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New exact travelling wave solutions for the shallow long wave approximate equations

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Abstract

Based upon a generally projective Riccati equation method, which is a direct and unified algebraic method for constructing more general form travelling wave solutions of nonlinear partial differential equations and implemented in a computer algebraic system, we consider the shallow long wave approximate equations. New and more general form solutions are obtained, including kink-shaped solitons, bell-shaped solitons, singular solitons and periodic solutions. The properties of the new formal solitary wave solutions are shown by some figures.

Keywords: Projective Riccati equation method; Shallow long wave approximate equation; Exact solutions

1. Introduction

In the field of nonlinear science, to find as many and general as possible exact solutions for a nonlinear system is one of the most fundamental and significant study. Many effective methods have been presented, such as inverse

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scattering transform method [1], Bäcklund transformation [1,2], Darboux transformation [3], the Hirota bilinear method [4], variable separation approach [5], various tanh methods [6–9], homogeneous balance method [10] and similarity reductions method [11,12] and so on.

Conte and Musette [13] presented a direct and unified algebraic method for building new solitary wave solutions of nonlinear partial differential equations (PDEs) that can be expressed as a polynomial in two elementary function which satisfy the coupled projective Riccati equations [14]. Recently, Yan [15] improved Conte's method and presented the general projective Riccati equation method to find more exact solutions of nonlinear differential equations based upon a more general form of the projective Riccati equations. However, in [15] Yan did not apply the method to system of differential equation. The present work is motivated by the desire to improve the generally projective Riccati equation method to study the system of nonlinear PDEs: the shallow long wave approximate (SLA) equations

$$\begin{cases} u_t - uu_x - v_x + \dfrac{u_{xx}}{2} = 0, \\ v_t - u_x v - uv_x - \dfrac{v_{xx}}{2} = 0, \end{cases} \tag{1}$$

which was found by Whitham [16] and Broer [17]. The symmetries and conservation laws of system (1), were discussed by Kuperschimidt [18]. By using the homogenous balance method, Zhang [19] obtained multiple soliton solutions of the system. Yan [20] used sine–cosine method to obtain three families of soliton solutions. Chen et al. used the generalized extended tanh-function method to construct news explicit exact solutions of SLA [21]. Here, by using the projective Riccati equation method, we can successfully recover the previously known solitary wave solutions that have been found by the extended tanh-function method and other more sophisticated method. More importantly, for the system of the approximate equation for long water waves, we also obtain other new and more general solutions at the same time, which include kink-profile solitary-wave solutions, bell-shaped solitary-wave solutions, singular solutions and new formal solutions. The properties of new formal soliton solutions for the system are shown by some figures.

This paper is organized as follows: in Section 2, we summarize the improved method. In Section 3, we apply the improved method to the generalized SLA equations and bring out many solutions. Conclusions will be presented in finally.

2. Summary of the improved method

Now we will improve the generalize projective Riccati equation method and apply it to system of differential equations.

Consider a given system of nonlinear PDEs, say, two variables:

$$\begin{cases} F(u,H,u_t,H_t,u_x,H_x,u_{xt},H_{x,t},u_{tt},H_{tt},u_{xx},H_{xx},\ldots)=0, \\ G(u,H,u_t,H_t,u_x,H_x,u_{xt},H_{x,t},u_{tt},H_{tt},u_{xx},H_{xx},\ldots)=0, \end{cases} \tag{2}$$

under the transformation $u(x,t)=u(\xi)$, $H(x,t)=H(\xi)$, $\xi=x-\lambda t$, (2) reduces to be

$$\begin{cases} F(u,H,u',H',u'',H'',u''',H''',\ldots)=0, \\ G(u,h',u',H',u'',H'',u''',H''',\ldots)=0, \end{cases} \tag{3}$$

where "′" denotes $\mathrm{d}/\mathrm{d}\xi$. In order to seek for the travelling wave solutions of (3), we need following steps:

Step 1. Balancing the highest order derivative term and the nonlinear terms in Eq. (3), we get balance constants m and n (m and n are usually positive integers). If m or n is a fraction or a negative integer, we make the following transformation:

$$\begin{cases} u(\xi)=v^m(\xi), \\ H(\xi)=w^n(\xi), \end{cases} \tag{4}$$

then return to determine balance constants m and n again.

