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Extended Jacobi elliptic function rational expansion method and abundant families of Jacobi elliptic function solutions to (1 + 1)-dimensional dispersive long wave equation

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Abstract

Our Jacobi elliptic function rational expansion method is extended to be a more powerful method, called the extended Jacobi elliptic function rational expansion method, by using more general ansatz. The (1 + 1)-dimensional dispersive long wave equation is chosen to illustrate the approach. As a consequence, we can successfully obtain the solutions found by most existing Jacobi elliptic function methods and find other new and more general solutions at the same time. When the modulus $m \to 1$, these doubly periodic solutions degenerate as soliton solutions. The method can be also applied to other nonlinear differential equations.

1. Introduction

With the development of soliton theory, there has been a great amount of activities aiming to find methods for exact solution of nonlinear differential equations, such as Bäcklund transformation, Darboux transformation, Cole-Hopf transformation, various tanh methods, various Jacobi elliptic function methods, variable separation approach, Painlevé method, homogeneous balance method, similarity reduction method and so on [1–12].

It is well known that the elliptic functions including Jacobi elliptic functions and Weierstrass elliptic functions, etc. are closely related to nonlinear differential equations [1]. Moreover many nonlinear evolution equations have been shown to possess the elliptic function solutions [6–16]. In [13], we present the Jacobi elliptic function rational expansion

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method to uniformly construct more new exact doubly-periodic solutions in terms of rational formal elliptic function of nonlinear evolution equations (NLEEs).

The present work is motivated by the desire to extended the Jacobi elliptic function rational expansion method, by using 12 Jacobi elliptic functions and their rational combination, to obtain more form and more general exact doubly-periodic solutions of NLEEs. The algorithm and its application are demonstrated later with the (1 + 1)-dimensional dispersive long wave equation.

This paper is organized as follows. In Section 2, we summarize the extended elliptic function rational expansion method. In Section 3, we apply the extended method to (1 + 1)-dimensional dispersive long wave equation and bring out many solutions. Conclusions will be presented in finally.

2. Summary of the extended Jacobi elliptic function rational expansion method

In the following we would like to outline the main steps of our general method:

Step 1. For a given nonlinear evolution equation system with some physical fields $u_i(x, y, t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0, \tag{2.1}$$

by using the wave transformation

$$u_i(x, y, t) = u_i(\xi), \quad \xi = k(x + ly + \lambda t), \tag{2.2}$$

where k, l and λ are constants to be determined later. Then the nonlinear evolution Eq. (2.1) is reduced to a nonlinear ordinary differential equation (ODE):

$$G_i(u_i, u_i', u_i'', \ldots) = 0. \tag{2.3}$$

Step 2. We introduce some ansätz in terms of finite Jacobi elliptic function rational expansion in the following forms:

1. $\mathrm{sn}\,\xi$ and $\mathrm{cn}\,\xi$ rational expansion

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\mathrm{sn}^j(\xi)}{(\mu_1\mathrm{sn}(\xi) + \mu_2\mathrm{cn}(\xi) + 1)^j} + b_{ij}\frac{\mathrm{sn}^{j-1}(\xi)\mathrm{cn}(\xi)}{(\mu_1\mathrm{sn}(\xi) + \mu_2\mathrm{cn}(\xi) + 1)^j}\right), \tag{2.4.1}$$

2. $\mathrm{sn}\,\xi$ and $\mathrm{dn}\,\xi$ rational expansion

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\mathrm{ns}^j(\xi)}{(\mu_1\mathrm{ns}(\xi) + \mu_2\mathrm{cs}(\xi) + 1)^j} + b_{ij}\frac{\mathrm{ns}^{j-1}(\xi)\mathrm{cs}(\xi)}{(\mu_1\mathrm{ns}(\xi) + \mu_2\mathrm{cs}(\xi) + 1)^j}\right), \tag{2.4.2}$$

3. $\mathrm{sc}\,\xi$ and $\mathrm{nc}\,\xi$ rational expansion

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\mathrm{sn}^j(\xi)}{(\mu_1\mathrm{sn}(\xi) + \mu_2\mathrm{dn}(\xi) + 1)^j} + b_{ij}\frac{\mathrm{sn}^{j-1}(\xi)\mathrm{dn}(\xi)}{(\mu_1\mathrm{sn}(\xi) + \mu_2\mathrm{dn}(\xi) + 1)^j}\right), \tag{2.4.3}$$

4. $\mathrm{sd}\,\xi$ and $\mathrm{nd}\,\xi$ rational expansion

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\mathrm{sd}^j(\xi)}{(\mu_1\mathrm{sd}(\xi) + \mu_2\mathrm{nd}(\xi) + 1)^j} + b_{ij}\frac{\mathrm{sd}^{j-1}(\xi)\mathrm{nd}(\xi)}{(\mu_1\mathrm{sd}(\xi) + \mu_2\mathrm{nd}(\xi) + 1)^j}\right), \tag{2.4.4}$$

where $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$, $\mathrm{dn}\,\xi$, $\mathrm{ns}\,\xi$, $\mathrm{cs}\,\xi$, $\mathrm{sd}\,\xi$ and $\mathrm{nd}\,\xi$ are the Jacobian elliptic sine function, the Jacobian elliptic cosine function, the Jacobian elliptic function of the third kind and other Jacobian functions which is denoted by Glaishers symbols and are generated by these three kinds of functions, namely [14–16],

$$\mathrm{ns}\,\xi = \frac{1}{\mathrm{sn}\,\xi}, \qquad \mathrm{nc}\,\xi = \frac{1}{\mathrm{cn}\,\xi}, \qquad \mathrm{nd}\,\xi = \frac{1}{\mathrm{dn}\,\xi}, \qquad \mathrm{sd}\,\xi = \frac{\mathrm{sn}\,\xi}{\mathrm{dn}\,\xi}, \tag{2.5.1}$$

$$\mathrm{sc}\,\xi = \frac{\mathrm{sn}\,\xi}{\mathrm{cn}\,\xi}, \qquad \mathrm{cs}\,\xi = \frac{\mathrm{cn}\,\xi}{\mathrm{sn}\,\xi}, \qquad \mathrm{ds}\,\xi = \frac{\mathrm{dn}\,\xi}{\mathrm{sn}\,\xi}, \tag{2.5.2}$$

which are double periodic and posses the following properties:

1. Properties of triangular function

$$\mathrm{cn}^2\xi + \mathrm{sn}^2\xi = \mathrm{dn}^2\xi + m^2\mathrm{sn}^2\xi = 1, \tag{2.6.1}$$

$$\mathrm{ns}^2\xi = 1 + \mathrm{cs}^2\xi, \qquad \mathrm{ns}^2\xi = m^2 + \mathrm{ds}^2\xi, \tag{2.6.2}$$

$$\mathrm{sc}^2\xi + 1 = \mathrm{nc}^2\xi, \qquad \mathrm{m}^2\mathrm{sd}^2\xi + 1 = \mathrm{nd}^2\xi. \tag{2.6.3}$$

2. Derivatives of the Jacobi elliptic functions

$$\mathrm{sn}'\xi = \mathrm{cn}\,\xi\,\mathrm{dn}\,\xi, \qquad \mathrm{cn}'\xi = -\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi, \qquad \mathrm{dn}'\xi = -m^2\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi, \tag{2.7.1}$$

$$\mathrm{ns}'\xi = -\mathrm{ds}\,\xi\,\mathrm{cs}\,\xi, \qquad \mathrm{ds}'\xi = -\mathrm{cs}\,\xi\,\mathrm{ns}\,\xi, \qquad \mathrm{cs}'\xi = -\mathrm{ns}\,\xi\,\mathrm{ds}\,\xi, \tag{2.7.2}$$

$$\mathrm{sc}'\xi = \mathrm{nc}\,\xi\,\mathrm{dc}\,\xi, \qquad \mathrm{nc}'\xi = \mathrm{sc}\,\xi\,\mathrm{dc}\,\xi, \qquad \mathrm{cd}'\xi = \mathrm{cd}\,\xi\,\mathrm{nd}\,\xi, \qquad \mathrm{nd}'\xi = m^2\,\mathrm{sd}\,\xi\,\mathrm{cd}\,\xi, \tag{2.7.3}$$

where m is a modulus. The Jacobi–Glaisher functions for elliptic function can be found in Refs. [14,15].

