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New Families of Rational Form Solitary Wave Solutions to (2+1)-Dimensional Broer–Kaup–Kupershmidt System*

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Abstract *Taking the (2+1)-dimensional Broer–Kaup–Kupershmidt system as a simple example, some families of rational form solitary wave solutions, triangular periodic wave solutions, and rational wave solutions are constructed by using the Riccati equation rational expansion method presented by us. The method can also be applied to solve more nonlinear partial differential equation or equations.*

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1 Introduction

In Ref. [1], the (2+1)-dimensional Broer–Kaup–Kupershmidt (BKK) system

$$
\begin{aligned}
&H_{ty} - H_{xxy} + 2(HH_x)_y + 2G_{xx} = 0\,,\\
&G_t + G_{xx} + 2(HG)_x = 0\,,
\end{aligned}
\tag{1}
$$

may be derived from the inner parameter-dependent symmetry constraint of the Kadomtsev–Petviashvili (KP) equation. Though the integrability of the BKK system can be guaranteed by the integrability of the KP equation (because it is a symmetry constraint of the KP equation), some authors have exactly proven its integrability in some different sense. For more details about the results of this system, the reader is advised to see the achievements in Refs. [1] ∼ [7].

The present work is motivated by the desire to present a new subequation method, named Riccati equation rational expansion (RERE) method, by proposing a more general ansatz so that it can be used to obtain more types and general formal solutions, which contain not only the results obtained by using the method[9–17] but also other types of solutions. For illustration, we apply the generalized method to solve the BKK system and successfully construct new and more general solutions including rational formal solitary wave solutions, rational solutions, and rational formal triangular periodic solutions for the BKK system.

2 New Families of Rational Form Solitary Wave Solutions to the (2+1)-dimensional Broer–Kaup–Kupershmidt System

In order to get some families of rational form solitary wave solutions to the $(2+1)$-dimensional BKK system, by considering the wave transformations $H(x,y,t) = U(\xi)$, $G(x,y,t) = V(\xi)$, and $\xi = k(x+ly+\lambda t)$, we change Eq. (1) to the form

$$
\begin{aligned}
&\lambda l\, U'' - lkU''' + 2l(UU')' + 2V'' = 0\,,\\
&\lambda V' + kV'' + 2(VU)' = 0\,.
\end{aligned}
\tag{2}
$$

We suppose that BKK system have the following formal travelling wave solution:

$$
\begin{aligned}
U(\xi) &= a_0 + \sum_{i=1}^{m_u} \frac{a_i\phi^i(\xi) + b_i\phi^{i-1}(\xi)\sqrt{R+\phi^2(\xi)}}{(\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1)^i}\,,\\
V(\xi) &= A_0 + \sum_{i=1}^{m_v} \frac{A_i\phi^i(\xi) + B_i\phi^{i-1}(\xi)\sqrt{R+\phi^2(\xi)}}{(\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1)^i}\,,
\end{aligned}
\tag{3}
$$

and the new variable $\phi = \phi(\xi)$ satisfying

$$
\phi' - (R+\phi^2) = \frac{\mathrm{d}\phi}{\mathrm{d}\xi} - (R+\phi^2) = 0\,,
\tag{4}
$$

where R, a_0, a_i, b_i, A_0, A_i, and B_i $(i = 1, 2, \ldots, m_i)$ are constants to be determined later. For the BKK system, by balancing the highest nonlinear terms and the highest order partial derivative terms in Eq. (2) (see Refs. [9] ∼ [17] for details), gives $m_u = 1$ and $m_v = 2$. So we have

$$
\begin{aligned}
U(\xi) &= a_0 + \frac{a_1\phi(\xi) + b_1\sqrt{R+\phi^2(\xi)}}{\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1}\,,\\
V(\xi) &= A_0 + \frac{A_1\phi(\xi) + B_1\sqrt{R+\phi^2(\xi)}}{\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1}\\
&\quad + \frac{A_2\phi^2(\xi) + B_2\phi(\xi)\sqrt{R+\phi^2(\xi)}}{(\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1)^2}\,,
\end{aligned}
\tag{5}
$$

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where $\phi(\xi)$ satisfies Eq. (4). With the aid of *Maple*, substituting Eq. (5) along with Eq. (4) into Eq. (2), yields a set of algebraic equations for $\phi^i(\xi)(\sqrt{R+\phi^2(\xi)})^j$, $(i=0,1,\ldots;\ j=0,1)$. Setting the coefficients of these terms $\phi^i(\xi)(\sqrt{R+\phi^2(\xi)})^j$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1 A_2, B_2, μ_1, μ_2, and k.

