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New Compacton-Like and Solitary Pattern-Like Solutions of (2+1)-Dimensional Generalization of Modified KdV Equation*

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Abstract *Recently some (1+1)-dimensional nonlinear wave equations with linearly dispersive terms were shown to possess compacton-like and solitary pattern-like solutions. In this paper, with the aid of Maple, new solutions of (2+1)-dimensional generalization of mKdV equation, which is of only linearly dispersive terms, are investigated using three new transformations. As a consequence, it is shown that this (2+1)-dimensional equation also possesses new compacton-like solutions and solitary pattern-like solutions.*

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1 Introduction

Many nonlinear wave equations have been shown to admit solitary wave solutions, which are of the hyperbolic functions.[1] Since 1993, two special types of localized solutions have been payed much attention. They are compacton solutions, which are solitary waves with the property that after colliding with other compacton solutions. They reemerge with the same coherent shapes,[2−4] i.e.

$$u(x,t)=\begin{cases} A\cos^\alpha k(x-\lambda t)\,, & |k(x-\lambda t)|\le \dfrac{\pi}{2}\,,\\ 0\,, & \text{otherwise}\,,\end{cases} \tag{1}$$

and solitary pattern solution having infinite slopes or cusps, i.e.

$$u(x,t)=A\cosh^\alpha k(x-\lambda t)\,,$$
$$u(x,t)=A\sinh^\alpha k(x-\lambda t)\,. \tag{2}$$

Generally speaking, it has been shown that many nonlinear wave equations with fully dispersive terms possess these two kinds of solutions, such as $K(m,n)$ equation,[2−4] $B(m,n)$ equation, $R(m,n)$ equation, $E(m,n)$ equation, $ES(m,n)$ equation, $mK(m,n,k)$ equation, etc.[2−11] More recently we showed that $B(1,n)$ equation with linear dispersive term and $R(1,n)$ equation with linearly dispersive term also possesses these two solutions.[8,12]

In 1999, Kumar and Panigrahi[13] presented that some nonlinear evolution equations with linear dispersive terms admitted the compacton-like solution in the form

$$u(x,t)=\begin{cases} \dfrac{K\cos^2 k(x+\lambda t)}{1-(2/3)\cos^2 k(x+\lambda t)}\,, & |k(x+\lambda t)|\le \dfrac{\pi}{2}\,,\\ 0\,, & \text{otherwise}\,,\end{cases} \tag{3}$$

whose width and amplitude both depend on the velocity. This behavior is reminiscent of the solitons. For example, the modified KdV equation, the five order KdV-like equation, and the phase function of the nonlinear Schrödinger equation with a source term all possess this kind of solutions. Recently, Wazwaz[14] presented that some nonlinear wave equations with linear dispersion term admitted the solitary pattern-like solutions in the form

$$u=\frac{K\cosh^2 k(x-\lambda t)}{1-(2/3)\cosh^2 k(x-\lambda t)}\,. \tag{4}$$

More recently, we[15] further presented that these equations (e.g. the modified KdV equation, the five order KdV-like equation, and the phase function of the nonlinear Schrödinger equation with a source) possess general compacton-like solutions and solitary pattern-like solutions,

$$u_1(x,t)=\begin{cases} \dfrac{A\cos k(x+\lambda t)+C}{1+B\cos k(x+\lambda t)}\,, & |k(x+\lambda t)|\le \dfrac{\pi}{2}\,,\\ 0\,, & \text{otherwise}\,,\end{cases}$$
$$u_2(x,t)=\frac{A\cosh k(x+\lambda t)+C}{1+B\cosh k(x+\lambda t)}\,,$$
$$u_3(x,t)=\frac{A\sinh k(x+\lambda t)+C}{1+B\sinh k(x+\lambda t)}\,.$$

It is easy to show that u_1 and u_2 are extensions of Eqs. (3) and (4), respectively, but u_3 is new.

A natural problem is whether other higher-dimensional, coupled nonlinear wave equations with linear dispersion term also possess compacton-like solutions and solitary pattern-like solutions.