Step 3. The parameter m_i can be found by balancing the highest order derivative term and the nonlinear terms in (2.1) or (2.3).

Step 4. Respectively substitute Eqs. (2.4) into Eq. (2.3) along with Eqs. (2.6) and (2.7) and then respectively set all coefficients of $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$, $\mathrm{ns}^i(\xi)\mathrm{cs}^j(\xi)$, $\mathrm{sn}^i(\xi)\mathrm{dn}^j(\xi)$ and $\mathrm{sd}^i(\xi)\mathrm{nd}^j(\xi)$ ($i = 1,2,\ldots;\ j = 0,1$) to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, μ_1, μ_2, a_{i0}, a_{ij} and b_{ij} ($i = 1,2,\ldots;\ j = 1,2,\ldots,m_i$).

Step 5. By use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method [24], solving the over-determined algebraic equations, we would end up with the explicit expressions for k, μ_1, μ_2, a_{i0}, a_{ij} and b_{ij} ($i = 1,2,\ldots;\ j = 1,2,\ldots,m_i$).

From which k, μ_1, μ_2, a_{i0}, a_{ij} and b_{ij} ($i = 1,2,\ldots;\ j = 1,2,\ldots,m_i$) can be obtained. In this way, we can get double periodic solutions with Jacobi elliptic function.

Since

$$\lim_{m\to 1}\mathrm{sn}\,\xi = \tanh\xi, \qquad \lim_{m\to 1}\mathrm{cn}\,\xi = \mathrm{sech}\,\xi, \qquad \lim_{m\to 1}\mathrm{dn}\,\xi = \mathrm{sech}\,\xi, \tag{2.8.1}$$

$$\lim_{m\to 1}\mathrm{ns}\,\xi = \coth\xi, \qquad \lim_{m\to 1}\mathrm{cs}\,\xi = \mathrm{csch}\,\xi, \qquad \lim_{m\to 1}\mathrm{ds}\,\xi = \mathrm{csch}\,\xi, \tag{2.8.2}$$

$$\lim_{m\to 0}\mathrm{sn}\,\xi = \sin\xi, \qquad \lim_{m\to 0}\mathrm{cn}\,\xi = \cos\xi, \qquad \lim_{m\to 0}\mathrm{dn}\,\xi = 1, \tag{2.8.3}$$

$$\lim_{m\to 0}\mathrm{ns}\,\xi = \csc\xi, \qquad \lim_{m\to 0}\mathrm{cs}\,\xi = \cot\xi, \qquad \lim_{m\to 0}\mathrm{ds}\,\xi = \csc\xi, \tag{2.8.4}$$

u_i degenerate respectively as the following form:

1. Solitary wave solutions:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\tanh^j(\xi)}{(\mu_1\tanh(\xi) + \mu_2\,\mathrm{sech}(\xi) + 1)^j} + b_{ij}\frac{\tanh^{j-1}(\xi)\,\mathrm{sech}(\xi)}{(\mu_1\tanh(\xi) + \mu_2\,\mathrm{sech}(\xi) + 1)^j}\right), \tag{2.9.1}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\frac{1}{\tanh^j(\xi)}}{\left(\mu_1\frac{1}{\tanh(\xi)} + \mu_2\frac{\mathrm{sech}(\xi)}{\tanh(\xi)} + 1\right)^j} + b_{ij}\frac{\left(\frac{\mathrm{sech}(\xi)}{\tanh(\xi)}\right)^{j-1}\frac{1}{\tanh(\xi)}}{\left(\mu_1\frac{1}{\tanh(\xi)} + \mu_2\frac{\mathrm{sech}(\xi)}{\tanh(\xi)} + 1\right)^j}\right), \tag{2.9.2}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\tanh^j(\xi)}{(\mu_1\tanh^j(\xi) + \mu_2\mathrm{sech}(\xi) + 1)^j} + b_{ij}\frac{\tanh^{j-1}(\xi)\mathrm{sech}(\xi)}{(\mu_1\tanh^j(\xi) + \mu_2\mathrm{sech}(\xi) + 1)^j}\right), \tag{2.9.3}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\left(\frac{\tanh(\xi)}{\mathrm{sech}(\xi)}\right)^j}{\left(\mu_1\left(\frac{\tanh(\xi)}{\mathrm{sech}(\xi)}\right) + \mu_2\frac{1}{\mathrm{sech}(\xi)} + 1\right)^j} + b_{ij}\frac{\left(\frac{\tanh(\xi)}{\mathrm{sech}(\xi)}\right)^{j-1}\frac{1}{\mathrm{sech}(\xi)}}{\left(\mu_1\left(\frac{\tanh(\xi)}{\mathrm{sech}(\xi)}\right) + \mu_2\frac{1}{\mathrm{sech}(\xi)} + 1\right)^j}\right). \tag{2.9.4}$$

2. Triangular function formal solution:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\sin^j(\xi)}{(\mu_1\sin(\xi)+\mu_2\cos(\xi)+1)^j} + b_{ij}\frac{\sin^{j-1}(\xi)\cos(\xi)}{(\mu_1\sin(\xi)+\mu_2\cos(\xi)+1)^j}\right), \tag{2.10.1}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\frac{1}{\sin^j(\xi)}}{\left(\mu_1\frac{1}{\sin(\xi)}+\mu_2\frac{\cos(\xi)}{\sin(\xi)}+1\right)^j} + b_{ij}\frac{\frac{\cos(\xi)}{\sin^j(\xi)}}{\left(\mu_1\frac{1}{\sin(\xi)}+\mu_2\frac{\cos(\xi)}{\sin(\xi)}+1\right)^j}\right), \tag{2.10.2}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\sin^j(\xi)}{(\mu_1\sin^j(\xi)+\mu_2+1)^j} + b_{ij}\frac{\sin^{j-1}(\xi)}{(\mu_1\sin^j(\xi)+\mu_2+1)^j}\right), \tag{2.10.3}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\sin^j(\xi)}{(\mu_1\sin(\xi)+\mu_2+1)^j} + b_{ij}\frac{\sin^{j-1}(\xi)}{(\mu_1\sin(\xi)+\mu_2+1)^j}\right). \tag{2.10.4}$$

So the extended Jacobi elliptic function rational expansion method is more powerful than the method by Liu et al. [10], the method by Fan [11], the method extended by Yan [12] and the Jacobi elliptic function rational expansion method by Wang et al. [13]. The solutions which contain solitary wave solutions, singular solitary solutions and triangular function formal solutions can be gotten by the extended method.