By making use of the *Maple* software package "Charsets" by Dongming Wang, which is based on the Wu-elimination method,[8] solving the over-determined algebraic equations, we get the following results.

Here we just consider the condition $R<0$.

Case 1

$$\begin{aligned}
&\mu_1 = A_1 = 0, \qquad \mu_2=\mu_2\,,\\
&k=\pm 2\frac{b_1}{\sqrt{1-\mu_2^2R}}, \quad a_0=-\frac{\lambda}{2}\,,\\
&a_1=\mp\frac{b_1}{\sqrt{1-{\mu_2}^2R}}, \qquad b_1=b_1\,,\\
&A_0=2\,\frac{b_1^2lR}{-1+\mu_2^2R}, \quad B_1=-2\,\frac{lb_1^2\mu_2R}{-1+\mu_2^2R}\,,\\
&A_2=2\,\frac{{b_1}^2l}{-1+{\mu_2}^2R}, \quad B_2=\pm 2\,\frac{{b_1}^2l}{\sqrt{1-{\mu_2}^2R}}\,.
\end{aligned}\tag{6}$$

Case 2

$$\begin{aligned}
&\mu_1=A_1=a_1=0, \quad k=\pm\frac{b_1}{\sqrt{1-\mu_2^2R}}\,,\\
&a_0=-\frac{\mu_2^2R\lambda+b_1R\mu_2-\lambda}{2(-1+\mu_2^2R)}, \quad b_1=b_1\,,\\
&A_0=\frac{lb_1^2R(2\,\mu_2^2R-1)}{4(-1+\mu_2^2R)^2}, \quad B_1=-\frac{lb_1^2\mu_2R}{2(-1+\mu_2^2R)}\,,\\
&\mu_2=\mu_2, \quad A_2=\frac{lb_1^2}{2(-1+\mu_2^2R)}\,,\\
&B_2=\pm\frac{lb_1^2}{2\sqrt{1-\mu_2^2R}}\,.
\end{aligned}\tag{7}$$

Case 3

$$\begin{aligned}
&\mu_2=a_1=0, \qquad \mu_1=\mu_1\,,\\
&k=\pm\frac{b_1}{\sqrt{1+\mu_1^2R}}\,,\\
&a_0=-\frac{1}{2}\lambda, \quad A_0=-\frac{(2\mu_1^2R+1)lb_1^2R}{4(1+\mu_1^2R)}\,,\\
&A_1=lb_1^2R\mu_1, \quad B_1=\mp\frac{lb_1^2R\mu_1}{2\sqrt{1+\mu_1^2R}}\,,\\
&b_1=b_1, \quad A_2=-\frac{1}{2}(1+\mu_1^2R)lb_1^2\,,\\
&B_2=\pm\frac{1}{2}\sqrt{1+\mu_1^2R}\,lb_1^2\,.
\end{aligned}\tag{8}$$

Case 4

$$\begin{aligned}
&\mu_2=0, \quad \mu_1=\mu_1, \quad k=\pm 2\frac{b_1}{\sqrt{1+\mu_1^2R}}\,,\\
&a_0=\frac{\pm 2\,b_1\mu_1R-\lambda\sqrt{1+\mu_1^2R}}{2\sqrt{1+\mu_1^2R}}\,,\\
&a_1=\mp\sqrt{1+\mu_1^2R}b_1, \quad b_1=b_1\,,\\
&A_0=-2\,lb_1^2R, \quad A_1=4\,lb_1^2R\mu_1\,,\\
&B_1=\mp 2\frac{lb_1^2R\mu_1}{\sqrt{1+\mu_1^2R}}, \quad A_2=-2b_1^2l(1+{\mu_1}^2R)\,,\\
&B_2=\pm 2\,\sqrt{1+{\mu_1}^2R}\,l{b_1}^2\,.
\end{aligned}\tag{9}$$