Step 2. We express the solutions of Eq. (3) to be the following forms:

Type 1

$$\begin{cases} u(\xi)=a_0+\sum_{i=1}^{m}\sigma^{i-1}[a_i\sigma(\xi)+b_i\tau(\xi)], \\ H(\xi)=A_0+\sum_{i=1}^{n}\sigma^{i-1}[A_i\sigma(\xi)+B_i\tau(\xi)], \end{cases} \tag{5}$$

where $\sigma(\xi)$ and $\tau(\xi)$ satisfy the following projective Riccati equations:

$$\sigma'(\xi)=\epsilon\sigma(\xi)\tau(\xi), \quad \tau'(\xi)=R+\epsilon\tau^2(\xi)-\mu\sigma(\xi), \quad \epsilon=\pm 1, \\ R,\mu=\text{constant}, \tag{6}$$

which admits the first integral with $R\neq 0$

$$\tau^2(\xi)=-\epsilon\left[R-2\mu\sigma(\xi)+\frac{\mu^2-1}{R}\sigma^2(\xi)\right] \quad (R\neq 0), \tag{7}$$

where "′" denotes $\mathrm{d}/\mathrm{d}\xi$. When $\epsilon=-1$, $R=1$, $\mu\to\mu/K$, (6) becomes a projective Riccati equation [13].

Type 2. When $R=\mu=0$ in Eqs. (6).

$$\begin{cases} u(\xi)=\sum_{i=0}^{m}a_i\tau^i(\xi), \\ H(\xi)=\sum_{i=0}^{n}A_i\tau^i(\xi), \end{cases} \tag{8}$$

where $\tau(\xi)$ satisfies

$$\tau'(\xi)=\tau^2(\xi). \tag{9}$$

Step 3. When $R \neq 0$, substituting (5) along with the conditions (6) and (7) into (3), and when $R = \mu = 0$, substituting (8) along with $\tau'(\xi) = \tau^2(\xi)$ into (3), then yields a set of algebraic equations for $\sigma^j(\xi)\tau^i(\xi)$, $(j = 0, 1, \ldots;\ i = 0, 1)(\tau^l\xi, l = 0, 1, \ldots)$. Setting the coefficients of these terms $\sigma^j\tau^i$ (or $\tau^l(\xi)$) to zero, yields a set of over-determined algebraic equations in λ, a_i, b_i, A_i, B_i, R and μ.

Step 4. With the aid of *Maple*, solving the above set of equations obtained in step 3, yields the values of a_i, b_i, A_i, B_i, R, λ, μ.

Step 5. We know that Eq. (6) admits the following solutions:

Case 1. When $\epsilon = -1$, $R \neq 0$,

$$\begin{cases} \sigma_1(\xi) = \dfrac{R\text{sech}(\sqrt{R}\xi)}{\mu\text{sech}(\sqrt{R}\xi) + 1}, & \tau_1(\xi) = \dfrac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\text{sech}(\sqrt{R}\xi) + 1}, \\ \sigma_2(\xi) = \dfrac{R\text{csch}(\sqrt{R}\xi)}{\mu\text{csch}(\sqrt{R}\xi) + 1}, & \tau_2(\xi) = \dfrac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\,\text{csch}(\sqrt{R}\xi) + 1}. \end{cases} \tag{10}$$

Case 2. When $\epsilon = 1$, $R \neq 0$,

$$\begin{cases} \sigma_3(\xi) = \dfrac{R\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi) + 1}, & \tau_3(\xi) = \dfrac{\sqrt{R}\tan(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi) + 1}, \\ \sigma_4(\xi) = \dfrac{R\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi) + 1}, & \tau_4(\xi) = -\dfrac{\sqrt{R}\cot(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi) + 1}. \end{cases} \tag{11}$$

Case 3. When $R = \mu = 0$,

$$\sigma_5(\xi) = \frac{C}{\xi} = C\epsilon\tau_5(\xi), \quad \tau_5(\xi) = \frac{1}{\epsilon\xi}, \tag{12}$$

where C is a constant.

Thus according to (5), (10)–(12) and the conclusions in *Step 4*, we can obtain many solutions for Eq. (2).

3. Exact solutions of SLA equation

Let us consider the SLA equations, i.e. Eq. (1). According to the above method, to seek travelling wave solutions of Eq. (1), we make the transformation

$$u(x,t) = \phi(\xi), \quad v(\xi) = \theta(\xi), \quad \xi = x + \lambda t, \tag{13}$$

where λ is constant to be determined later, and thus Eq. (3) becomes

$$\begin{cases} \lambda\phi' - \phi\phi' - \theta' + \dfrac{\phi''}{2} = 0, \\ \lambda\phi' - \phi'\theta - \phi\theta' - \dfrac{\theta'}{2} = 0. \end{cases} \tag{14}$$

Integrating the above equation once with regard to ξ, we obtain

$$\begin{cases} \lambda\phi - \dfrac{\phi^2}{2} - \theta + \dfrac{\phi'}{2} = 0, \\ \lambda\theta - \phi\theta - \dfrac{\theta'}{2} = 0 \end{cases} \tag{15}$$

with the integration constants taken to be zero. According to *Step 1* in Section 2, by balancing $\phi'(\xi)$ and $\phi^2(\xi)$ in Eq. (15), we get $m = 1$ and by balancing $\theta'(\xi)$ and $\phi(\xi)\theta(\xi)$ in Eq. (15), we get $n = 2$.