Remark. It is necessary to point out that above four rational expansion combinations only require solving the recurrent coefficient relation or derivative relation for the terms of polynomial for computation closed. Therefore other the Jacobi elliptic functions can be chosen to combinating, for simplicity, we omit them here.

3. Exact solutions of the (1 + 1)-dimensional dispersive long wave equation

Let us consider the (1 + 1)-dimensional dispersive long wave equation (DLWE), i.e.,

$$\begin{cases} v_t + vv_x + w_x = 0, \\ w_t + (wv)_x + \frac{1}{3}v_{xxx} = 0, \end{cases} \tag{3.1}$$

where $w-1$ is the elevation of the water wave, v is the surface velocity of water along x-direction. There is an amount of paper devoted to this equations [17–23]. According to the above method, to seek travelling wave solutions of Eqs. (3.1), we make the transformation

$$w(x,t) = \sigma(\xi), \qquad v(x,t) = \phi(\xi), \qquad \xi = k(x+\lambda t), \tag{3.2}$$

where λ is a constant to be determined later, and thus Eqs. (3.1) becomes

$$\begin{cases} \lambda\phi' + \phi\phi' + \sigma' = 0, \\ \lambda\sigma' + (\sigma\phi)' + \frac{k^2}{3}\phi''' = 0. \end{cases} \tag{3.3}$$

By balancing $\phi'''(\xi)$ and $(\sigma(\xi)\phi(\xi))'$ in Eq. (3.3) and by balancing $\sigma'(\xi)$ and $\phi(\xi)\phi'(\xi)$ in Eq. (3.3), we can obtain that $m_\phi = 1$ and $m_\sigma = 2$. Now we consider the system (3.3) in the above four cases, i.e. (2.4.1)–(2.4.4).

3.1. $\mathrm{sn}\,\xi$ and $\mathrm{cn}\,\xi$ rational expansion

Now we consider the ansätz (2.4.1). For Eq. (3.3), the ansätz (2.4.1) becomes

$$\begin{cases} \phi(\xi) = a_0 + a_1\frac{\mathrm{sn}(\xi)}{\mu_1\mathrm{sn}(\xi)+\mu_2\mathrm{cn}(\xi)+1} + b_1\frac{\mathrm{cn}(\xi)}{\mu_1\mathrm{sn}(\xi)+\mu_2\mathrm{cn}(\xi)+1}, \\ \sigma(\xi) = A_0 + A_1\frac{\mathrm{sn}(\xi)}{\mu_1\mathrm{sn}(\xi)+\mu_2\mathrm{cn}(\xi)+1} + B_1\frac{\mathrm{cn}(\xi)}{\mu_1\mathrm{sn}(\xi)+\mu_2\mathrm{cn}(\xi)+1} \\ \qquad + A_2\frac{\mathrm{sn}^2(\xi)}{(\mu_1\mathrm{sn}(\xi)+\mu_2\mathrm{cn}(\xi)+1)^2} + B_2\frac{\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{(\mu_1\mathrm{sn}(\xi)+\mu_2\mathrm{cn}(\xi)+1)^2}, \end{cases} \tag{3.4}$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 are constants to be determined later.

With the aid of *Maple*, substituting (3.4) along with (2.6) and (2.7) into (3.3), yields a set of algebraic equations for $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$ $(i = 0, 1, \ldots; j = 0, 1)$. Setting the coefficients of these terms $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2 and k.

By use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method [24], solving the over-determined algebraic equations, we get the following results:

Case 1

$$\begin{aligned} &k = \pm\frac{\sqrt{3}a_1}{2m}, \qquad a_0 = -\lambda, \qquad A_0 = \frac{a_1^2(1+m^2)}{4m^2}, \\ &A_2 = -\frac{1}{2}a_1^2, \qquad b_1 = A_1 = B_1 = B_2 = \mu_1 = \mu_2 = 0. \end{aligned} \tag{3.5}$$

Case 2

$$\begin{aligned} &B_1 = -\frac{1}{3}k^2\mu_2, \qquad A_2 = -\frac{1}{3}(k^2\mu_2^2 - k^2\mu_2^2m^2 + k^2m^2), \qquad A_0 = \frac{1}{3}\frac{k^2m^2}{\mu_2^2 - \mu_2^2m^2 + m^2}, \\ &b_1 = \pm\frac{1}{3}\sqrt{-3\mu_2^4m^2 + 3\mu_2^4 + 6\mu_2^2m^2 - 3\mu_2^2 - 3m^2}k, \qquad a_1 = \pm\frac{1}{3}\sqrt{3\mu_2^2 - 3\mu_2^2m^2 + 3m^2}k, \\ &B_2 = \pm\frac{k^2}{3}\sqrt{\mu_2^2 - \mu_2^2m^2 + m^2}\sqrt{-\mu_2^4m^2 + \mu_2^4 + 2\mu_2^2m^2 - \mu_2^2 - m^2}, \\ &\mu_1 = 0, \qquad A_1 = \frac{\sqrt{\mu_2^2 - \mu_2^2m^2 + m^2}k^2\mu_2(\mu_2^2m^2 - \mu_2^2 - m^2 + 1)}{3\sqrt{-\mu_2^4m^2 + \mu_2^4 + 2\mu_2^2m^2 - \mu_2^2 - m^2}}, \\ &a_0 = \frac{k\mu_2^3m^2 - k\mu_2^3 - k\mu_2m^2 + k\mu_2 \pm \lambda\sqrt{-3\mu_2^4m^2 + 3\mu_2^4 + 6\mu_2^2m^2 - 3\mu_2^2 - 3m^2}}{\sqrt{-3\mu_2^4m^2 + 3\mu_2^4 + 6\mu_2^2m^2 - 3\mu_2^2 - 3m^2}}. \end{aligned} \tag{3.6}$$

Case 3

$$\begin{aligned} &A_2 = -\frac{1}{3}(k^2\mu_1^4 - k^2\mu_1^2 - k^2\mu_1^2m^2 + k^2m^2), \qquad A_1 = -\frac{1}{3}(k^2\mu_1 - 2k^2\mu_1^3 + k^2\mu_1m^2), \\ &a_0 = -\frac{-k\mu_1 \pm \lambda\sqrt{3\mu_1^4 - 3\mu_1^2 - 3\mu_1^2m^2 + 3m^2} + k\mu_1^3}{\sqrt{3\mu_1^4 - 3\mu_1^2 - 3\mu_1^2m^2 + 3m^2}}, \qquad b_1 = \frac{k}{3}\sqrt{3\mu_1^2 - 3m^2}, \\ &\mu_2 = 0, \qquad a_1 = \frac{k}{3}\sqrt{3\mu_1^4 - 3\mu_1^2 - 3\mu_1^2m^2 + 3m^2}, \qquad B_1 = \frac{1}{3}\frac{\sqrt{\mu_1^2 - m^2}k^2\mu_1(-1 + \mu_1^2)}{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2m^2 + m^2}}, \\ &B_2 = -\frac{1}{3}\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2m^2 + m^2}k^2\sqrt{\mu_1^2 - m^2}, \qquad A_0 = -\frac{k^2(\mu_1^4 - 2\mu_1^2m^2 + m^2)}{3(\mu_1^2 - m^2)}. \end{aligned} \tag{3.7}$$