Case 5

$$\begin{aligned}
&\mu_2=b_1=B_1=B_2=0, \quad a_0=-\frac{{\mu_1}^2R\lambda+2\,a_1R\mu_1+\lambda}{2(1+{\mu_1}^2R)}\,,\\
&\mu_1=\mu_1, \quad b_1=b_1, \quad a_1=a_1\,,\\
&A_0=-\frac{Rl{a_1}^2}{1+{\mu_1}^2R}, \quad k=-\frac{a_1}{1+{\mu_1}^2R}\,,\\
&A_1=2\frac{{a_1}^2lR\mu_1}{1+{\mu_1}^2R}, \quad A_2=-l{a_1}^2\,.
\end{aligned}\tag{10}$$

Case 6

$$\begin{aligned}
&\mu_2=\pm1, \quad \mu_1=\pm1, \quad k=-2\frac{a_1}{R+1}\,,\\
&a_0=-\frac{2a_1R+R\lambda+\lambda}{2(R+1)}, \quad a_1=a_1, \quad b_1=b_1\,,\\
&A_0=2\frac{l{a_1}^2R}{(R+1)(R-1)}, \quad B_2=2\frac{l{a_1}^2}{R-1}\,,\\
&A_1=-4\frac{l{a_1}^2R}{(R+1)(R-1)}, \quad B_1=-4\frac{l{a_1}^2R}{(R+1)(R-1)}\,,\\
&A_2=2\frac{l{a_1}^2}{R-1}\,.
\end{aligned}\tag{11}$$

Case 7

$$\begin{aligned}
&\mu_2=\pm1, \quad \mu_1=\pm1, \quad k=-b_1\,,\\
&a_0=\frac{1}{2}(b_1R-\lambda), \quad a_1=0\,,\\
&b_1=b_1, \quad A_0=\frac{(R+1)^2l{b_1}^2R}{4(R-1)}\,,\\
&A_1=-\frac{(R+1)l{b_1}^2R}{R-1}, \quad A_2=\frac{{b_1}^2l(R+1)^2}{2(R-1)}\,,\\
&B_1=-\frac{(R+1)l{b_1}^2R}{R-1}, \quad B_2=\frac{{b_1}^2l(R+1)^2}{2(R-1)}\,.
\end{aligned}\tag{12}$$

It is well known that the general solutions of Eq. (4) are

(i) When $R<0$,

$$\begin{aligned}
&\phi(\xi)=-\sqrt{-R}\tanh(\sqrt{-R}\xi)\,,\\
&\phi(\xi)=-\sqrt{-R}\coth(\sqrt{-R}\xi)\,;
\end{aligned}\tag{13}$$

(ii) When $R=0$,

$$\phi(\xi)=-\frac{1}{\xi}\,;\tag{14}$$

(iii) When $R > 0$,

$$\phi(\xi) = \sqrt{R}\tan(\sqrt{R}\,\xi), \quad \phi(\xi) = -\sqrt{R}\cot(\sqrt{R}\,\xi)\,. \quad (15)$$

Thus according to Eqs. (5), (13) $\sim$ (15) and Cases 1 $\sim$ 7, we can obtain the following rational formal travelling-wave solutions of Eq. (2).

Family 1

$$H_{11} = -\frac{\lambda}{2} \pm \frac{b_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi) \pm b_1\sqrt{1-\mu_2{}^2R}\sqrt{-R}\ \mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{\sqrt{1-\mu_2{}^2R}(1 \pm \mu_2\sqrt{-R}\ \mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi))}\,, \tag{16a}$$