Therefore we suppose that Eq. (15) has the following formal solutions:

$$\begin{cases} \phi(\xi) = a_0 + a_1\sigma(\xi) + b_1\tau(\xi), \\ \tau(\xi) = A_0 + A_1\sigma(\xi) + B_1\tau(\xi) + A_2\sigma^2(\xi) + B_2\sigma(\xi)\tau(\xi), \end{cases} \tag{16}$$

where $\sigma(\xi)$, $\tau(\xi)$ satisfies (6) and (7), where a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2 are constants to be determined later.

With the aid of *Maple*, substituting (16) along with (6) and (7) into (15), yields a set of algebraic equations for $\sigma^j(\xi)\tau^i(\xi)(j = 0, 1, \ldots;\ i = 0, 1)$. Setting the coefficients of these terms $\sigma^j\tau^i$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2, R, μ and λ. (*Note* 1. Here we take $\epsilon = -1$.)

$$-a_0^2R + 2\lambda a_0R - 2A_0R - b_1^2R^2 = 0, \tag{17}$$

$$-b_1^2\mu^2 - a_1^2R + b_1^2 - 2A_2R - b_1\mu^2 + b_1 = 0, \tag{18}$$

$$-Ra_1 - 2RB_2 - 2Ra_1b_1 = 0, \tag{19}$$

$$b_1\mu R - 2a_0a_1R + 2b_1^2\mu R + 2\lambda a_1R - 2A_1R = 0, \tag{20}$$

$$2R\lambda b_1 - 2B_1R - 2Ra_0b_1 = 0, \tag{21}$$

$$2\lambda A_0R - 2b_1B_1R^2 - 2a_0A_0R = 0, \tag{22}$$

$$4b_1B_1\mu R - B_1\mu R + B_2R^2 - 2b_1B_2R^2 - 2a_1A_0R + 2\lambda A_1R - 2a_0A_1R = 0, \tag{23}$$

$$-2Ra_1B_1 - 2Ra_0B_2 + 2R\lambda B_2 + A_1R - 2Rb_1A_1 = 0, \tag{24}$$

$$\begin{aligned} &-3B_2\mu R - 2a_1A_1R + B_1\mu^2 - B_1 - 2b_1B_1\mu^2 + 4b_1B_2\mu R + 2\lambda A_2R \\ &\quad - 2a_0A_2R + 2b_1B_1 = 0, \end{aligned} \tag{25}$$

$$-2Ra_1B_2 - 2Rb_1A_2 + 2A_2R = 0, \tag{26}$$

$$-2a_1A_2R - 2B_2 + 2b_1B_2 + 2B_2\mu^2 - 2b_1B_2\mu^2 = 0, \tag{27}$$

$$-2Rb_1A_0 - 2Ra_0B_1 + 2R\lambda B_1 = 0. \tag{28}$$

By use of the *Maple* soft package "charsets" by Dongming Wang, which based on the Wu-elimination method [22], solving Eqs. (17)–(28), we get the following results:

Case 1

$$\begin{aligned} &a_1 = A_0 = A_1 = B_1 = B_2 = \mu = 0, \quad a_0 = \pm\sqrt{R}, \quad b_1 = 1, \\ &\lambda = \pm\sqrt{R}, \quad A_2 = \frac{1}{R}. \end{aligned} \tag{29}$$

Case 2

$$\begin{aligned} &b_1 = A_1 = B_1 = \mu = 0, \quad a_0 = \frac{\sqrt{-2R}}{2}, \quad \lambda = \frac{\sqrt{-2R}}{2}, \\ &A_2 = \frac{1}{2R}, \quad B_2 = \mp\frac{\sqrt{-R}}{2R}, \quad a_1 = \pm\frac{\sqrt{-R}}{R}, \quad A_0 = -\frac{R}{4}. \end{aligned} \tag{30}$$

Case 3

$$\begin{aligned} &b_1 = A_1 = B_1 = \mu = 0, \quad a_0 = -\frac{\sqrt{-2R}}{2}, \quad \lambda = -\frac{\sqrt{-2R}}{2}, \\ &A_2 = \frac{1}{2R}, \quad B_2 = \mp\frac{\sqrt{-R}}{2R}, \quad a_1 = \pm\frac{\sqrt{-R}}{R}, \quad A_0 = -\frac{R}{4}. \end{aligned} \tag{31}$$