Case 4

$$\begin{aligned} &a_1 = \pm(2a_0 + 2\lambda), \qquad A_1 = \pm 2\frac{2a_0\lambda m^2 + A_0 - A_0m^2 + \lambda^2m^2 + a_0^2m^2}{1 - m^2}, \qquad \mu_1 = \mu_2 = \pm 1, \\ &B_1 = \pm 2\frac{A_0 - A_0m^2 + 2a_0\lambda + a_0^2 + \lambda^2}{1 - m^2}, \qquad A_2 = -2\lambda^2 - 4a_0\lambda - 2a_0^2, \\ &k = 3\frac{\lambda + a_0}{\sqrt{3 - 3m^2}}, \qquad B_2 = 2\frac{A_0 - A_0m^2 + 2a_0\lambda + a_0^2 + \lambda^2}{1 - m^2}, \qquad b_1 = 0. \end{aligned} \tag{3.8}$$

From (3.2), (3.4) and Cases 1–4, we obtain the following solutions for Eqs. (3.1):

Family 1. From Eqs. (3.5), we obtain the following $\mathrm{sn}\,\xi$ and $\mathrm{cn}\,\xi$ rational formal doubly periodic solutions for the DLWE, as follows:

$$v_1 = -\lambda + a_1\,\mathrm{sn}\left(\pm\frac{1}{2}\frac{\sqrt{3}a_1(x + \lambda t)}{m}\right), \tag{3.9.1}$$

$$w_1 = \frac{a_1^2(1 + m^2)}{4m^2} - \frac{1}{2}a_1^2\,\mathrm{sn}^2\left(\pm\frac{1}{2}\frac{\sqrt{3}a_1(x + \lambda t)}{m}\right), \tag{3.9.2}$$

where a_1 and λ are arbitrary constants.

Family 2. From Eqs. (3.6), we obtain the following $\operatorname{sn}\xi$ and $\operatorname{cn}\xi$ rational formal doubly periodic solutions for the DLWE, as follows:

$$v_2 = \frac{k\mu_2^3 m^2 - k\mu_2^3 - k\mu_2 m^2 + k\mu_2 \pm \lambda\sqrt{-3\mu_2^4 m^2 + 3\mu_2^4 + 6\mu_2^2 m^2 - 3\mu_2^2 - 3m^2}}{\sqrt{-3\mu_2^4 m^2 + 3\mu_2^4 + 6\mu_2^2 m^2 - 3\mu_2^2 - 3m^2}} \pm \frac{\sqrt{3\mu_2^2 - 3\mu_2^2 m^2 + 3m^2}k\operatorname{sn}(k(x+\lambda t))}{3(1+\mu_2\operatorname{cn}(k(x+\lambda t)))} \pm \frac{\sqrt{-3\mu_2^4 m^2 + 3\mu_2^4 + 6\mu_2^2 m^2 - 3\mu_2^2 - 3m^2}k\operatorname{cn}(k(x+\lambda t))}{3(1+\mu_2\operatorname{cn}(k(x+\lambda t)))}, \tag{3.10.1}$$

$$w_2 = \frac{1}{3}\frac{k^2 m^2}{\mu_2^2 - \mu_2^2 m^2 + m^2} + \frac{(-k^2\mu_2^2 + k^2\mu_2^2 m^2 - k^2 m^2)\operatorname{sn}^2(k(x+\lambda t))}{3(1+\mu_2\operatorname{cn}(k(x+\lambda t)))^2} - \frac{1}{3}\frac{k^2\mu_2\operatorname{cn}(k(x+\lambda t))}{1+\mu_2\operatorname{cn}(k(x+\lambda t))} + \frac{1}{3}\frac{\sqrt{\mu_2^2 - \mu_2^2 m^2 + m^2}k^2\mu_2(\mu_2^2 m^2 - \mu_2^2 - m^2 + 1)\operatorname{sn}(k(x+\lambda t))}{\sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2\mu_2^2 m^2 - \mu_2^2 - m^2}(1+\mu_2\operatorname{cn}(k(x+\lambda t)))} \pm \frac{\sqrt{\mu_2^2 - \mu_2^2 m^2 + m^2}k^2\sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2\mu_2^2 m^2 - \mu_2^2 - m^2}\operatorname{sn}(k(x+\lambda t))\operatorname{cn}(k(x+\lambda t))}{3(1+\mu_2\operatorname{cn}(k(x+\lambda t)))^2}, \tag{3.10.2}$$

where k, μ_2 and λ are arbitrary constants.

Family 3. From Eqs. (3.7), we obtain the following $\operatorname{sn}\xi$ and $\operatorname{cn}\xi$ rational formal doubly periodic solutions for the DLWE, as follows:

$$v_3 = -\frac{-k\mu_1 \pm \lambda\sqrt{3\mu_1^4 - 3\mu_1^2 - 3\mu_1^2 m^2 + 3m^2} + k\mu_1^3}{\sqrt{3\mu_1^4 - 3\mu_1^2 - 3\mu_1^2 m^2 + 3m^2}} \pm \frac{\sqrt{3\mu_1^4 - 3\mu_1^2 - 3\mu_1^2 m^2 + 3m^2}k\operatorname{sn}(k(x+\lambda t))}{3(\mu_1\operatorname{sn}(k(x+\lambda t))+1)} \pm \frac{\sqrt{3\mu_1^2 - 3m^2}k\operatorname{cn}(k(x+\lambda t))}{3(\mu_1\operatorname{sn}(k(x+\lambda t))+1)}, \tag{3.11.1}$$

$$w_3 = -\frac{k^2(\mu_1^4 - 2\mu_1^2 m^2 + m^2)}{3(\mu_1^2 - m^2)} - \frac{(k^2\mu_1 - 2k^2\mu_1^3 + k^2\mu_1 m^2)\operatorname{sn}(k(x+\lambda t))}{3(\mu_1\operatorname{sn}(k(x+\lambda t))+1)} \pm \frac{1}{3} \times \frac{\sqrt{\mu_1^2 - m^2}k^2\mu_1(-1+\mu_1^2)\operatorname{cn}(k(x+\lambda t))}{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}(\mu_1\operatorname{sn}(k(x+\lambda t))+1)} - \frac{(k^2\mu_1^4 - k^2\mu_1^2 - k^2\mu_1^2 m^2 + k^2 m^2)\operatorname{sn}^2(k(x+\lambda t))}{3(\mu_1\operatorname{sn}(k(x+\lambda t))+1)^2} \pm \frac{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}k^2\sqrt{\mu_1^2 - m^2}\operatorname{sn}(k(x+\lambda t))\operatorname{cn}(k(x+\lambda t))}{3(\mu_1\operatorname{sn}(k(x+\lambda t))+1)^2}, \tag{3.11.2}$$

where k, μ_1 and λ are arbitrary constants.