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In this paper we would like to extend our transformations to seek new types of solutions of the (2+1)-dimensional generalization of mKdV equation,

$$u_t = \alpha(u_{yyy} + \beta u_{xxy} + 6u^2 u_y) + (\psi_x u)_y + \psi_y u_x\,, \qquad (5a)$$

$$\psi_{yy} + \gamma\psi_{xx} = 2\mu u u_x\,, \qquad (5b)$$

where α, β, γ, and μ are constants, which was presented by Imai and Nozaki.[16] Recently, we gave its doubly periodic solutions in terms of Weierstrass elliptic function.[17]

2 Generalized Compacton-Like and Solitary Pattern-Like Solutions of Eqs. (5)

In the following we consider the system (5). We consider the travelling wave solution in the from,

$$u(x,y,t) = U(\xi)\,, \quad \psi(x,y,t) = \Psi(\xi)\,, \quad \xi = kx + ly - \lambda t\,. \qquad (6)$$

The substitution of Eq. (6) into Eq. (5) yields

$$\lambda U' + \alpha[(l^3 + \beta k^2 l)U''' + 6lU^2U'] + kl(\Psi' U)' + kl\Psi' U' = 0\,, \qquad (7a)$$

$$(l^2 + \gamma k^2)\Psi'' = \mu k (U^2)'\,, \qquad (7b)$$

where the prime denotes the derivative with respect to ξ.

From Eq. (7b) we have

$$\Psi' = \frac{\mu k}{l^2 + \gamma k^2} U^2 + c\,, \qquad (8)$$

where c is a constant of integration. Substituting Eq. (8) into Eq. (7a) we get

$$\alpha(l^3 + \beta k^2 l)U''' + \Big(6\alpha l + \frac{4\mu l k^2}{l^2 + \gamma k^2}\Big)U^2U' + (\lambda + 2klc)U' = 0\,. \qquad (9)$$

In what follows we mainly consider the solutions of Eq. (9) by using our above-mentioned transformations.

2.1 *Generalized Compacton-Like Solutions*

We assume that equation (9) admits the solution

$$U(\xi) = \begin{cases} \dfrac{A\cos(\xi) + C}{1 + B\cos(\xi)}\,, & |\xi| \le \dfrac{\pi}{2}\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \qquad (10)$$

As is well know, there exists the relationship $\cos^2(\xi) = [1+\cos(2\xi)]/2$. Thus the transformation (10) is more general than the known transformation (3).

With the aid of *Maple*, the substitution of the solution (10) into Eq. (9) yields a polynomial equation in $\cos^i\xi \sin^j\xi$ $(i = 0,1,2,3\ldots, j = 0,1)$. Setting their coefficients to zero yields a set of algebraic equations in unknowns A, B, C, k, λ as

$$\begin{aligned} &- 6\alpha l^3\beta k^2AB^2 + 6\alpha l^3\beta k^2B^3C - 6\alpha l\beta k^4AB^2\gamma + 6\alpha l\beta k^4B^3C\gamma + \lambda CB\gamma k^2 - 2k^3lcA\gamma \\ &- \alpha l\beta k^4CB\gamma + \alpha l^5A - \lambda Al^2 - 6\alpha l^3AB^2\gamma k^2 + \alpha l^3A\gamma k^2 + 4l\mu k^2C^3B - 4l\mu k^2C^2A + 2kl^3cCB \\ &+ 6\alpha l^3B^3C\gamma k^2 - \alpha l^5BC + 6\alpha l^3C^3B - 6\alpha l^3C^2A + \lambda CBl^2 - 2kl^3cA - \lambda A\gamma k^2 - 6\alpha l^5AB^2 \\ &+ 6l\alpha\gamma k^2C^3B - 6l\alpha\gamma k^2C^2A - \alpha l^3BC\gamma k^2 + 2k^3lcCB\gamma + \alpha l\beta k^4A\gamma \\ &+ 6\alpha l^5B^3C + \alpha l^3\beta k^2A - \alpha l^3\beta k^2CB = 0\,, \end{aligned} \qquad (11)$$