$$\begin{aligned} G_{11} = {} & \frac{2b_1{}^2 lR}{-1+\mu_2{}^2R} \pm \frac{2b_1{}^2 l\mu_2 R\sqrt{-R}\ \mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{(-1+\mu_2{}^2R)(1 \pm \mu_2\sqrt{-R}\ \mathrm{i}\,\mathrm{sech}(\sqrt{-R}\xi))} \\ & - \frac{2b_1{}^2 lR\tanh^2(\sqrt{R}\,\xi)}{(1 \pm \mu_2\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi))^2(-1+\mu_2{}^2R)} \mp \frac{2b_1{}^2 l\tanh(\sqrt{R}\,\xi)R\mathrm{i}\,\mathrm{sech}(\sqrt{R}\,\xi)}{(1 \pm \mu_2\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi))^2\sqrt{1-\mu_2{}^2R}}\,, \end{aligned} \tag{16b}$$

$$H_{12} = -\frac{\lambda}{2} \pm \frac{b_1\sqrt{-R}\coth(\sqrt{-R}\,\xi) \pm b_1\sqrt{1-\mu_2{}^2R}\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{\sqrt{1-\mu_2{}^2R}(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi))}\,, \tag{17a}$$

$$\begin{aligned} G_{12} = {} & \frac{2b_1{}^2 lR}{-1+\mu_2{}^2R} \pm \frac{2b_1{}^2 l\mu_2 R\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{(-1+\mu_2{}^2R)(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{R}\,\xi))} \\ & - \frac{2b_1{}^2 lR\coth^2(\sqrt{-R}\,\xi)}{(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi))^2(-1+\mu_2{}^2R)} \pm \frac{2b_1{}^2 l\coth(\sqrt{-R}\,\xi)R\,\mathrm{csch}(\sqrt{-R}\,\xi)}{(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi))^2\sqrt{1-\mu_2{}^2R}}\,, \end{aligned} \tag{17b}$$

where $\xi = k(x+ly+\lambda t)$, k is determined by Eq. (6), $R < 0$, μ_2, b_1, l, and λ are arbitrary constants.

Family 2

$$H_{21} = -\frac{\mu_2{}^2R\lambda + b_1R\mu_2 - \lambda}{2(-1+\mu_2{}^2R)} \pm \frac{b_1\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{1 \pm \mu_2\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}\,, \tag{18a}$$

$$\begin{aligned} G_{21} = {} & \frac{lb_1{}^2R(2\,\mu_2{}^2R-1)}{4(-1+\mu_2{}^2R)^2} \pm \frac{1}{2}\frac{lb_1{}^2\mu_2R\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{(-1+\mu_2{}^2R)(1 \pm \mu_2\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi))} \\ & - \frac{1}{2}\frac{lb_1{}^2R\tanh^2(\sqrt{-R}\,\xi)}{(1 \pm \mu_2\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi))^2(-1+\mu_2{}^2R)} \mp \frac{1}{2}\frac{lb_1{}^2\tanh(\sqrt{-R}\,\xi)R\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{(1 \pm \mu_2\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi))^2\sqrt{1-\mu_2{}^2R}}\,, \end{aligned} \tag{18b}$$

$$H_{22} = -\frac{\mu_2{}^2R\lambda + b_1R\mu_2 - \lambda}{2(-1+\mu_2{}^2R)} \pm \frac{b_1\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}\,, \tag{19a}$$

$$\begin{aligned} G_{22} = {} & \frac{lb_1{}^2R(2\,\mu_2{}^2R-1)}{4(-1+\mu_2{}^2R)^2} \pm \frac{1}{2}\frac{lb_1{}^2\mu_2R\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{(-1+\mu_2{}^2R)(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi))} \\ & - \frac{1}{2}\frac{lb_1{}^2R\coth^2(\sqrt{-R}\,\xi)}{(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi))^2(-1+\mu_2{}^2R)} \mp \frac{1}{2}\frac{lb_1{}^2\coth(\sqrt{-R}\,\xi)R\,\mathrm{csch}(\sqrt{-R}\,\xi)}{(1 \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi))^2\sqrt{1-\mu_2{}^2R}}\,, \end{aligned} \tag{19b}$$

where $\xi = k(x+ly+\lambda t)$, k is determined by (7), $R < 0$, μ_2, b_1, l, and λ are arbitrary constants.