Family 4. From Eqs. (3.8), we obtain the following $\operatorname{sn}\xi$ and $\operatorname{cn}\xi$ rational formal doubly periodic solutions for the DLWE, as follows:

$$v_4 = a_0 \pm \frac{(2\lambda + 2a_0)\operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)}{\pm\operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) \pm \operatorname{cn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) + 1}, \tag{3.12.1}$$

$$w_4(x,t) = A_0 \pm \frac{2(A_0 m^2 - A_0 + \lambda^2 m^2 - 2a_0\lambda m^2 - a_0^2 m^2)\operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)}{(m^2-1)\left(\pm\operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) \pm \operatorname{cn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) + 1\right)} \pm \frac{2(2a_0\lambda a_0^2 - A_0 m^2 + A_0 + \lambda^2)\operatorname{cn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)}{(m^2-1)\left(\pm\operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) \pm \operatorname{cn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) + 1\right)}$$

$$
-\frac{(2\lambda^2+4a_0\lambda+2a_0^2)\mathrm{sn}^2\left(\pm 3\,\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)}{\left(\pm\mathrm{sn}\left(\pm 3\,\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)\pm\mathrm{cn}\left(\pm 3\,\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)+1\right)^2}
\pm 2\,\frac{(2a_0\lambda+a_0^2-A_0m^2+A_0+\lambda^2)\mathrm{sn}\left(\pm 3\,\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)\mathrm{cn}\left(\pm 3\,\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)}{(m^2-1)\left(\pm\mathrm{sn}\left(\pm 3\,\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)\pm\mathrm{cn}\left(\pm 3\,\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)+1\right)^2}, \tag{3.12.2}
$$

where a_0, A_0 and λ are arbitrary constants.

3.2. ns ξ and cs ξ rational expansion

Now we consider the ansätz (2.4.2). For Eq. (3.3), the ansätz (2.4.2) becomes

$$
\begin{cases}
\phi(\xi)=a_0+a_1\frac{\mathrm{ns}(\xi)}{\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1}+b_1\frac{\mathrm{cs}(\xi)}{\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1},\\
\sigma(\xi)=A_0+A_1\frac{\mathrm{ns}(\xi)}{\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1}+B_1\frac{\mathrm{cs}(\xi)}{\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1}\\
\qquad +A_2\frac{\mathrm{ns}^2(\xi)}{(\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1)^2}+B_2\frac{\mathrm{ns}(\xi)\mathrm{cs}(\xi)}{(\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1)^2},
\end{cases} \tag{3.13}
$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 are constants to be determined later. Following the same steps in Section 3.1, we can obtain the following ns ξ and cs ξ rational formal doubly periodic solution:

Family 5

$$
v_5=-\lambda+b_1\,\mathrm{cs}\left(\pm\frac{1}{2}\sqrt{3}b_1(x+\lambda t)\right), \tag{3.14.1}
$$

$$
w_5=\frac{1}{4}b_1^2m^2-\frac{1}{2}b_1^2\,\mathrm{ns}^2\left(\pm\frac{1}{2}\sqrt{3}b_1(x+\lambda t)\right), \tag{3.14.2}
$$

where b_1 and λ are arbitrary constants.

Family 6

$$
v_6=-\lambda+a_1\,\mathrm{ns}\left(\pm\frac{1}{2}\sqrt{3}a_1(x+\lambda t)\right), \tag{3.15.1}
$$

$$
w_6=\frac{1}{4}a_1^2(m^2+1)-\frac{1}{2}a_1^2\,\mathrm{ns}^2\left(\pm\frac{1}{2}\sqrt{3}a_1(x+\lambda t)\right), \tag{3.15.2}
$$

where a_1 and λ are arbitrary constants.

Family 7

$$
v_7=-\lambda+a_1\,\mathrm{ns}(\pm\sqrt{3}a_1(x+\lambda t))-a_1\,\mathrm{cs}(\pm\sqrt{3}a_1(x+\lambda t)), \tag{3.16.1}
$$

$$
w_7=a_1^2m^2-a_1^2\,\mathrm{ns}^2(\pm\sqrt{3}a_1(x+\lambda t))+a_1^2\,\mathrm{ns}(\pm\sqrt{3}a_1(x+\lambda t))\mathrm{cs}(\pm\sqrt{3}a_1(x+\lambda t)), \tag{3.16.2}
$$

where a_1 and λ are arbitrary constants.

Family 8

$$
v_8=-\frac{k\mu_2^3-k\mu_2^3m^2+k\mu_2-k\mu_2m^2\pm\lambda\sqrt{-3\mu_2^4m^2+3\mu_2^4+6\mu_2^2-3\mu_2^2m^2+3}}{\sqrt{-3\mu_2^4m^2+3\mu_2^4+6\mu_2^2-3\mu_2^2m^2+3}}
\pm\frac{\sqrt{-3\mu_2^4m^2+3\mu_2^4+6\mu_2^2-3\mu_2^2m^2+3}\,k\,\mathrm{cs}(k(x+\lambda t))}{3(1+\mu_2\,\mathrm{cs}(k(x+\lambda t)))}\pm\frac{\sqrt{-3\mu_2^2m^2+3\mu_2^2+3}\,k\,\mathrm{ns}(k(x+\lambda t))}{3(1+\mu_2\,\mathrm{cs}(k(x+\lambda t)))}, \tag{3.17.1}
$$

$$w_8 = -\frac{\frac{1}{3}k^2m^2}{\mu_2^2m^2-\mu_2^2-1} \pm \frac{\sqrt{1-\mu_2^2m^2+\mu_2^2}k^2\mu_2(m^2-\mu_2^2+\mu_2^2m^2-1)\mathrm{ns}(k(x+\lambda t))}{3\sqrt{1-\mu_2^4m^2+\mu_2^4+2\mu_2^2-\mu_2^2m^2}(1+\mu_2\,\mathrm{cs}(k(x+\lambda t)))} - \frac{1}{3}\frac{k^2\mu_2m^2\,\mathrm{cs}(k(x+\lambda t))}{1+\mu_2\,\mathrm{cs}(k(x+\lambda t))} \pm \frac{(k^2\mu_2^2m^2-k^2\mu_2^2-k^2)\mathrm{ns}^2(k(x+\lambda t))}{3(1+\mu_2\,\mathrm{cs}(k(x+\lambda t)))^2} \pm \frac{\sqrt{1-\mu_2^2m^2+\mu_2^2+1}k^2\sqrt{\mu_2^4-\mu_2^4m^2+2\mu_2^2-\mu_2^2m^2}\,\mathrm{ns}(k(x+\lambda t))\mathrm{cs}(k(x+\lambda t))}{3(1+\mu_2\,\mathrm{cs}(k(x+\lambda t)))^2}, \tag{3.17.2}$$

where k, μ_2 and λ are arbitrary constants.

Family 9

$$v_9 = \frac{k\mu_1^3m^2 \pm \lambda\sqrt{3\mu_1^4m^2-3\mu_1^2-3\mu_1^2m^2+3}-k\mu_1m^2}{\sqrt{3\mu_1^4m^2-3\mu_1^2-3\mu_1^2m^2+3}} \pm \frac{\sqrt{1-\mu_1^2m^2}k\,\mathrm{cs}(k(x+\lambda t))}{\sqrt{3}(\mu_1\,\mathrm{ns}(k(x+\lambda t))+1)} \pm \frac{\sqrt{\mu_1^4m^2-\mu_1^2-\mu_1^2m^2+1}k\,\mathrm{ns}(k(x+\lambda t))}{\sqrt{3}(\mu_1\,\mathrm{ns}(k(x+\lambda t))+1)}, \tag{3.18.1}$$

$$w_9 = \frac{k^2m^2(\mu_1^4m^2-2\mu_1^2+1)}{3(1-\mu_1^2m^2)} - \frac{(k^2\mu_1m^2-2k^2\mu_1^3m^2+k^2\mu_1)\mathrm{ns}(k(x+\lambda t))}{3(\mu_1\,\mathrm{ns}(k(x+\lambda t))+1)} \pm \frac{1}{3}\frac{\sqrt{-\mu_1^2m^2+1}k^2\mu_1m^2(\mu_1^2-1)\mathrm{cs}(k(x+\lambda t))}{\sqrt{\mu_1^4m^2-\mu_1^2-\mu_1^2m^2+1}(\mu_1\,\mathrm{ns}(k(x+\lambda t))+1)} - \frac{(k^2\mu_1^4m^2-k^2\mu_1^2-k^2\mu_1^2m^2+k^2)\mathrm{ns}^2(k(x+\lambda t))}{(\mu_1\,\mathrm{ns}(k(x+\lambda t))+1)^2} \pm \frac{\sqrt{\mu_1^4m^2-\mu_1^2-\mu_1^2m^2+1}k^2\sqrt{1-\mu_1^2m^2}\,\mathrm{ns}(k(x+\lambda t))\mathrm{cs}(k(x+\lambda t))}{(\mu_1\,\mathrm{ns}(k(x+\lambda t))+1)^2}, \tag{3.18.2}$$

where k, μ_1 and λ are arbitrary constants.