Case 4

$$\begin{aligned} &A_0 = B_1 = 0, \quad \mu = \pm\sqrt{1 + 4a_1^2R}, \quad b_1 = \frac{1}{2}, \quad a_0 = \frac{\sqrt{R}}{2}, \\ &A_2 = -2a_1^2, \quad B_2 = -a_1, \quad \lambda = \frac{\sqrt{R}}{2}, \quad A_1 = \pm\frac{\sqrt{1 + 4a_1^2R}}{2}. \end{aligned} \tag{32}$$

Case 5

$$\begin{aligned} &A_0 = B_1 = 0, \quad \mu = \pm\sqrt{1 + 4a_1^2R}, \quad b_1 = \frac{1}{2}, \quad a_0 = -\frac{\sqrt{R}}{2}, \\ &A_2 = -2a_1^2, \quad B_2 = -a_1, \quad \lambda = -\frac{\sqrt{R}}{2}, \quad A_1 = \pm\frac{\sqrt{1 + 4a_1^2R}}{2}. \end{aligned} \tag{33}$$

From (13) and (16) and Cases 1–5, we obtain the following solutions for Eq. (1):

Family 1. From Eq. (29), we obtain the following solutions for the SLA equations, as follows:

$$\begin{cases} u_{11} = \pm\sqrt{R} + \sqrt{R}\tanh(\sqrt{R}\xi), \\ v_{11} = R\mathrm{sech}^2(\sqrt{R}\xi), \end{cases} \tag{34}$$

$$\begin{cases} u_{12} = \pm\sqrt{R} + \sqrt{R}\coth(\sqrt{R}\xi), \\ v_{12} = R\,\mathrm{csch}^2(\sqrt{R}\xi), \end{cases} \tag{35}$$

where R is an arbitrary constant and $\xi = x + \lambda t$, $\lambda = \pm\sqrt{R}$.

Family 2. From (30), the SLA equations (1) have the following solutions:

$$\begin{cases} u_{21} = \dfrac{\sqrt{-2R}}{2} \pm \sqrt{-R}\mathrm{sech}(\sqrt{R}\xi), \\ v_{21} = -\dfrac{R}{4} + \dfrac{R}{2}\mathrm{sech}^2(\sqrt{-R}\xi) \mp \dfrac{R}{2}\mathrm{i}\,\mathrm{sech}(\sqrt{R}\xi)\tanh(\sqrt{R}\xi), \end{cases} \tag{36}$$

$$\begin{cases} u_{22} = \dfrac{\sqrt{-2R}}{2} \pm \sqrt{-R}\,\mathrm{csch}(\sqrt{R}\xi), \\ v_{22} = -\dfrac{R}{4} + \dfrac{R}{2}\mathrm{csch}^2\left(\sqrt{R}\xi\right) \mp \dfrac{R}{2}\mathrm{csch}(\sqrt{R}\xi)\coth(\sqrt{R}\xi), \end{cases} \tag{37}$$

where R is an arbitrary constant and $\xi = x + \lambda t$, $\lambda = \frac{\sqrt{-2R}}{2}$.

Family 3. From (31), the SLA equations (3) have the following solutions:

$$\begin{cases} u_{31} = -\dfrac{\sqrt{-2R}}{2} \pm \sqrt{-R}\mathrm{sech}\left(\sqrt{R}\xi\right), \\ v_{31} = -\dfrac{R}{4} + \dfrac{R}{2}\mathrm{sech}^2(\sqrt{R}\xi) \mp \dfrac{R}{2}\mathrm{i}\,\mathrm{sech}(\sqrt{R}\xi)\tanh(\sqrt{R}\xi), \end{cases} \tag{38}$$

$$\begin{cases} u_{32} = -\dfrac{\sqrt{-2R}}{2} \pm \sqrt{-R}\mathrm{csch}(\sqrt{R}\xi), \\ v_{32} = -\dfrac{R}{4} + \dfrac{R}{2}\mathrm{csch}^2(\sqrt{R}\xi) \mp \dfrac{R}{2}\mathrm{csch}(\sqrt{R}\xi)\coth(\sqrt{R}\xi), \end{cases} \tag{39}$$

where R is an arbitrary constant and $\xi = x + \lambda t$, $\lambda = -\frac{\sqrt{-2R}}{2}$.