Family 3

$$H_{31} = -\frac{1}{2}\lambda \pm \frac{b_1\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)+1}\,, \tag{20a}$$

$$\begin{aligned} G_{31} = A_0 & - \frac{lb_1{}^2R\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)}{1-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)} \mp \frac{(1/2)lb_1{}^2R\mu_1\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{(1-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi))\sqrt{1+\mu_1{}^2R}} \\ & + \frac{\sqrt{1+\mu_1{}^2R}lb_1{}^2R\tanh^2(\sqrt{-R}\,\xi) \mp lb_1{}^2\tanh(\sqrt{-R}\,\xi)R\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{2\sqrt{1+\mu_1{}^2R}(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)+1)^2}\,, \end{aligned} \tag{20b}$$

$$H_{32} = -\frac{1}{2}\lambda \pm \frac{b_1\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{-\mu_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)+1}\,, \tag{21a}$$

$$G_{32} = A_0 - \frac{lb_1{}^2R\mu_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)}{1-\mu_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)} \pm \frac{(1/2)lb_1{}^2R\mu_1\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{(1-\mu_1\sqrt{-R}\coth(\sqrt{-R}\,\xi))\sqrt{1+\mu_1{}^2R}}$$

$$+\frac{\sqrt{1+\mu_1{}^2R}\, lb_1{}^2R\coth^2(\sqrt{-R}\,\xi)\pm lb_1{}^2\coth(\sqrt{-R}\,\xi)R\,\mathrm{csch}(\sqrt{-R}\,\xi)}{2\sqrt{1+\mu_1{}^2R}(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)+1)^2}\,, \tag{21b}$$

where $\xi=k(x+ly+\lambda t)$, k and A_0 are determined by Eq. (8), $R<0$, μ_1, b_1, l, and λ are arbitrary constants.

Family 4

$$H_{41}=\frac{\pm 2\,b_1\mu_1R-\lambda\sqrt{1+\mu_1{}^2R}}{2\sqrt{1+\mu_1{}^2R}}\pm\frac{\sqrt{1+\mu_1{}^2R}\,b_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)\pm b_1\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)+1}\,, \tag{22a}$$

$$G_{41}=A_0-\frac{4\,lb_1{}^2R\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)}{-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)+1}\mp 2\,\frac{lb_1{}^2R\,\mu_1\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)+1)\sqrt{1+\mu_1{}^2R}}$$
$$+\frac{(2\,lb_1{}^2R\,\mu_1{}^2+2\,lb_1{}^2)R\tanh^2(\sqrt{-R}\,\xi)\mp 2\sqrt{1+\mu_1{}^2R}\,lb_1{}^2\tanh(\sqrt{-R}\,\xi)R\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)+1)^2}\,, \tag{22b}$$

$$H_{42}=\frac{\pm 2\,b_1\mu_1R-\lambda\sqrt{1+\mu_1{}^2R}}{2\sqrt{1+\mu_1{}^2R}}\pm\frac{\sqrt{1+\mu_1{}^2R}\,b_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)\pm b_1\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{-\mu_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)+1}\,, \tag{23a}$$

$$G_{42}=A_0-\frac{4\,lb_1{}^2R\,\mu_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)}{-\mu_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)+1}\pm 2\frac{lb_1{}^2R\,\mu_1\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)+1)\sqrt{1+\mu_1{}^2R}}$$
$$+\frac{(2\,lb_1{}^2R\,\mu_1{}^2+2\,lb_1{}^2)R\coth^2(\sqrt{-R}\,\xi)\pm 2\sqrt{1+\mu_1{}^2R}\,lb_1{}^2\coth(\sqrt{-R}\,\xi)R\,\mathrm{csch}(\sqrt{-R}\,\xi)}{(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)+1)^2}\,, \tag{23b}$$

where $\xi=k(x+ly+\lambda t)$, k, and A_0 are determined by Eq. (9), $R<0$, μ_1, b_1, l, and λ are arbitrary constants.