Family 10

$$v_{10} = \pm\frac{1}{3}\sqrt{3-3m^2}k - \lambda \pm \frac{2}{3}\frac{\sqrt{3-3m^2}k\,\mathrm{ns}(k(x+\lambda t))}{\pm\mathrm{ns}(k(x+\lambda t))\pm\mathrm{cs}(k(x+\lambda t))+1} \pm \frac{2}{3}\frac{\sqrt{3-3m^2}k\,\mathrm{cs}(k(x+\lambda t))}{\pm\mathrm{ns}(k(x+\lambda t))\pm\mathrm{cs}(k(x+\lambda t))+1}, \tag{3.19.1}$$

$$w_{10} = \frac{1}{3}k^2m^2 - \frac{2}{3}\frac{(k^2-k^2m^2)\mathrm{ns}^2(k(x+\lambda t))}{(\pm\mathrm{ns}(k(x+\lambda t))\pm\mathrm{cs}(k(x+\lambda t))+1)^2} - \frac{2}{3}\frac{(k^2-k^2m^2)\mathrm{ns}(k(x+\lambda t))\mathrm{cs}(k(x+\lambda t))}{(\pm\mathrm{ns}((x+\lambda t))\pm\mathrm{cs}(k(x+\lambda t))+1)^2}, \tag{3.19.2}$$

where k and λ are arbitrary constants.

3.3. sn ξ and dn ξ rational expansion

Now we consider the ansätz (2.4.3). For Eq. (3.3), the ansätz (2.4.3) becomes

$$\begin{cases} \phi(\xi) = a_0 + a_1\frac{\mathrm{sn}(\xi)}{\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1} + b_1\frac{\mathrm{dn}(\xi)}{\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1}, \\ \sigma(\xi) = A_0 + A_1\frac{\mathrm{sn}(\xi)}{\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1} + B_1\frac{\mathrm{dn}(\xi)}{\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1} + A_2\frac{\mathrm{sn}^2(\xi)}{(\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1)^2} + B_2\frac{\mathrm{sn}(\xi)\mathrm{dn}(\xi)}{(\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1)^2}, \end{cases} \tag{3.20}$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 are constants to be determined later. Following the same steps in Section 3.1, we can obtain the following sn ξ and dn ξ rational formal doubly periodic solution:

Family 11

$$v_{11} = -\lambda + a_1\,\mathrm{sn}\left(\frac{\sqrt{3}a_1(x+\lambda t)}{m}\right) \pm \frac{ia_1\mathrm{dn}\left(\frac{\sqrt{3}a_1(x+\lambda t)}{m}\right)}{m}, \tag{3.21.1}$$

$$w_{11}(\xi) = a_1^2 - a_1^2\mathrm{sn}^2\left(\pm\frac{\sqrt{3}a_1(x+\lambda t)}{m}\right) \pm \frac{ia_1^2\mathrm{sn}\left(\pm\frac{\sqrt{3}a_1(x+\lambda t)}{m}\right)\mathrm{dn}\left(\pm\frac{\sqrt{3}a_1(x+\lambda t)}{m}\right)}{m}, \tag{3.21.2}$$

where a_1 and λ are arbitrary constants.

Family 12

$$v_{12} = -\lambda + a_1\,\mathrm{sn}\left(\pm\frac{\sqrt{3}a_1(x+\lambda t)}{2m}\right), \tag{3.22.1}$$

$$w_{12} = \frac{a_1^2(m^2+1)}{4m^2} - \frac{1}{2}a_1^2\mathrm{sn}^2\left(\pm\frac{\sqrt{3}a_1(x+\lambda t)}{2m}\right), \tag{3.22.2}$$

where a_1 and λ are arbitrary constants.

Family 13

$$v_{13} = -\lambda + b_1\mathrm{dn}\left(\pm\frac{1}{2}i\sqrt{3}b_1(x+\lambda t)\right), \tag{3.23.1}$$

$$w_{13} = -\frac{1}{4}b_1^2m^2 + \frac{1}{2}b_1^2m^2\mathrm{sn}^2\left(\pm\frac{1}{2}i\sqrt{3}b_1(x+\lambda t)\right), \tag{3.23.2}$$

where b_1 and λ are arbitrary constants.

Family 14

$$v_{14} = a_0 \pm \frac{(\mu_2^2m^2-\mu_2^2+1)(\lambda+a_0)\mathrm{msn}(\xi)}{\sqrt{\mu_2^2-1}(m^2-1)\mu_2(1+\mu_2\mathrm{dn}(\xi))} - \frac{(\lambda+a_0)(\mu_2^2m^2-\mu_2^2+1)\mathrm{dn}(\xi)}{(m^2-1)\mu_2(1+\mu_2\mathrm{dn}(\xi))}, \tag{3.24.1}$$

$$\begin{aligned} w_{14} = {} & \frac{m^2(\lambda+a_0)}{(\mu_2^2-1)(m^2-1)^2\mu_2^2} \pm \frac{(\mu_2^2m^2-\mu_2^2+1)(\lambda+a_0)^2\mathrm{msn}(\xi)}{\sqrt{\mu_2^2-1}(m^2-1)\mu_2(1+\mu_2\mathrm{dn}(\xi))} - \frac{(\lambda+a_0)^2(\mu_2^2m^2-\mu_2^2+1)m^2\mathrm{dn}(\xi)}{(m^2-1)^2\mu_2(\mu_2^2-1)(1+\mu_2\mathrm{dn}(\xi))} \\ & \times \frac{(\lambda+a_0)^2(\mu_2^2m^2-\mu_2^2+1)^2m^2\mathrm{sn}^2(\xi)}{(m^2-1)^2\mu_2^2(\mu_2^2-1)^2(1+\mu_2\mathrm{dn}(\xi))^2} \pm \frac{(\mu_2^2m^2-\mu_2^2+1)(\lambda+a_0)^2m^2\mathrm{sn}(\xi)\mathrm{dn}(\xi)}{\sqrt{\mu_2^2-1}(m^2-1)^2\mu_2^2(1+\mu_2\mathrm{dn}(\xi))^2}, \end{aligned} \tag{3.24.2}$$

where $\xi = k(x+\lambda t)$, $k = \pm\frac{\sqrt{3}\sqrt{(\mu_2^2-1)(\mu_2^2m^2-\mu_2^2+1)(\lambda+a_0)}}{(\mu_2^2-1)(m^2-1)\mu_2}$, μ_2, λ and a_0 are arbitrary constants.