Family 4. From Eq. (32), we obtain the following solutions for the SLA equations:

$$\begin{cases} u_{41} = \dfrac{\sqrt{R}}{2} + \dfrac{a_1 R \operatorname{sech}(\sqrt{R}\xi)}{\mu \operatorname{sech}\left(\sqrt{R}\xi\right) + 1} + \dfrac{\sqrt{R}\tanh(\sqrt{R}\xi)}{2\mu \operatorname{sech}(\sqrt{R}\xi) + 2}, \\ v_{41} = \dfrac{\sqrt{1+4a_1^2 R} R \operatorname{sech}(\sqrt{R}\xi)}{2\mu \operatorname{sech}(\sqrt{R}\xi) + 2} - 2\dfrac{a_1^2 R^2 \operatorname{sech}^2(\sqrt{R}\xi)}{\left(\mu \operatorname{sech}\left(\sqrt{R}\xi\right) + 1\right)^2} \\ \qquad - \dfrac{a_1 R^{\frac{3}{2}} \operatorname{sech}(\sqrt{R}\xi)\tanh(\sqrt{R}\xi)}{(\mu \operatorname{sech}(\sqrt{R}\xi) + 1)^2}, \end{cases} \tag{40}$$

$$\begin{cases} u_{42} = \dfrac{\sqrt{R}}{2} + \dfrac{a_1 R \operatorname{csch}(\sqrt{R}\xi)}{\mu \operatorname{csch}(\sqrt{R}\xi) + 1} + \dfrac{\sqrt{R}\coth(\sqrt{R}\xi)}{2\mu \operatorname{csch}(\sqrt{R}\xi) + 2}, \\ v_{42} = \dfrac{\sqrt{1+4a_1^2 R} R \operatorname{csch}(\sqrt{R}\xi)}{2\mu \operatorname{csch}(\sqrt{R}\xi) + 2} - 2\dfrac{a_1^2 R^2 \operatorname{csch}^2(\sqrt{R}\xi)}{(\mu \operatorname{csch}(\sqrt{R}\xi) + 1)^2} \\ \qquad - \dfrac{a_1 R^{\frac{3}{2}} \operatorname{csch}(\sqrt{R}\xi)\coth(\sqrt{R}\xi)}{(\mu \operatorname{csch}(\sqrt{R}\xi) + 1)^2}, \end{cases} \tag{41}$$

where R and a_1 arbitrary constant and $\xi = x + \lambda t$, $\lambda = \frac{\sqrt{R}}{2}$, $\mu = \pm\sqrt{1 + 4a_1^2 R}$. The properties of the solutions u_{41} and $v_{91} v_{41}$ are shown by Fig. 1.

Family 5. From Eq. (33), we obtain the following solutions for the SLA equations:

$$\begin{cases} u_{51} = -\dfrac{\sqrt{R}}{2} + \dfrac{a_1 R \operatorname{sech}(\sqrt{R}\xi)}{\mu \operatorname{sech}(\sqrt{R}\xi) + 1} + \dfrac{\sqrt{R}\tanh(\sqrt{R}\xi)}{2\mu \operatorname{sech}(\sqrt{R}\xi) + 2}, \\ v_{51} = \dfrac{\sqrt{1+4a_1^2 R} R \operatorname{sech}(\sqrt{R}\xi)}{2\mu \operatorname{sech}(\sqrt{R}\xi) + 2} - 2\dfrac{a_1^2 R^2 \operatorname{sech}^2(\sqrt{R}\xi)}{(\mu \operatorname{sech}(\sqrt{R}\xi) + 1)^2} \\ \qquad - \dfrac{a_1 R^{\frac{3}{2}} \operatorname{sech}(\sqrt{R}\xi)\tanh(\sqrt{R}\xi)}{(\mu \operatorname{sech}(\sqrt{R}\xi) + 1)^2}, \end{cases} \tag{42}$$

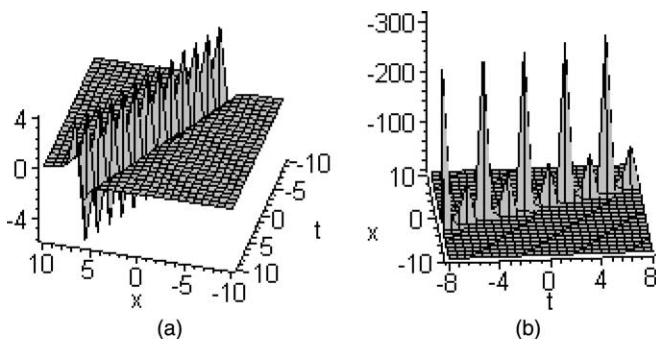

Fig. 1. The solitary solution u_{41} and v_{41}, where $a_1 = 1$ and $R = 1$.

$$\begin{cases} u_{52} = -\dfrac{\sqrt{R}}{2} + \dfrac{a_1 R\operatorname{csch}(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1} + \dfrac{\sqrt{R}\coth(\sqrt{R}\xi)}{2\mu\operatorname{csch}(\sqrt{R}\xi)+2}, \\ v_{52} = \dfrac{\sqrt{1+4a_1^2R}R\operatorname{csch}(\sqrt{R}\xi)}{2\mu\operatorname{csch}(\sqrt{R}\xi)+2} - 2\dfrac{a_1^2R^2\operatorname{csch}^2(\sqrt{R}\xi)}{(\mu\operatorname{csch}(\sqrt{R}\xi)+1)^2} \\ \qquad - \dfrac{a_1 R^{\frac{3}{2}}\operatorname{csch}(\sqrt{R}\xi)\coth(\sqrt{R}\xi)}{(\mu\operatorname{csch}(\sqrt{R}\xi)+1)^2}, \end{cases} \tag{43}$$

where R is an arbitrary constant and $\xi = x + \lambda t$, $\lambda = -\frac{\sqrt{R}}{2}$, $\mu = \pm\sqrt{1+4a_1^2R}$.