$$\begin{aligned} &6\alpha l^3A^2CB + 4l\mu k^2A^2CB + \lambda CB^3\gamma k^2 + \lambda CB^3l^2 - 6\alpha l^3A^3 - 6l\alpha\gamma k^2A^3 + \alpha l^3AB^2\gamma k^2 \\ &+ 6l\alpha\gamma k^2A^2CB - 2kl^3cAB^2 + \alpha l\beta k^4AB^2\gamma - 4l\mu k^2A^3 - 2k^3lcA\gamma B^2 - \alpha l^3\beta k^2B^3C \\ &- \alpha l\beta k^4B^3C\gamma - \lambda Al^2B^2 + \alpha l^5AB^2 - \lambda A\gamma k^2B^2 - \alpha l^3B^3C\gamma k^2 \\ &+ 2kl^3cCB^3 - \alpha l^5B^3C + 2k^3lcCB^3\gamma + \alpha l^3\beta k^2AB^2 = 0\,, \end{aligned} \qquad (12)$$

$$\begin{aligned} &4\alpha l^5B^2C + 8l\mu k^2AC^2B - 12l\alpha\gamma k^2A^2C + 2\lambda CB^2\gamma k^2 + 12l\alpha\gamma k^2AC^2B + 4\alpha l^3\beta k^2B^2C \\ &+ 2\lambda CB^2l^2 - 4\alpha l\beta k^4AB\gamma - 2\lambda A\gamma k^2B - 4\alpha l^3AB\gamma k^2 - 2\lambda Al^2B - 4\alpha l^5AB + 4k^3lcCB^2\gamma \\ &- 4k^3lcA\gamma B + 4kl^3cCB^2 - 12\alpha l^3A^2C + 4\alpha l\beta k^4B^2C\gamma + 4\alpha l^3B^2C\gamma k^2 \\ &- 8l\mu k^2A^2C + 12\alpha l^3AC^2B - 4\alpha l^3\beta k^2AB - 4kl^3cAB = 0\,. \end{aligned} \qquad (13)$$

Solving the set of algebraic equations, we have

$$A = -\frac{3(l^4 + l^2\beta k^2 + l^2\gamma k^2 + \beta k^4\gamma)B\alpha}{4(3\alpha l^2 + 3\alpha\gamma k^2 + 2\mu k^2)(B^2 - 1)}\frac{1}{R}\,, \quad C = (2B^2 - 1)R\,,$$

$$c = \frac{2k^2\alpha l\beta B^2 + \alpha l\beta k^2 + \alpha l^3 - 2\lambda B^2 + 2l^3\alpha B^2 + 2\lambda}{4lk\Big(-1 + B^2\Big)}\,, \qquad (14)$$

where

$$R = \sqrt{-\frac{3\alpha l^2\beta k^2 + 3\alpha\beta k^4\gamma + 3\alpha l^2\gamma k^2 + 3\alpha l^4}{-12\alpha l^2 + 12B^2\alpha l^2 - 12\alpha\gamma k^2 + 12B^2\alpha\gamma k^2 - 8\mu k^2 + 8B^2\mu k^2}}\,. \qquad (15)$$

Thus we know that the function $u(x,y,t)$ in system (5) possesses the comacton-like solutions,

$$u(x,y,t)=\begin{cases}\dfrac{A\cos(kx+ly-\lambda t)+C}{1+B\cos(kx+ly-\lambda t)}\,, & |kx+ly-\lambda t|\leq\dfrac{\pi}{2}\,,\\ 0\,, & \text{otherwise}\,,\end{cases}\tag{16}$$

where A, B, C are defined by Eq. (14), and the variable $\Psi'(\xi)$ in system (5) possesses the generalized comacton-like solutions,

$$\Psi'(\xi)=\begin{cases}\dfrac{\mu k}{l^2+\gamma k^2}\dfrac{[A\cos(\xi)+C]^2}{[1+B\cos(\xi)]^2}+c\,, & |\xi|\leq\dfrac{\pi}{2}\,,\\ c\,, & \text{otherwise}\,,\end{cases}\tag{17}$$

where A, B, C, c are defined by Eq. (14).

