu_1^2 + 2k^2)R\coth^2(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1)^2}\,, \tag{32b}$$

where $\xi = k(x+\lambda t)$, $R<0$, μ_1, k and λ are arbitrary constants.

Family 4.

$$v_{41} = -\lambda \pm \frac{2i\sqrt{3\mu_1^2R+3}k\sqrt{-R}\operatorname{sech}(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)}\,, \tag{33a}$$

$$w_{41} = -\frac{1}{3}Rk^2(2\mu_1^2R+1) - \frac{4R(\mu_1^2R+1)k^2\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)} + \frac{(2k^2\mu_1^4R^2 + 4k^2R\mu_1^2 + 2k^2)R\tanh^2(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)^2}\,, \tag{33b}$$

$$v_{42} = -\lambda \pm \frac{2\sqrt{3\mu_1^2R+3}k\sqrt{-R}\operatorname{csch}(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1)}\,, \tag{34a}$$

$$w_{42} = -\frac{1}{3}Rk^2(2\mu_1^2R+1) - \frac{4R(\mu_1^2R+1)k^2\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1)} + \frac{(2k^2\mu_1^4R^2 + 4k^2R\mu_1^2 + 2k^2)R\coth^2(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1)^2}\,, \tag{34b}$$

where $\xi = k(x+\lambda t)$, $R<0$, μ_1, k and λ are arbitrary constants.

Family 5.

$$v_{51} = a_0 \pm \frac{\sqrt{3}(\mu_1^2R+1)k\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{3\mu_1^2R+3}k\sqrt{-R}\operatorname{sech}(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)}\,, \tag{35a}$$

$$w_{51} = A_0 - \frac{2(k^2\mu_1^3R^2 + k^2R\mu_1)\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm ik^2\sqrt{\mu_1^2R+1}\mu_1R\sqrt{-R}\operatorname{sech}(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)} + \frac{k^2(\mu_1^4R^2 + 2R\mu_1^2+1)R\tanh^2(\sqrt{-R}\xi) \pm ik^2(\mu_1^2R+1)^{3/2}R\tanh(\sqrt{-R}\xi)\operatorname{sech}(\sqrt{-R}\xi)}{(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)^2}\,, \tag{35b}$$

$$v_{52} = a_0 \pm \frac{\sqrt{3}(\mu_1^2 R + 1)k\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{3\mu_1^2 R + 3}k\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)}{3(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi) + 1)}, \tag{36a}$$

$$w_{52} = A_0 - \frac{\begin{array}{c} 2(k^2\mu_1^3 R^2 + k^2 R\mu_1)\sqrt{-R}\coth(\sqrt{-R}\xi) \\ \pm k^2\sqrt{\mu_1^2 R + 1}\mu_1 R\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi) \end{array}}{3(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi) + 1)} + \frac{\begin{array}{c} k^2(\mu_1^4 R^2 + 2R\mu_1^2 + 1)R\coth^2(\sqrt{-R}\xi) \\ \pm k^2(\mu_1^2 R + 1)^{3/2} R\coth(\sqrt{-R}\xi)\,\mathrm{csch}(\sqrt{-R}\xi) \end{array}}{(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi) + 1)^2}, \tag{36b}$$

where $\xi = k(x + \lambda t)$, a_0 and A_0 are determined by Eq. (23), $R < 0$, μ_1, k and λ are arbitrary constants.

Family 6.

$$v_{61} = \pm\frac{1}{3}\sqrt{3}kR - \lambda \pm \frac{2i\sqrt{3}k\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)}{3(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi) + 1)}, \tag{37a}$$

$$w_{61} = \frac{k^2 R(1 + R^2)}{3(R - 1)} + \frac{4Rk^2\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm 2ik^2 R(1 + R)\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)}{3(R - 1)(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi) + 1)} - \frac{2k^2 R(1 + R)\tanh^2(\sqrt{-R}\xi) \pm 4ik^2 R^2\tanh(\sqrt{-R}\xi)\,\mathrm{sech}(\sqrt{-R}\xi)}{3(R - 1)(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi) + 1)^2}, \tag{37b}$$

$$v_{62} = \pm\frac{1}{3}\sqrt{3}kR - \lambda \pm \frac{2\sqrt{3}k\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)}{3(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm i\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi) + 1)}, \tag{38a}$$