The following periodic wave solutions obtained is under $\epsilon = 1$.

Family 6. The SLA equations have the following periodic solutions:

$$\begin{cases} u_{61} = \pm\sqrt{-R} - \sqrt{-R}\tan(\sqrt{R}\xi), \\ v_{61} = -R\sec^2(\sqrt{R}\xi), \end{cases} \tag{44}$$

$$\begin{cases} u_{62} = \pm\sqrt{-R} + \sqrt{-R}\cot(\sqrt{R}\xi), \\ v_{62} = -R\csc^2(\sqrt{R}\xi), \end{cases} \tag{45}$$

where R is an arbitrary constant and $\xi = x + \lambda t$, $\lambda = \pm\sqrt{R}$.

Family 7. The SLA equations (1) have the following solutions:

$$\begin{cases} u_{71} = \dfrac{\sqrt{2R}}{2} \pm \sqrt{R}\sec(\sqrt{R}\xi), \\ v_{71} = \dfrac{R}{4} - \dfrac{R}{2}\sec^2(\sqrt{R}\xi) \pm \dfrac{R}{2}\sec(\sqrt{R}\xi)\tan(\sqrt{R}\xi), \end{cases} \tag{46}$$

$$\begin{cases} u_{72} = \dfrac{\sqrt{2R}}{2} \pm \sqrt{R}\csc(\sqrt{R}\xi), \\ v_{72} = \dfrac{R}{4} - \dfrac{R}{2}\csc^2(\sqrt{R}\xi) \mp \dfrac{R}{2}\csc(\sqrt{R}\xi)\cot(\sqrt{R}\xi), \end{cases} \tag{47}$$

where R is an arbitrary constant and $\xi = x + \lambda t$, $\lambda = \frac{\sqrt{2R}}{2}$.

Family 8. The SLA equations (1) possess the following solutions:

$$\begin{cases} u_{81} = -\dfrac{\sqrt{2R}}{2} \pm \sqrt{R}\sec(\sqrt{R}\xi), \\ v_{81} = \dfrac{R}{4} - \dfrac{R}{2}\sec^2(\sqrt{R}\xi) \pm \dfrac{R}{2}\sec(\sqrt{R}\xi)\tan(\sqrt{R}\xi), \end{cases} \tag{48}$$

$$\begin{cases} u_{82} = -\dfrac{\sqrt{2R}}{2} \pm \sqrt{R}\csc(\sqrt{R}\xi), \\ v_{82} = \dfrac{R}{4} - \dfrac{R}{2}\csc^2(\sqrt{R}\xi) \mp \dfrac{R}{2}\csc(\sqrt{R}\xi)\cot(\sqrt{R}\xi), \end{cases} \tag{49}$$

where R is an arbitrary constant and $\xi = x + \lambda t$, $\lambda = -\frac{\sqrt{2R}}{2}$.

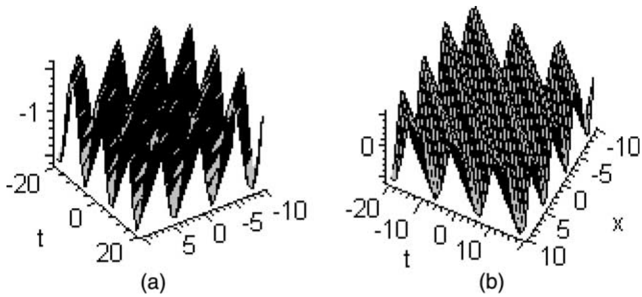

Fig. 2. The solitary solution u_{91} and v_{92}, where $a_1 = 15$ and $R = -2$.