Family 5

$$H_{51}=-\frac{\mu_1{}^2R\lambda+2\,a_1R\,\mu_1+\lambda}{2(1+\mu_1{}^2R)}-\frac{a_1\sqrt{-R}\,\tanh(\sqrt{-R}\,\xi)}{-\mu_1\sqrt{-R}\,\tanh(\sqrt{-R}\,\xi)+1}\,, \tag{24a}$$

$$G_{51}=A_0-\frac{2R\,la_1{}^2\mu_1\sqrt{-R}\,\tanh(\sqrt{-R}\,\xi)}{(1+\mu_1{}^2R)(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)+1)}+\frac{R\,la_1{}^2\tanh^2(\sqrt{-R}\,\xi)}{(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi)+1)^2}\,, \tag{24b}$$

$$H_{52}=-\frac{\mu_1{}^2R\,\lambda+2\,a_1R\,\mu_1+\lambda}{2(1+\mu_1{}^2R)}-\frac{a_1\sqrt{-R}\,\coth(\sqrt{-R}\,\xi)}{-\mu_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)+1}\,, \tag{25a}$$

$$G_{52}=A_0-\frac{2R\,la_1{}^2\mu_1\sqrt{-R}\,\coth(\sqrt{-R}\,\xi)}{(1+\mu_1{}^2R)(-\mu_1\sqrt{-R}\,\coth(\sqrt{-R}\,\xi)+1)}+\frac{Rla_1{}^2\coth^2(\sqrt{-R}\,\xi)}{(-\mu_1\sqrt{-R}\,\coth(\sqrt{-R}\,\xi)+1)^2}\,, \tag{25b}$$

where $\xi=k(x+ly+\lambda t)$, k, and A_0 are determined by Eq. (10), $R<0$, μ_1, b_1, l, and λ are arbitrary constants.

Family 6

$$H_{61}=-\frac{2\,a_1R+R\lambda+\lambda}{2(R+1)}+\frac{-a_1\sqrt{-R}\,\tanh(\sqrt{-R}\,\xi)\pm a_1\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{\mp\sqrt{-R}\,\tanh(\sqrt{-R}\,\xi)\pm\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)+1}\,, \tag{26a}$$

$$G_{61}=\frac{2la_1{}^2R}{(R+1)(R-1)}+\frac{4la_1{}^2R\sqrt{-R}\tanh(\sqrt{-R}\,\xi)\pm 4la_1{}^2R\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{(R^2-1)(\mp\sqrt{-R}\tanh(\sqrt{-R}\,\xi)\pm\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)+1)}$$
$$-\frac{2la_1{}^2R\tanh^2(\sqrt{-R}\,\xi)\pm 2la_1{}^2R\tanh(\sqrt{-R}\,\xi)\mathrm{sech}(\sqrt{-R}\,\xi)}{(R-1)(\mp\sqrt{-R}\tanh(\sqrt{-R}\,\xi)\pm\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)+1)^2}\,, \tag{26b}$$

$$H_{62}=-\frac{2\,a_1R+R\lambda+\lambda}{2(R+1)}+\frac{-a_1\sqrt{-R}\coth(\sqrt{-R}\,\xi)\pm a_1\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{\mp\sqrt{-R}\,\coth(\sqrt{-R}\,\xi)\pm\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)+1}\,, \tag{27a}$$

$$G_{62}=\frac{2la_1{}^2R}{(R+1)(R-1)}+\frac{4la_1{}^2R\sqrt{-R}\coth(\sqrt{-R}\,\xi)\pm 4la_1{}^2R\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{(R^2-1)(\mp\sqrt{-R}\,\coth(\sqrt{-R}\,\xi)\mp\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)+1)}$$
$$-\frac{2la_1{}^2R\tanh^2(\sqrt{-R}\,\xi)\pm 2la_1{}^2R\coth(\sqrt{-R}\,\xi)\mathrm{csch}(\sqrt{-R}\,\xi)}{(R-1)(\mp\sqrt{-R}\coth(\sqrt{-R}\,\xi)\mp\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)+1)^2}\,, \tag{27b}$$

where $\xi=k(x+ly+\lambda t)$, k is determined by Eq. (11), $R<0$, a_1, l, and λ are arbitrary constants.

