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An extended Jacobi elliptic function rational expansion method and its application to (2 + 1)-dimensional dispersive long wave equation

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Abstract

With the aid of computerized symbolic computation, a new elliptic function rational expansion method is presented by means of a new general ansatz, in which periodic solutions of nonlinear partial differential equations that can be expressed as a finite Laurent series of some of 12 Jacobi elliptic functions, is more powerful than exiting Jacobi elliptic function methods and is very powerful to uniformly construct more new exact periodic solutions in terms of rational formal Jacobi elliptic function solution of nonlinear partial differential equations. As an application of the method, we choose a (2 + 1)-dimensional dispersive long wave equation to illustrate the method. As a result, we can successfully obtain the solutions found by most existing Jacobi elliptic function methods and find other new and more general solutions at the same time. Of course, more shock wave solutions or solitary wave solutions can be gotten at their limit condition.

Keywords: (2 + 1)-dimensional dispersive long wave equation; Jacobi elliptic functions; Travelling wave solution; Soliton solution; Periodic solution

1. Introduction

In recent years, the nonlinear partial differential equations (NPDEs) are widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, biology, etc. Many authors have paid

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attention to this subject [1–12]. It is well known that the elliptic functions including Jacobi elliptic functions and Weierstrass elliptic functions, etc. are closely related to nonlinear differential equations, which have been shown to possess the elliptic function solutions [10–12]. Some methods were presented to seek the periodic solutions, such as the Jacobi elliptic function expansion method [10], in which Liu et al. used three Jacobi elliptic functions to express exact solutions of some NPDEs, and the extended Jacobi elliptic function expansion method [12] in which Yan developed an extended Jacobi elliptic function expansion method by using 12 Jacobi elliptic functions, etc. When the modulus $m \to 1$ or 0, the Jacobi elliptic functions degenerate as soliton solutions or trigonometric function solutions. Therefore, seeking the Jacobi elliptic function solutions of NPDEs is significant. However, the above existing methods seek only periodic solutions of NPDEs that can be expressed as a finite Tailor series of some of 12 Jacobi elliptic functions. Recently, we present the elliptic function rational expansion method [13], in which periodic solutions of NPDEs that can be expressed as a finite Laurent series of some of 12 Jacobi elliptic functions. In this Letter a new Jacobi elliptic function rational expansion method is presented by means of a new general ansatz, and is more powerful than above exiting Jacobi elliptic function methods [10–12] to uniformly construct more new exact periodic solutions in terms of rational formal elliptic function of NPDEs. The algorithm and its applications are demonstrated later.

This Letter is organized as follows. In Section 2, we summarize the elliptic function rational expansion method. In Section 3, we apply the generalized method to $(2+1)$-dimensional dispersive long wave equation and bring out many solutions. Conclusions will be presented in finally.

2. Summary of the extended Jacobi elliptic function rational expansion method

In the following we would like to outline the main steps of our general method:

Step 1. For a given nonlinear evolution equation system with some physical fields $u_i(x, y, t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0, \tag{2.1}$$

by using the wave transformation

$$u_i(x, y, t) = u_i(\xi), \quad \xi = k(x + ly + \lambda t), \tag{2.2}$$

where k, l and λ are constants to be determined later. Then the nonlinear evolution Eq. (2.1) is reduced to a nonlinear ordinary differential equation (ODE):

$$G_i\left(u_i, u_i', u_i'', \ldots\right) = 0. \tag{2.3}$$

Step 2. We introduce a new ansatz in terms of finite Jacobi elliptic function rational expansion in the following forms:

1. sn ξ *and* cn ξ *rational expansion*:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\mathrm{sn}^j(\xi)}{(\mu_1\,\mathrm{sn}(\xi) + \mu_2\,\mathrm{cn}(\xi) + 1)^j} + b_{ij}\frac{\mathrm{sn}^{j-1}(\xi)\,\mathrm{cn}(\xi)}{(\mu_1\,\mathrm{sn}(\xi) + \mu_2\,\mathrm{cn}(\xi) + 1)^j}\right), \tag{2.4.1}$$