Family 9. The SLA equations have the following solutions:

$$\begin{cases} u_{91} = -\dfrac{\sqrt{-R}}{2} - \dfrac{a_1 R \sec(\sqrt{R}\xi)}{\mu \sec(\sqrt{R}\xi) + 1} - \dfrac{\sqrt{R}\tan(\sqrt{R}\xi)}{2\mu \sec(\sqrt{R}\xi) + 2}, \\ v_{91} = \dfrac{\sqrt{1 - 4a_1^2 R} R \sec(\sqrt{R}\xi)}{2\mu \sec(\sqrt{R}\xi) + 2} - 2\dfrac{a_1^2 R^2 \sec^2(\sqrt{R}\xi)}{(\mu \sec(\sqrt{R}\xi) + 1)^2} \\ \qquad + \dfrac{a_1 R^{\frac{3}{2}} \sec(\sqrt{R}\xi)\tan(\sqrt{R}\xi)}{(\mu \sec(\sqrt{R}\xi) + 1)^2}, \end{cases} \tag{50}$$

$$\begin{cases} u_{92} = -\dfrac{\sqrt{-R}}{2} - \dfrac{a_1 R \csc(\sqrt{R}\xi)}{\csc(\sqrt{R}\xi) + 1} + \dfrac{\sqrt{R}\cot(\sqrt{R}\xi)}{2\mu \csc(\sqrt{R}\xi) + 2}, \\ v_{92} = \dfrac{\sqrt{1 - 4a_1^2 R} R \csc(\sqrt{R}\xi)}{2\mu \csc(\sqrt{R}\xi) + 2} - 2\dfrac{a_1^2 R^2 \csc^2(\sqrt{R}\xi)}{(\mu \csc(\sqrt{R}\xi) + 1)^2} \\ \qquad - \dfrac{a_1 R^{\frac{3}{2}} \csc(\sqrt{R}\xi)\cot(\sqrt{R}\xi)}{(\mu \csc(\sqrt{R}\xi) + 1)^2}, \end{cases} \tag{51}$$

where R is an arbitrary constant and $\xi = x + \lambda t$, $\lambda = -\frac{\sqrt{-R}}{2}$, $\mu = \pm\sqrt{1 - 4a_1^2 R}$. The properties of the solutions u_{91} and v_{91} are shown by Fig. 2.

Family 10. The SLA equations have the following solutions:

$$\begin{cases} u_{101} = -\dfrac{\sqrt{-R}}{2} + \dfrac{a_1 R \sec(\sqrt{R}\xi)}{\mu \sec(\sqrt{R}\xi) + 1} - \dfrac{\sqrt{R}\tan(\sqrt{R}\xi)}{2\mu \sec(\sqrt{R}\xi) + 2}, \\ v_{101} = \dfrac{\sqrt{1 - 4a_1^2 R} R \sec(\sqrt{R}\xi)}{2\mu \sec(\sqrt{R}\xi) + 2} - 2\dfrac{a_1^2 R^2 \sec^2(\sqrt{R}\xi)}{(\mu \sec(\sqrt{R}\xi) + 1)^2} \\ \qquad + \dfrac{a_1 R^{\frac{3}{2}} \sec(\sqrt{R}\xi)\tan(\sqrt{R}\xi)}{(\mu \sec(\sqrt{R}\xi) + 1)^2}, \end{cases} \tag{52}$$

$$\begin{cases} u_{102} = -\dfrac{\sqrt{-R}}{2} - \dfrac{a_1 R \csc(\sqrt{-R}\xi)}{\mu\csc(\sqrt{R}\xi)+1} + \dfrac{\sqrt{R}\cot(\sqrt{R}\xi)}{2\mu\csc(\sqrt{R}\xi)+2}, \\ v_{102} = \dfrac{\sqrt{1-4a_1^2 RR\csc(\sqrt{R}\xi)}}{2\mu\csc(\sqrt{R}\xi)+2} - 2\dfrac{a_1^2R^2\csc^2(\sqrt{R}\xi)}{(\mu\csc(\sqrt{R}\xi)+1)^2} \\ \qquad + \dfrac{a_1 R^{\frac{3}{2}}\csc(\sqrt{R}\xi)\cot(\sqrt{R}\xi)}{(\mu\csc(\sqrt{R}\xi)+1)^2}, \end{cases} \tag{53}$$

where R is an arbitrary constant and $\xi = x + \lambda t$, $\lambda = -\frac{\sqrt{R}}{2}$, $\mu = \pm\sqrt{1-4a_1^2R}$.

Remark

(1) The solutions (41) and (42) reproduce the solutions case 1 in [21], when $\mu = 0$, $R = 4a_{10}^2$ and $a_1 = \frac{\sqrt{R}}{2R}$; the solutions (36) and (37) reproduce the solutions case 5 in [21], when $R = -b_{11}$, and reproduce the solutions case 7 in [21], when $R = a_{10}^2$; the solutions (38) and (39) reproduce the solutions case 6 in [21] when $R = -2a_{10}^2$, and reproduce the solutions case 9 in [21], when $R = -d_{11}^2$; the solutions (40) and (43) reproduce the solutions case 8 in [21], when $\mu = 0$, $R = -4d_{11}^2$ and $a_1 = \pm\frac{\sqrt{R}}{4R^2}$.
(2) The other solutions obtained here, to our knowledge, are all new families of exact solutions of the SLA equations.
(3) We corrected some errors Eqs. (10) and (11) in [15].