Family 15

$$v_{15} = a_0 - \frac{(\mu_1^2-1)(\lambda+a_0)\mathrm{sn}(\xi)}{\mu_1(1+\mu_1\mathrm{sn}(\xi))} \pm \frac{(\mu_1^2-1)(\lambda+a_0)\mathrm{dn}(\xi)}{\sqrt{\mu_1^2-m^2}\mu_1(1+\mu_1\mathrm{sn}(\xi))}, \tag{3.25.1}$$

$$\begin{aligned} w_{15} = {} & \frac{(\lambda+a_0)^2(m^2+\mu_1^4-2\mu_1^2)}{\mu_1^2(m^2-\mu_1^2)} + \frac{(\mu_1^2-1)(\lambda+a_0)^2(m^2+1-2\mu_1^2)\mathrm{sn}(\xi)}{\mu_1(m^2-\mu_1^2)(1+\mu_1\mathrm{sn}(\xi))} \pm \frac{(\mu_1^2-1)(\lambda+a_0)^2\mathrm{dn}(\xi)}{\sqrt{\mu_1^2-m^2}\mu_1(1+\mu_1\mathrm{sn}(\xi))} \\ & - \frac{(\mu_1^2-1)^2(\lambda+a_0)^2\mathrm{sn}^2(\xi)}{\mu_1^2(1+\mu_1\mathrm{sn}(\xi))^2} \pm \frac{(\mu_1^2-1)^2(\lambda+a_0)^2\mathrm{sn}(\xi)\mathrm{dn}(\xi)}{\mu_1^2\sqrt{\mu_1^2-m^2}(1+\mu_1\mathrm{sn}(\xi))^2}, \end{aligned} \tag{3.25.2}$$

where $\xi = k(x+\lambda t)$, $k = \pm\frac{\sqrt{3(\mu_1^2-m^2)(\mu_1^2-1)(\lambda+a_0)}}{(\mu_1^2-m^2)\mu_1}$, μ_1, λ and a_0 are arbitrary constants.

Family 16

$$v_{16} = a_0 - \frac{(3+m^2)(\lambda+a_0)\mathrm{sn}(\xi)}{1\pm\mathrm{sn}(\xi)\pm\mathrm{d}n(\xi)}, \tag{3.26.1}$$

$$
\begin{aligned}
w_{16} = A_0 &- \frac{2(4a_0^2+4\lambda^2+8\lambda a_0+A_0m^2-A_0)\mathrm{sn}(\xi)}{(m^2-1)(1\pm \mathrm{sn}(\xi)\pm \mathrm{dn}(\xi))} \\
&- \frac{(4\lambda m^2a_0+10\lambda a_0+2a_0^2m^2+5a_0^2+a_0^2m^4+2\lambda a_0m^4+5\lambda^2-2A_0+2\lambda^2m^2+\lambda^2m^4)\mathrm{dn}(\xi)}{(m^2-1)(1\pm \mathrm{sn}(\xi)\pm \mathrm{dn}(\xi))} \\
&- \frac{(-a_0^2m^4-2\lambda a_0m^4-\lambda^2m^4-A_0m^2-4\lambda^2m^2-8\lambda m^2a_0-4a_0^2m^2-7a_0^2+A_0-7\lambda^2-14\lambda a_0)\mathrm{sn}^2(\xi)}{1\pm \mathrm{sn}(\xi)\pm \mathrm{dn}(\xi))^2} \\
&\times \frac{(4\lambda m^2a_0+10\lambda a_0+2a_0^2m^2+5a_0^2+a_0^2m^4+2\lambda a_0m^4+5\lambda^2+2A_0m^2-2A_0+2\lambda^2m^2+\lambda^2m^4)\mathrm{sn}(\xi)(\mathrm{dn})\xi}{(m^2-1)(1\pm \mathrm{sn}(\xi)\pm \mathrm{dn}(\xi))^2}
\end{aligned} \tag{3.26.2}
$$

where $\xi = k(x+\lambda t)$, $k = \pm\frac{\sqrt{3(m^2-1)(3+m^2)}\lambda+a_0}{(m^2-1)}$ λ and a_0 are arbitrary constants.

3.4. sdξ and ndξ rational expansion

Now we consider the ansätz (2.4.4). For Eq. (3.3), the ansätz (2.4.4) becomes

$$
\begin{cases}
\phi(\xi) = a_0 + a_1\frac{\mathrm{sd}(\xi)}{\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1} + b_1\frac{\mathrm{nd}(\xi)}{\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1}, \\
\sigma(\xi) = A_0 + A_1\frac{\mathrm{sd}(\xi)}{\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1} + B_1\frac{\mathrm{nd}(\xi)}{\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1} \\
\qquad +A_2\frac{\mathrm{sd}^2(\xi)}{(\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1)^2} + B_2\frac{\mathrm{sd}(\xi)\mathrm{nd}(\xi)}{(\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1)^2},
\end{cases} \tag{3.27}
$$

where $a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2$ are constants to be determined later. Following the same steps in Section 3.1, we can obtain the following sdξ and ndξ rational formal doubly periodic solution:

Family 17

$$v_{17} = -\lambda + b_1\,\mathrm{nd}\left(\pm\frac{3}{2}\frac{b_1(x+\lambda t)}{\sqrt{3m^2-3}}\right), \tag{3.28.1}$$

$$w_{17} = -\frac{1}{4}\frac{b_1^2m^2}{m^2-1} - \frac{1}{2}b_1^2m^2\,\mathrm{sd}^2\left(\pm\frac{3}{2}\frac{b_1(x+\lambda t)}{\sqrt{3m^2-3}}\right), \tag{3.28.2}$$

where b_1 and λ are arbitrary constants.

Family 18

$$v_{18} = -\lambda + a_1\,\mathrm{sd}\left(\pm\frac{3}{2}\frac{a_1(x+\lambda t)}{\sqrt{3m^2-3m}}\right), \tag{3.29.1}$$

$$w_{18} = -\frac{1}{4}\frac{a_1^2(2m^2-1)}{m^2(m^2-1)} - \frac{1}{2}a_1^2\,\mathrm{sd}^2\left(\pm\frac{3}{2}\frac{a_1(x+\lambda t)}{\sqrt{3m^2-3m}}\right), \tag{3.29.2}$$

where a_1 and λ are arbitrary constants.

Family 19

$$v_{19} = -\lambda + a_1\,\mathrm{sd}\left(3\pm\frac{a_1(x+\lambda t)}{\sqrt{3m^2-3m}}\right) - \frac{a_1}{m}\,\mathrm{nd}\left(\pm 3\frac{a_1(x+\lambda t)}{\sqrt{3m^2-3m}}\right), \tag{3.30.1}$$

$$w_{19} = \frac{a_1^2}{1-m^2} - a_1^2\,\mathrm{sd}^2\left(\pm 3\frac{a_1(x+\lambda t)}{\sqrt{3m^2-3m}}\right) + \frac{a_1^2}{m}\,\mathrm{nd}\left(\pm 3\frac{a_1(x+\lambda t)}{\sqrt{3m^2-3m}}\right)\mathrm{sd}\left(\pm 3\frac{a_1(x+\lambda t)}{\sqrt{3m^2-3m}}\right), \tag{3.30.2}$$

where a_1 and λ are arbitrary constants.