2. ns ξ *and* cs ξ *rational expansion*:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\mathrm{ns}^j(\xi)}{(\mu_1\,\mathrm{ns}(\xi) + \mu_2\,\mathrm{cs}(\xi) + 1)^j} + b_{ij}\frac{\mathrm{ns}^{j-1}(\xi)\,\mathrm{cs}(\xi)}{(\mu_1\,\mathrm{ns}(\xi) + \mu_2\,\mathrm{cs}(\xi) + 1)^j}\right), \tag{2.4.2}$$

3. $\mathrm{sn}\,\xi$ *and* $\mathrm{dn}\,\xi$ *rational expansion*:

$$u_i(\xi)=a_{i0}+\sum_{j=1}^{m_i}\left(a_{ij}\frac{\mathrm{sn}^j(\xi)}{(\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1)^j}+b_{ij}\frac{\mathrm{sn}^{j-1}(\xi)\,\mathrm{dn}(\xi)}{(\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1)^j}\right), \tag{2.4.3}$$

4. $\mathrm{sd}\,\xi$ *and* $\mathrm{nd}\,\xi$ *rational expansion*:

$$u_i(\xi)=a_{i0}+\sum_{j=1}^{m_i}\left(a_{ij}\frac{\mathrm{sd}^j(\xi)}{(\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1)^j}+b_{ij}\frac{\mathrm{sd}^{j-1}(\xi)\,\mathrm{nd}(\xi)}{(\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1)^j}\right), \tag{2.4.4}$$

where $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$, $\mathrm{dn}\,\xi$, $\mathrm{ns}\,\xi$, $\mathrm{cs}\,\xi$, $\mathrm{sd}\,\xi$ and $\mathrm{nd}\,\xi$ are the Jacobian elliptic sine function, the Jacobian elliptic cosine function and the Jacobian elliptic function of the third kind and other Jacobian functions which is denoted by Glaishers symbols and are generated by these three kinds of functions, namely [14–16],

$$\mathrm{ns}\,\xi=\frac{1}{\mathrm{sn}\,\xi},\qquad \mathrm{nc}\,\xi=\frac{1}{\mathrm{cn}\,\xi},\qquad \mathrm{nd}\,\xi=\frac{1}{\mathrm{dn}\,\xi},\qquad \mathrm{sd}\,\xi=\frac{\mathrm{sn}\,\xi}{\mathrm{dn}\,\xi}, \tag{2.5.1}$$

$$\mathrm{sc}\,\xi=\frac{\mathrm{sn}\,\xi}{\mathrm{cn}\,\xi},\qquad \mathrm{cs}\,\xi=\frac{\mathrm{cn}\,\xi}{\mathrm{sn}\,\xi},\qquad \mathrm{ds}\,\xi=\frac{\mathrm{dn}\,\xi}{\mathrm{sn}\,\xi}, \tag{2.5.2}$$

which are periodic and possess the following properties:

1. Properties of triangular functions:

$$\mathrm{cn}^2\,\xi+\mathrm{sn}^2\,\xi=\mathrm{dn}^2\,\xi+m^2\,\mathrm{sn}^2\,\xi=1, \tag{2.6.1}$$

$$\mathrm{ns}^2\,\xi=1+\mathrm{cs}^2\,\xi,\qquad \mathrm{ns}^2\,\xi=m^2+\mathrm{ds}^2\,\xi, \tag{2.6.2}$$

$$\mathrm{sc}^2\,\xi+1=\mathrm{nc}^2\,\xi,\qquad m^2\,\mathrm{sd}^2\,\xi+1=nd^2\xi. \tag{2.6.3}$$