4. Summary and conclusions

In summary, by use of the improved general projective Riccati method, more general forms of solutions for the system SLA equations are obtained. Of course, this method can be extended to other system of nonlinear differential equations. On the other hand, we will extend this method to seek soliton-like solutions for some PDEs in the forthcoming works.

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References

[1] M.J. Ablowitz, P.A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scatting, Cambridge University Press, New York, 1991.

[2] Y. Chen, B. Li, H.Q. Zhang, Auto-Bäcklund transformation and exact solutions for modified nonlinear dispersive $mK(m,n)$ equations, Chaos Solitons Fract. 17 (2003) 693;
Y. Chen, Z.Y. Yan, H.Q. Zhang, Exact solutions for a family variable coefficient reaction-Duffing equation via the Bäcklund tranformation, Theor. Math. Phys. 132 (2002) 970;
B. Li, Y. Chen, H.Q. Zhang, Auto-Bäcklund transformation and exact solutions for compound KdV-type and compound KdVCBurgers-type equations with nonlinear terms of any order, Phys. Lett. A 305 (2002) 377.

[3] V.B. Matveev, M.A. Salle, Darbooux Transformation and Soliton, Spinger, Berlin, 1991.

[4] H.W. Tam, W.X. Ma, X.B. Hu, D.L. Wang, The Hirota–Satsuma coupled KdV equation and a coupled Ito system revisited, J. Phys. Soc. Japan 69 (2000) 45.

[5] S.Y. Lou, J.Z. Lu, Special solutions from variable separation approach: Davey–Stewartson equation, J. Phys. A: Math. Gen. 29 (1996) 4029.

[6] E.J. Parkes, B.R. Duffy, An automated tanh-function method for finding solitary wave solutions to non-linear evolution equations, Comp. Phys. Commun. 98 (1996) 288;
E.J. Parkes, B.R. Duffy, Travelling solitary wave solutions to a compound KdV–Burgers equation, Phys. Lett. A 229 (1997) 217.

[7] E. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000) 212.

[8] Y.T. Gao, B. Tian, Generalized hyperbolic-function method with computerized symbolic computation to construct the solitonic solutions to nonlinear equations of mathematical physics, Comput. Phys. Commun. 133 (2001) 158.

[9] Z.Y. Yan, New explicit travelling wave solutions for two new integrable coupled nonlinear evolution equations, Phys. Lett. A 292 (2001) 100.

[10] M.L. Wang, Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, Phys. Lett. A 216 (1996) 67;
M.L. Wang, Y.M. Wang, Y.B. Zhou, An auto-Bäcklund transformation and exact solutions to a generalized KdV equation with variable coefficients and their applications, Phys. Lett. A 303 (2002) 45.

[11] P.A. Clarkson, Kruskal, New similiarity reductions of Boussinesq equation, J. Math. Phys. 30 (1989) 2202.

[12] S.Y Lou, X.Y. Tang, J. Lin, Similarity and conditional similarity reductions of a $(2+1)$-dimensional KdV equation via a direct method, J. Math. Phys. 41 (2000) 8286.

[13] R. Conte, M. Musette, Link between solitary waves and projective Riccati equations, J. Phys. A 25 (1992) 5609.

[14] T.C. Bountis, V. Papageorgiou, P. Winternitz, On the integrability of systems nonlinear ordinary differential equations with superposition principles, J. Math. Phys. 27 (1986) 1215.

[15] Z.Y. Yan, Generalized method and its application in the higher-order nonlinear Schrodinger equation in nonlinear optical fibres, Chaos Solitons Fract. 16 (2003) 759.

[16] G.B Whitham, Proc. R. Soc. A 299 (1967) 6.

[17] L.T.F. Broer, Appl. Sci. Res. 31 (1957) 377.

[18] B.A. Kuperschmidt, Mathematics of dispersive water waves, Comm. Math. Phys. 99 (1985) 51.

[19] J.F. Zhang, Multiple soliton solutions for the approximate equation of long water wave, Acta Phys. Sin. 47 (1998) 1416 (in Chinese).

[20] Z.Y. Yan, H.Q. Zhang, Explicit exact solutions for nonlinear approximate equations with long waves in shallow water, Acta Phys. Sin. 48 (1999) 1962 (in Chinese).

[21] W. Wu, in: D.Z. Du et al. (Eds.), Algorithms and Computation, Springer, Berlin, 1994, p. 1.

[22] Y. Chen, Y. Zheng, Generalized extended tanh-function method to construct new explicit exact solutions for the approximate equations for long water waves, IJMPC 14 (4) (2003).