Family 7

$$H_{71}=\frac{1}{2}(\,b_1R-\,\lambda)\pm\frac{b_1\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{R}\,\xi)}{\mp\sqrt{-R}\,\tanh(\sqrt{-R}\,\xi)\pm\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)+1}\,, \tag{28a}$$

$$G_{71} = \frac{(R^2+2\,R+1)lb_1{}^2R}{4(R-1)} + \frac{(R+1)lb_1{}^2(R\sqrt{-R}\,\tanh(\sqrt{-R}\,\xi) - R\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi))}{(R-1)(\mp\sqrt{-R}\,\tanh(\sqrt{-R}\,\xi) \pm \sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)+1)}$$
$$- \frac{(R^2+2\,R+1)lb_1{}^2(R\tanh^2(\sqrt{-R}\,\xi) \pm \tanh(\sqrt{-R}\,\xi)R\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi))}{2(R-1)(\mp\sqrt{-R}\,\tanh(\sqrt{-R}\,\xi) \pm \sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)+1)^2}\,, \tag{28b}$$

$$H_{72} = \frac{1}{2}(\,b_1R - \lambda) \pm \frac{b_1\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{\mp\sqrt{-R}\,\coth(\sqrt{-R}\,\xi) \mp \sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)+1}\,, \tag{29a}$$

$$G_{72} = \frac{(R^2+2\,R+1)lb_1{}^2R}{4(R-1)} + \frac{(R+1)lb_1{}^2(R\sqrt{-R}\,\coth(\sqrt{-R}\,\xi) \pm R\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi))}{(R-1)(\mp\sqrt{-R}\,\coth(\sqrt{-R}\,\xi) \mp \sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)+1)}$$
$$- \frac{(R^2+2\,R+1)lb_1{}^2(R\,\coth^2(\sqrt{-R}\,\xi) \pm \coth(\sqrt{-R}\,\xi)R\,\mathrm{csch}(\sqrt{-R}\,\xi))}{2(R-1)(\mp\sqrt{-R}\,\coth(\sqrt{-R}\,\xi) \mp \sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)+1)^2}\,, \tag{29b}$$

where $\xi = k(x+ly+\lambda t)$, k is determined by Eq. (12), $R<0$, b_1, l, and λ are arbitrary constants.

Note: Since tan- and cot-type solutions appear in pairs with tanh- and coth-type solutions, respectively, we omit them in this paper. In addition, some rational solutions are also omitted.

3 Summary of Riccati Equation Rational Expansion Method

In the following we would like to outline the main steps of our method:

Step 1 A given nonlinear partial differential equation system (30) with some physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0\,, \tag{30}$$

by using the wave transformation

$$u_i(x,y,t) = U_i(\xi), \qquad \xi = k(x+ly+\lambda\,t)\,, \tag{31}$$

where k, l, and λ are constants to be determined later. Then the nonlinear partial differential Eq. (30) is reduced to a nonlinear ordinary differential equation (ODE):

$$G_i(\,U_i,\ U_i',\ U_i'', \ldots) = 0\,. \tag{32}$$

Step 2 We introduce a new ansätz in terms of finite rational formal expansion in the following forms:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}\phi^j(\xi) + b_{ij}\phi^{j-1}(\xi)\sqrt{R+\phi^2(\xi)}}{(\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)}+1)^j} \tag{33}$$

and the new vriable $\phi = \phi(\xi)$ satisfying Eq. (4), where R, a_{i0}, a_{ij}, and b_{ij} $(i=1,2,\ldots;\ j=1,2,\ldots,m_i)$ are constants to be determined later.

Step 3 To determine the m_i of the rational formal polynomial solutions (33) by respectively balancing the highest nonlinear terms and the highest-order partial derivative terms in the given system equations (see Ref. [9] $\sim$ [17] for details), and then give the formal solutions.

Step 4 Substitute Eq. (33) into Eq. (32) along with Eq. (4) and then set all coefficients of $\phi^i(\xi)(\sqrt{R+\phi^2(\xi)})^j$ of the resulting system's numerator, $(i=1,2,\ldots;\ j=0,1)$ to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, μ_1, μ_2, a_{i0}, a_{ij}, and b_{ij} $(i=1,2,\ldots;\ j=1,2,\ldots,m_i)$.