2. Derivatives of the Jacobi elliptic functions:

$$\mathrm{sn}'\,\xi=\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi,\qquad \mathrm{cn}'\,\xi=-\,\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi,\qquad \mathrm{dn}'\,\xi=-m^2\,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi, \tag{2.7.1}$$

$$\mathrm{ns}'\,\xi=-\,\mathrm{ds}\,\xi\,\mathrm{cs}\,\xi,\qquad \mathrm{ds}'\,\xi=-\,\mathrm{cs}\,\xi\,\mathrm{ns}\,\xi,\qquad \mathrm{cs}'\,\xi=-\,\mathrm{ns}\,\xi\,\mathrm{ds}\,\xi, \tag{2.7.2}$$

$$\mathrm{sc}'\,\xi=\mathrm{nc}\,\xi\,\mathrm{dc}\,\xi,\qquad \mathrm{nc}'\,\xi=\mathrm{sc}\,\xi\,\mathrm{dc}\,\xi,\qquad \mathrm{cd}'\,\xi=\mathrm{cd}\,\xi\,\mathrm{nd}\,\xi,\qquad \mathrm{nd}'\,\xi=m^2\,\mathrm{sd}\,\xi\,\mathrm{cd}\,\xi, \tag{2.7.3}$$

where m is a modulus. The Jacobi–Glaisher functions for elliptic function can be found in Refs. [14–16]. It is necessary to point out that above combinations only require solving the recurrent coefficient relation or derivative relation for the terms of polynomial for computation closed. Therefore other Jacobi elliptic functions can be chosen to combine as new ansatz. For simplicity, we omit them here.

Step 3. The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass periodic solutions to occur is that differ effects that act to change wave forms in many nonlinear equations, i.e., dispersion, dissipation and nonlinearity, either separately or various combination are able to balance out. We define the degree of $u_i(\xi)$ as $D[u_i(\xi)]=n_i$, which gives rise to the degrees of other expressions as

$$D\big[u_i^{(\alpha)}\big]=n_i+\alpha,\qquad D\big[u_i^{\beta}\big(u_j^{(\alpha)}\big)^s\big]=n_i\beta+(\alpha+n_j)s. \tag{2.8}$$

Therefore we can get the value of m_i in Eqs. (2.4). If n_i is a nonnegative integer, then we first make the transformation $u_i=\omega^{n_i}$.

Step 4. Respectively substitute four cases of Eqs. (2.4) into Eq. (2.3) along with Eqs. (2.6) and (2.7) and then respectively set all coefficients of $\mathrm{sn}^i(\xi)\,\mathrm{cn}^j(\xi)$, $\mathrm{ns}^i(\xi)\,\mathrm{cs}^j(\xi)$, $\mathrm{sn}^i(\xi)\,\mathrm{dn}^j(\xi)$ and $\mathrm{sd}^i(\xi)\,\mathrm{nd}^j(\xi)$ $(i=1,2,\ldots;\ j=0,1)$ to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, μ_1, μ_2, a_{i0}, a_{ij} and b_{ij} $(i = 1, 2, \ldots;\ j = 1, 2, \ldots, m_i)$.

Step 5. By use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method [17], solving the over-determined algebraic equations, we would end up with the explicit expressions for k, μ_1, μ_2, a_{i0}, a_{ij} and b_{ij} $(i = 1, 2, \ldots;\ j = 1, 2, \ldots, m_i)$. In this way, we can get periodic solutions with Jacobi elliptic function.

Since

$$\lim_{m\to 1} \operatorname{sn}\xi = \tanh\xi, \qquad \lim_{m\to 1} \operatorname{cn}\xi = \operatorname{sech}\xi, \qquad \lim_{m\to 1} \operatorname{dn}\xi = \operatorname{sech}\xi, \tag{2.9.1}$$

$$\lim_{m\to 1} \operatorname{ns}\xi = \coth\xi, \qquad \lim_{m\to 1} \operatorname{cs}\xi = \operatorname{csch}\xi, \qquad \lim_{m\to 1} \operatorname{ds}\xi = \operatorname{csch}\xi, \tag{2.9.2}$$