Family 20

$$v_{20}=\frac{\mu_1^3k+k\mu_1m^2\pm\lambda\sqrt{3\mu_1^4+6\mu_1^2m^2-3\mu_1^2+3m^4-3m^2}}{\sqrt{3\mu_1^4+6\mu_1^2m^2-3\mu_1^2+3m^4-3m^2}}\pm\frac{\sqrt{\mu_1^4+2\mu_1^2m^2-\mu_1^2+m^4-m^2}k\,\mathrm{sd}(k(x+\lambda t))}{\sqrt{3}(\mu_1\,\mathrm{sd}(k(x+\lambda t))+1)}\pm\frac{\sqrt{\mu_1^2+m^2-1}k\,\mathrm{nd}(k(x+\lambda t))}{\sqrt{3}(\mu_1\,\mathrm{sd}(k(x+\lambda t))+1)},\tag{3.31.1}$$

$$\begin{aligned}w_{20}={}&\frac{k^2(m^2-\mu_1^4+2\mu_1^2-2\mu_1^2m^2-m^4)}{3(\mu_1^2+m^2-1)}+\frac{(2k^2\mu_1^3+2k^2\mu_1m^2-k^2\mu_1)\mathrm{sd}(k(x+\lambda t))}{3(\mu_1\,\mathrm{sd}(k(x+\lambda t))+1)}\\&\pm\frac{1}{3}\frac{\sqrt{\mu_1^2+m^2-1}k^2\mu_1(m^2+\mu_1^2)\mathrm{nd}(k(x+\lambda t))}{\sqrt{\mu_1^4+2\mu_1^2m^2-\mu_1^2+m^4-3m^2}(\mu_1\,\mathrm{sd}(k(x+\lambda t))+1)}\\&+\frac{(k^2m^2-\mu_1^4k^2-2k^2\mu_1^2m^2+k^2\mu_1^2-k^2m^4)\mathrm{sd}^2(k(x+\lambda t))}{3(\mu_1\,\mathrm{sd}(k(x+\lambda t))+1)^2}\\&\pm\frac{\sqrt{\mu_1^4+2\mu_1^2m^2-\mu_1^2+m^4-m^2}k^2\sqrt{\mu_1^2+m^2-1}\,\mathrm{nd}(k(x+\lambda t))\mathrm{sd}(k(x+\lambda t))}{3(\mu_1\,\mathrm{sd}(k(x+\lambda t))+1)^2},\end{aligned}\tag{3.31.2}$$

where k, μ_1 and λ are arbitrary constants.

Family 21

$$v_{21}=\frac{k\mu_2-k\mu_2^3\pm\lambda\sqrt{-3\mu_2^2m^2-3+6\mu_2^2-3\mu_2^4+3m^2}}{\sqrt{-3\mu_2^2m^2-3+6\mu_2^2-3\mu_2^4+3m^2}}\pm\frac{\sqrt{m^2+\mu_2^2-1}mk\,\mathrm{sd}(k(x+\lambda t))}{\sqrt{3}(1+\mu_2\,\mathrm{nd}(k(x+\lambda t)))}\pm\frac{\sqrt{m^2-\mu_2^2m^2-1+2\mu_2^2-\mu_2^4}k\,\mathrm{nd}(k(x+\lambda t))}{\sqrt{3}(1+\mu_2\,\mathrm{nd}(k(x+\lambda t)))},\tag{3.32.1}$$

$$\begin{aligned}w_{21}={}&-\frac{1}{3}\frac{(m^2-1)m^2k^2}{m^2+\mu_2^2-1}\pm\frac{1}{3}\frac{\sqrt{m^2+\mu_2^2-1}mk^2\mu_2(\mu_2^2-1)\mathrm{sd}(k(x+\lambda t))}{\sqrt{m^2-\mu_2^2m^2-1+2\mu_2^2-\mu_2^4}(1+\mu_2\,\mathrm{nd}(k(x+\lambda t)))}\\&+\frac{1}{3}\frac{k^2\mu_2m^2\,\mathrm{nd}(k(x+\lambda t))}{1+\mu_2\,\mathrm{nd}(k(x+\lambda t))}+\frac{(k^2m^2-k^2m^4-k^2\mu_2^2m^2)\mathrm{sd}^2(k(x+\lambda t))}{3(1+\mu_2\,\mathrm{nd}(k(x+\lambda t)))^2}\\&\pm\frac{\sqrt{m^2+\mu_2^2-1}mk^2\sqrt{m^2-\mu_2^2m^2-1+2\mu_2^2-\mu_2^4}\,\mathrm{nd}(k(x+\lambda t))\mathrm{sd}(k(x+\lambda t))}{3(1+\mu_2\,\mathrm{nd}(k(x+\lambda t)))^2},\end{aligned}\tag{3.32.2}$$

where k, μ_2 and λ are arbitrary constants.

Family 22

$$v_{22}=\frac{k-km^2\pm\lambda\sqrt{-9+3m^4+6m^2}}{\sqrt{-9+3m^4+6m^2}}\pm\frac{1}{3}\frac{\sqrt{-9+3m^4+6m^2}k\,\mathrm{sd}(k(x+\lambda t))}{\pm\mathrm{sd}(k(x+\lambda t))\pm\mathrm{nd}(k(x+\lambda t))+1},\tag{3.33.1}$$

$$\begin{aligned}w_{22}={}&-\frac{3A_1m^2+9A_1+8k^2}{6(m^2+3)}+\frac{A_1\,\mathrm{sd}(k(x+\lambda t))}{\pm\mathrm{sd}(k(x+\lambda t))\pm\mathrm{nd}(k(x+\lambda t))+1}\\&+\frac{(3A_1+k^2-k^2m^2)\mathrm{nd}(k(x+\lambda t))}{3(\pm\mathrm{sd}(k(x+\lambda t))\pm\mathrm{nd}(k(x+\lambda t))+1)}+\frac{(2k^2-2k^2m^2-3A_1m^2-3A_1)\mathrm{sd}^2(k(x+\lambda t))}{6(\pm\mathrm{sd}(k(x+\lambda t))\pm\mathrm{nd}(k(x+\lambda t))+1)^2}\\&-\frac{(3A_1+k^2-k^2m^2)\mathrm{nd}(k(x+\lambda t))\mathrm{sd}(k(x+\lambda t))}{3(\pm\mathrm{sd}(k(x+\lambda t))\pm\mathrm{nd}(k(x+\lambda t))+1)^2},\end{aligned}\tag{3.33.2}$$

where k, A_1 and λ are arbitrary constants.

Remark. From Section 3.1, we can easily recover all the solutions obtained in [13], when $\mu_2 = 0$ and $k = 1$. The other solutions obtained here, to our knowledge, are all new families of rational form Jacobi elliptic function solutions of the DLWE.

4. Summary and conclusions

In this paper, we have extended the new Jacobi elliptic function rational expansion method. The method is more powerful than the method proposed recently by Liu et al. [10], Fan [11], Yan [12] and Wang et al. [13]. The (1 + 1)-dimensional dispersive long wave equation is chosen to illustrate the method such that many families of new Jacobi elliptic function solutions are obtained. When the modulus $m \to 1$, some of these obtained solutions degenerate as solitary wave solutions. The algorithm can be also applied to many nonlinear differential equations in mathematical physics. Further work about various extensions and improvement of Jacobi function method need us to find the more general ansätzes or the more general subequation.

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