Note that we can obtain $\psi(x,y,t)=\Psi(\xi)$ from Eq. (17). But it is complicated, we omit it here. Moreover we have the potentials ψ_x, ψ_y of $\psi(x,y,t)$ in the forms

$$\psi_x(x,y,t)=\begin{cases}\dfrac{\mu k^2}{l^2+\gamma k^2}\dfrac{[A\cos(kx+ly-\lambda t)+C]^2}{[1+B\cos(kx+ly-\lambda t)]^2}+c\,, & |kx+ly-\lambda t|\leq\dfrac{\pi}{2}\,,\\ c\,, & \text{otherwise}\,,\end{cases}\tag{18}$$

$$\psi_y(x,y,t)=\begin{cases}\dfrac{\mu kl}{l^2+\gamma k^2}\dfrac{[A\cos(kx+ly-\lambda t)+C]^2}{[1+B\cos(kx+ly-\lambda t)]^2}+c\,, & |kx+ly-\lambda t|\leq\dfrac{\pi}{2}\,,\\ c\,, & \text{otherwise}\,.\end{cases}\tag{19}$$

Remark 1 Particularly, we have the formal compacton-like solution

$$u=\begin{cases}\dfrac{4(\alpha l^4+\alpha l^2\gamma k^2+\alpha\beta k^4\gamma\alpha l^2\beta k^2}{27\alpha\gamma k^2+18\mu k^2+27\alpha l^2}\dfrac{\cos^2(kx+ly-\lambda t)}{1-(2/3)\cos^2(kx+ly-\lambda t)}\,, & |kx+ly-\lambda t|\leq\dfrac{\pi}{2}\,,\\ 0\,, & \text{otherwise}\,,\end{cases}\tag{20}$$

where $\lambda=-4\alpha l\beta k^2-2klc-4\alpha l^3$.

2.2 *Solitary Pattern-Like Solutions*

We assume that equation admits the solution

$$U(\xi)=\frac{A\cosh(\xi)+C}{1+B\cosh(\xi)}\,.\tag{21}$$

As is well known, there exists the relationship $\cosh^2(\xi)=[1+\cosh(2\xi)]/2$. Thus the transformation (10) is more general than Eq. (4).

With the aid of *Maple*, the substitution of solution (21) into Eq. (9) yields a polynomial equation in $\cosh^i\xi\sinh^j\xi$ $(i=0,1,2,3,\ldots,j=0,1)$. Setting their coefficients to zero yields a set of algebraic equations in unknowns A, B, C, k, λ. Solving the set of equations, we can determine them as follows:

$$A=\frac{3(l^4+l^2\beta k^2+l^2\gamma k^2+\beta k^4\gamma)\alpha B}{4(B^2-1)(3\alpha l^2+3\alpha\gamma k^2+2\mu k^2)}\frac{1}{Q}\,,\quad C=(2B^2-1)Q\,,$$
$$c=-\frac{2k^2\alpha l\beta B^2+\alpha l\beta k^2+\alpha l^3+2\lambda B^2+2l^3\alpha B^2-2\lambda}{4lk\Big(-1+B^2\Big)}\,,\tag{22}$$

where

$$Q=\sqrt{\frac{3\alpha l^2\beta k^2+3\alpha\beta k^4\gamma+3\alpha l^2\gamma k^2+3\alpha l^4}{-12\alpha l^2+12B^2\alpha l^2-12\alpha\gamma k^2+12B^2\alpha\gamma k^2-8\mu k^2+8B^2\mu k^2}}\,.\tag{23}$$

Therefore we have solitary pattern-like solution of Eq. (5)

$$u(x,y,t)=\frac{A\cosh(kx+ly-\lambda t)+C}{1+B\cosh(kx+ly-\lambda t)}\,,\tag{24}$$

$$\Psi'(\xi)=\frac{\mu k}{l^2+\gamma k^2}\frac{[A\cosh(\xi)+C]^2}{[1+B\cosh(\xi)]^2}+c\,,\tag{25}$$

from which we have the potentials of $\psi(x,y,t)$ as

$$\psi_x(x,y,t)=\frac{\mu k^2}{l^2+\gamma k^2}\frac{[A\cosh(kx+ly-\lambda t)+C]^2}{[1+B\cosh(kx+ly-\lambda t)]^2}+c\,,\tag{26}$$

$$\psi_y(x,y,t)=\frac{\mu kl}{l^2+\gamma k^2}\frac{[A\cosh(kx+ly-\lambda t)+C]^2}{[1+B\cosh(kx+ly-\lambda t)]^2}+c\,,\tag{27}$$

where A, C, c are given by Eqs. (22) and (23).