$$\lim_{m\to 0} \operatorname{sn}\xi = \sin\xi, \qquad \lim_{m\to 0} \operatorname{cn}\xi = \cos\xi, \qquad \lim_{m\to 0} \operatorname{dn}\xi = 1, \tag{2.9.3}$$

$$\lim_{m\to 0} \operatorname{ns}\xi = \csc\xi, \qquad \lim_{m\to 0} \operatorname{cs}\xi = \cot\xi, \qquad \lim_{m\to 0} \operatorname{ds}\xi = \csc\xi, \tag{2.9.4}$$

u_i degenerate respectively as the following form:

1. Solitary wave solutions:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\tanh^j(\xi)}{(\mu_1\tanh(\xi) + \mu_2\operatorname{sech}(\xi) + 1)^j} + b_{ij}\frac{\tanh^{j-1}(\xi)\operatorname{sech}(\xi)}{(\mu_1\tanh(\xi) + \mu_2\operatorname{sech}(\xi) + 1)^j}\right), \tag{2.10.1}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\frac{1}{\tanh^j(\xi)}}{(\mu_1\frac{1}{\tanh(\xi)} + \mu_2\frac{\operatorname{sech}(\xi)}{\tanh(\xi)} + 1)^j} + b_{ij}\frac{(\frac{\operatorname{sech}(\xi)}{\tanh(\xi)})^{j-1}\frac{1}{\tanh(\xi)}}{(\mu_1\frac{1}{\tanh(\xi)} + \mu_2\frac{\operatorname{sech}(\xi)}{\tanh(\xi)} + 1)^j}\right), \tag{2.10.2}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\tanh^j(\xi)}{(\mu_1\tanh(\xi) + \mu_2\operatorname{sech}(\xi) + 1)^j} + b_{ij}\frac{\tanh^{j-1}(\xi)\operatorname{sech}(\xi)}{(\mu_1\tanh(\xi) + \mu_2\operatorname{sech}(\xi) + 1)^j}\right), \tag{2.10.3}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{(\frac{\tanh(\xi)}{\operatorname{sech}(\xi)})^j}{(\mu_1(\frac{\tanh(\xi)}{\operatorname{sech}(\xi)}) + \mu_2\frac{1}{\operatorname{sech}(\xi)} + 1)^j} + b_{ij}\frac{(\frac{\tanh(\xi)}{\operatorname{sech}(\xi)})^{j-1}\frac{1}{\operatorname{sech}(\xi)}}{(\mu_1(\frac{\tanh(\xi)}{\operatorname{sech}(\xi)}) + \mu_2\frac{1}{\operatorname{sech}(\xi)} + 1)^j}\right). \tag{2.10.4}$$

2. Triangular function formal solution:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\sin^j(\xi)}{(\mu_1\sin(\xi) + \mu_2\cos(\xi) + 1)^j} + b_{ij}\frac{\sin^{j-1}(\xi)\cos(\xi)}{(\mu_1\sin(\xi) + \mu_2\cos(\xi) + 1)^j}\right), \tag{2.11.1}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\frac{1}{\sin^j(\xi)}}{(\mu_1\frac{1}{\sin(\xi)} + \mu_2\frac{\cos(\xi)}{\sin(\xi)} + 1)^j} + b_{ij}\frac{(\frac{\cos(\xi)}{\sin(\xi)})^{j-1}\frac{1}{\sin(\xi)}}{(\mu_1\frac{1}{\tanh(\xi)} + \mu_2\frac{\cos(\xi)}{\sin(\xi)} + 1)^j}\right), \tag{2.11.2}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\sin^j(\xi)}{(\mu_1\sin(\xi) + \mu_2 + 1)^j} + b_{ij}\frac{\sin^{j-1}(\xi)}{(\mu_1\sin(\xi) + \mu_2 + 1)^j}\right), \tag{2.11.3}$$

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\sin^j(\xi)}{(\mu_1\sin(\xi) + \mu_2 + 1)^j} + b_{ij}\frac{\sin^{j-1}(\xi)}{(\mu_1\sin(\xi) + \mu_2 + 1)^j}\right). \tag{2.11.4}$$

So the extended Jacobi elliptic function rational expansion method is more powerful than the methods [10–13]. The solutions which contain solitary wave solutions, singular solitary solutions and triangular function formal solutions can be gotten by the extended method.