Step 5 Solving the over-determined system of nonlinear algebraic equations by use of *Maple*, we would end up with the explicit expressions for k, μ_1, μ_2, a_{i0}, a_{ij}, and b_{ij} $(i=1,2,\ldots;\ j=1,2,\ldots,m_i)$.

Step 6 It is well known that the general solutions of Eqs. (4) are (13) $\sim$ (15). Thus according to Eqs. (31), (33), (13) $\sim$ (15) and the conclusions in *Step 5*, we can obtain the following rational formal travelling-wave solutions of Eq. (30).

(i) When $R<0$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(-\sqrt{-R}\,\tanh(\sqrt{-R}\,\xi))^j \pm b_{ij}(-\sqrt{-R}\,\tanh(\sqrt{-R}\,\xi))^{j-1}\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)}{(1-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\,\xi) \pm \mu_2\sqrt{-R}\,\mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi))^j}\,, \tag{34a}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(-\sqrt{-R}\,\coth(\sqrt{-R}\,\xi))^j \pm b_{ij}(-\sqrt{-R}\,\coth(\sqrt{-R}\,\xi))^{j-1}\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi)}{(1-\mu_1\sqrt{-R}\,\coth(\sqrt{-R}\,\xi) \pm \mu_2\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi))^j}\,; \tag{34b}$$

(ii) When $R=0$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{(-1)^j(a_{ij} \mp b_{ij})}{(\xi - \mu_1 \pm \mu_2)^j}\,; \tag{35}$$

(iii) When $R > 0$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(\sqrt{R}\,\tan(\sqrt{R}\,\xi))^j \pm b_{ij}(\sqrt{R}\,\tan(\sqrt{R}\,\xi))^{j-1}\sqrt{R}\sec(\sqrt{R}\,\xi)}{(1+\mu_1\sqrt{R}\tan(\sqrt{R}\,\xi) \pm \mu_2\sqrt{R}\sec(\sqrt{R}\,\xi))^j}\,, \tag{36a}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(-\sqrt{R}\cot(\sqrt{R}\,\xi))^j \pm b_{ij}(-\sqrt{R}\cot(\sqrt{R}\,\xi))^{j-1}\sqrt{R}\csc(\sqrt{R}\,\xi)}{(1-\mu_1\sqrt{R}\cot(\sqrt{R}\,\xi) \pm \mu_2\sqrt{R}\csc(\sqrt{R}\,\xi))^j}\,, \tag{36b}$$

where $\mathrm{i} = \sqrt{-1}$ and $\xi = k(x + ly + \lambda\, t)$.

Remark The ansatz proposed here is more general than the ansatz in the tanh function method,[9,10] extended tanh function method,[11] improved extended tanh function method,[12−14] projective Riccati equations method[15] and general projective Riccati equations method.[16,17] If we set the parameters in Eq. (33) to different values, the above methods can be recovered by the RERE method. The concrete cases are as follows:

(i) Setting $\mu_1 = \mu_2 = b_1 = 0$, we just recover the solutions obtained by extended tanh function method;[11]

(ii) Setting $\mu_1 = \mu_2 = 0$, we just recover the solutions obtained by the improved extended tanh function method;[12−14]

(iii) Setting $\mu_1 = 0$ and $\mu_2 \neq 0$, we just recover the solutions obtained by the Projective Riccati method;[15−17]

(iv) The other solutions obtained here, to our knowledge, are all new formal exact solutions of NPDEs.

4 Summary and Conclusion

Taking the (2+1)-dimensional Broer–Kaup–Kuper-shmidt (BKK) system as a simple example, some families of rational form solitary wave solutions, triangular periodic wave solutions and rational wave solutions are constructed by using the RERE method. The method can also be applied to solve more nonlinear partial differential equation or equations. In fact, we naturally present a more general ansatz, in the rational formal polynomial solutions (3) taking a_0, a_i, b_i, A_0, A_i, B_i $(i = 1, 2, \ldots, m_i)$, and ξ as differentiable function to be determined later. Therefore, for some nonlinear equations, more types of non-travelling solutions, such as soliton-like solutions, would be expected. We would like to study the more general ansatz for finding new formal soliton-like solutions of some nonlinear partial differential equation or equations further.

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