Similarly we assume that equation (9) admits another solution in the form

$$U(\xi) = \frac{A\sinh(\xi) + C}{1 + B\sinh(\xi)}\,, \tag{28}$$

which is a new transformation.

With the aid of *Maple*, the substitution of the solution (28) into Eq. (9) yields a polynomial equation in $\cosh^i \xi \sinh^j \xi$ ($i = 0, 1, 2, 3, \ldots, j = 0, 1$). Setting their coefficients to zero yields a set of algebraic equations in unknowns A, B, C, k, λ. Solving the set of equations, we can determine them as follows:

$$C = \frac{3\alpha\left(\beta k^4\gamma + 2\beta k^4 B^2\gamma + l^2\gamma k^2 + 2l^2 B^2\gamma k^2 + l^2\beta k^2 + 2l^2\beta k^2 B^2 + l^4 + 2l^4 B^2\right)}{4(B^2+1)(3\alpha l^2 + 3\alpha\gamma k^2 + 2\mu k^2)W}\,,$$

$$A = BW\,, \quad \lambda = -\frac{(-k^2\alpha\beta + 2k^2\alpha\beta B^2 + 4kc + 4kcB^2 - \alpha l^2 + 2B^2\alpha l^2)l}{2(1+B^2)}\,, \tag{29}$$

where

$$W = \sqrt{-\frac{3\alpha l^2\beta k^2 + 3\alpha\beta k^4\gamma + 3\alpha l^2\gamma k^2 + 3\alpha l^4}{12\alpha l^2 + 12B^2\alpha l^2 + 12\alpha\gamma k^2 + 12B^2\alpha\gamma k^2 + 8\mu k^2 + 8B^2\mu k^2}}\,. \tag{30}$$

Therefore we have another solitary pattern-like solution of Eq. (5)

$$u(x,y,t) = \frac{A\sinh(kx+ly-\lambda t) + C}{1 + B\sinh(kx+ly-\lambda t)}\,, \tag{31}$$

$$\Psi'(\xi) = \frac{\mu k}{l^2+\gamma k^2}\frac{[A\sinh(\xi)+C]^2}{[1+B\sinh(\xi)]^2} + c\,, \tag{32}$$

from which we have the potentials of $\psi(x,y,t)$ as

$$\psi_x(x,y,t) = \frac{\mu k^2}{l^2+\gamma k^2}\frac{[A\sinh(kx+ly-\lambda t)+C]^2}{[1+B\sinh(kx+ly-\lambda t)]^2} + c\,, \tag{33}$$

$$\psi_y(x,y,t) = \frac{\mu kl}{l^2+\gamma k^2}\frac{[A\sinh(kx+ly-\lambda t)+C]^2}{[1+B\sinh(kx+ly-\lambda t)]^2} + c\,, \tag{34}$$

where A, C, c are given by Eqs. (22) and (23).

3 Conclusions and Discussions

In summary, we have extended our new transformations to the higher-dimenonal coupled system, i.e., (2+1)-dimensional generalization of mKdV equation. As a result, it is shown that this equation with linearly dispersive terms also possesses some new types of solutions, which include compacton-like solutions and solitary pattern-like solutions. These solutions may be useful to explaining some physical phenomena. In addition, these transformations can be also extended to other higher-dimensional nonlinear equations, such as the (2+1)-dimensional mKP equation[18]

$$q_t = \frac{1}{8}(q_{xxx} - 6q^2 q_x + 6q_x\partial_x^{-1}q_y + 3\partial_x^{-1}q_{yy}).$$

Similarly we can also obtain the compatcon-like solutions and solitary pattern-like solutions.

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