3. Exact solutions of the (2 + 1)-dimensional dispersive long wave equation

Let us consider the (2 + 1)-dimensional dispersive long wave equation (DLWE), i.e.,

$$\begin{cases} u_{yt} + v_{xx} + (uu_x)_y = 0, \\ v_t + u_x + (uv)_x + u_{xxy} = 0. \end{cases} \tag{3.1}$$

The (2 + 1)-dimensional DLWE (3.1) was first derived by Boiti et al. [18] as a compatibility for a “weak” Lax pair. Recently considerable effort has been devoted to the study of this system. In [19], Paquin and Winternitz showed that the symmetry algebra of (2 + 1)-dimensional DLWE (3.1) is infinite-dimensional and possesses a Kac–Moody–Virasoro structure. Some special similarity solutions are also given in [19] by using symmetry algebra and the classical theoretical analysis. The more general symmetry algebra, w_∞ symmetry algebra, is given in [20]. Lou [21] has given nine types of the two-dimensional partial differential equation reductions and thirteen types of the ordinary differential equation reductions by means of the direct and nonclassical method. The system (3.1) have no Painleve property though they are Lax or IST integrable [22]. More recently, Tang et al. [23], by means of the variable separation approach, the abundant localized coherent structures of the system (3.1) are given out. In [24], the possible chaotic and fractal localized structures are revealed for the system (3.1).

According to the above method, to seek travelling wave solutions of Eqs. (3.1), we make the following transformation

$$u(x,t) = \phi(\xi), \qquad v(x,t) = \sigma(\xi), \quad \xi = k(x + ly + \lambda t), \tag{3.2}$$

where k, l, λ is a constant to be determined later, and thus Eqs. (3.1) becomes

$$\begin{cases} \lambda l\phi'' + \sigma'' + l\phi'^2 + l\phi\phi'' = 0, \\ \lambda\sigma' + \phi' + (\phi\sigma)' + k^2 l\phi''' = 0. \end{cases} \tag{3.3}$$

Now we consider the system (3.3) in the above four cases, i.e., (2.4.1)–(2.4.4). According to Step 1 in Section 2, by balancing ϕ''' and $(\sigma\phi)'$ in Eq. (3.3) and by balancing σ'' and $\phi\phi''$ in Eq. (3.3), we suppose that Eq. (3.3) has the following formal solutions.

3.1. $\operatorname{sn}\xi$ *and* $\operatorname{cn}\xi$ *rational expansion*

Now we consider the ansatz (2.4.1). For Eq. (3.3), the ansatz (2.4.1) becomes

$$\begin{cases} \phi(\xi) = a_0 + a_1 \frac{\operatorname{sn}(\xi)}{\mu_1 \operatorname{sn}(\xi) + \mu_2 \operatorname{cn}(\xi) + 1} + b_1 \frac{\operatorname{cn}(\xi)}{\mu_1 \operatorname{sn}(\xi) + \mu_2 \operatorname{cn}(\xi) + 1}, \\ \sigma(\xi) = A_0 + A_1 \frac{\operatorname{sn}(\xi)}{\mu_1 \operatorname{sn}(\xi) + \mu_2 \operatorname{cn}(\xi) + 1} + B_1 \frac{\operatorname{cn}(\xi)}{\mu_1 \operatorname{sn}(\xi) + \mu_2 \operatorname{cn}(\xi) + 1} \\ \qquad + A_2 \frac{\operatorname{sn}^2(\xi)}{(\mu_1 \operatorname{sn}(\xi) + \mu_2 \operatorname{cn}(\xi) + 1)^2} + B_2 \frac{\operatorname{sn}(\xi)\operatorname{cn}(\xi)}{(\mu_1 \operatorname{sn}(\xi) + \mu_2 \operatorname{cn}(\xi) + 1)^2}, \end{cases} \tag{3.4}$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 are constants to be determined later.