$$w_{62} = \frac{k^2 R(1 + R^2)}{3(R - 1)} + \frac{4Rk^2\sqrt{-R}\coth(\sqrt{-R}\xi) \pm 2k^2 R(1 + R)\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)}{3(R - 1)(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi) + 1)} - \frac{2k^2 R(1 + R)\coth^2(\sqrt{-R}\xi) \pm 4k^2 R^2\coth(\sqrt{-R}\xi)\,\mathrm{csch}(\sqrt{-R}\xi)}{3(R - 1)(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi) + 1)^2}, \tag{38b}$$

where $\xi = k(x + \lambda t)$, $R < 0$, k and λ are arbitrary constants.

Family 7.

$$v_{71} = a_0 \pm \frac{(\sqrt{3}k + \sqrt{3}kR)\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i(1 + R)k\sqrt{3}\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)}{3(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi) + 1)} \tag{39a}$$

$$w_{71} = A_0 \pm \frac{2k^2R(1+R)\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm 2ik^2R(1+R)\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)}{3(R-1)(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)+1)} - \frac{k^2(1+R)^2R\tanh^2(\sqrt{-R}\xi) \pm ik^2(1+R)^2R\tanh(\sqrt{-R}\xi)\,\mathrm{sech}(\sqrt{-R}\xi)}{3(R-1)(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)+1)^2}, \tag{39b}$$

$$v_{72} = a_0 \pm \frac{(\sqrt{3}k+\sqrt{3}kR)\sqrt{-R}\coth(\sqrt{-R}\xi) \pm (1+R)k\sqrt{3}\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)}{3(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)+1)}, \tag{40a}$$

$$w_{72} = A_0 \pm \frac{2k^2R(1+R)\sqrt{-R}\coth(\sqrt{-R}\xi) \pm 2k^2R(1+R)\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)}{3(R-1)(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)+1)} - \frac{k^2(1+R)^2R\coth^2(\sqrt{-R}\xi) \pm k^2(1+R)^2R\coth(\sqrt{-R}\xi)\,\mathrm{csch}(\sqrt{-R}\xi)}{3(R-1)(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)+1)^2}, \tag{40b}$$

where $\xi = k(x+\lambda t)$, a_0 and A_0 are determined by Eq. (25), $R < 0$, k and λ are arbitrary constants.

Family 8.

$$v_{81} = a_0 \pm \frac{(1+R)k\sqrt{3}\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{3}(-1+R)k\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)}{3(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)+1)}, \tag{41a}$$

$$w_{81} = -\frac{1}{3}k^2R - \frac{2k^2R\sqrt{-R}\tanh(\sqrt{-R}\xi)}{3(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)+1)} + \frac{(k^2R+k^2)R\tanh^2(\sqrt{-R}\xi) \pm i(k^2R-k^2)R\tanh(\sqrt{-R}\xi)\,\mathrm{sech}(\sqrt{-R}\xi)}{3(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm i\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)+1)^2}, \tag{41b}$$

$$v_{82} = a_0 \pm \frac{(1+R)k\sqrt{3}\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{3}(-1+R)k\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)}{3(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)+1)}, \tag{42a}$$

$$w_{82} = -\frac{1}{3}k^2R - \frac{2k^2R\sqrt{-R}\coth(\sqrt{-R}\xi)}{3(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)+1)} + \frac{(k^2R+k^2)R\coth^2(\sqrt{-R}\xi) \pm (k^2R-k^2)R\coth(\sqrt{-R}\xi)\,\mathrm{csch}(\sqrt{-R}\xi)}{3(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\xi)+1)^2}, \tag{42b}$$

where $\xi = k(x + \lambda t)$, a_0 is determined by Eq. (26), $R < 0$, k and λ are arbitrary constants.

4. Summary and Conclusions

Based on the RERE method and symbolic computation, we obtain many types of solutions including rational form solitary wave solutions, triangular periodic wave solutions and rational wave solutions for $(1+1)$-DLWE. The success of the RERE method lies in the fact one circumvents integration to get explicit solutions based on the fact that soliton solutions are essentially of a localized nature. Writing the soliton solutions of a nonlinear equation as the polynomials of auxiliary variables of the Riccati equation, the equation can changed into a nonlinear system of algebraic equations. The system can be solved with help of symbolic computation.

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