With the aid of *Maple*, substituting (3.4) along with (2.6) and (2.7) into (3.3), yields a set of algebraic equations for $\operatorname{sn}^i(\xi)\operatorname{cn}^j(\xi)$ $(i = 0, 1, \ldots;\ j = 0, 1)$. Setting the coefficients of these terms $\operatorname{sn}^i(\xi)\operatorname{cn}^j(\xi)$ to be zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2 and k.

By use of the *Maple* soft package “Charsets” by Dongming Wang, which is based on the Wu-elimination method [17], solving the over-determined algebraic equations, we get the following results.

Case 1.

$$k = \pm\frac{1}{2}\frac{a_1}{m}, \qquad a_0 = -\lambda, \qquad A_0 = \frac{-4m^2 + la_1^2 + la_1^2 m^2}{4m^2},$$

$$A_2 = -\frac{1}{2}la_1^2, \qquad b_1 = A_1 = B_1 = B_2 = \mu_1 = \mu_2 = 0. \tag{3.5}$$

Case 2.

$$A_0 = -\frac{lb_1^2 + 4m^2}{4m^2}, \qquad a_0 = -\lambda, \qquad A_2 = \frac{1}{2}lb_1^2,$$
$$k = \pm\frac{ib_1}{2m}, \qquad a_1 = A_1 = B_1 = B_2 = \mu_1 = \mu_2 = 0. \tag{3.6}$$

Case 3.

$$A_2 = -la_1^2, \qquad a_0 = -\lambda, \qquad B_2 = -ila_1^2, \qquad b_1 = ia_1,$$
$$k = \frac{a_1}{m}, \qquad A_0 = \frac{-m^2 + la_1^2}{m^2}, \qquad A_1 = B_1 = \mu_1 = \mu_2 = 0. \tag{3.7}$$

Case 4.

$$b_1 = \pm 2\sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2m^2\mu_2^2 - \mu_2^2 - m^2}\,k, \qquad a_1 = A_1 = B_2 = \mu_1 = 0,$$
$$A_0 = \frac{-\mu_2^2 + \mu_2^4 - \mu_2^4 m^2 - m^2 - 2lk^2\mu_2^2 m^2 + lk^2 m^2 + 2m^2\mu_2^2 + \mu_2^4 lk^2 m^2 - \mu_2^4 lk^2}{\mu_2^4 m^2 - \mu_2^4 - 2m^2\mu_2^2 + \mu_2{}^2 + m^2},$$
$$a_0 = \frac{\pm(2k\mu_2^3 m^2 - 2k\mu_2^3 - 2k\mu_2 m^2 + k\mu_2) - \sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2m^2\mu_2^2 - \mu_2^2 - m^2}\,\lambda}{\sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2m^2\mu_2^2 - \mu_2^2 - m^2}},$$
$$B_1 = -2lk^2\mu_2, \qquad A_2 = 2lk^2\mu_2^2 m^2 - 2lk^2\mu_2^2 - 2lk^2 m^2. \tag{3.8}$$

Case 5.

$$B_2 = \pm l\sqrt{-m^2\mu_2^2 + \mu_2^2 + m^2}\,k^2\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}, \qquad \mu_1 = 0,$$
$$a_1 = \pm\sqrt{-m^2\mu_2^2 + \mu_2^2 + m^2}\,k, \qquad b_1 = \pm\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}\,k,$$
$$A_2 = -lk^2\mu_2^2 + lk^2\mu_2^2 m^2 - lk^2 m^2, \qquad A_0 = -\frac{-\mu_2^2 + m^2\mu_2^2 - m^2 + lk^2 m^2}{m^2\mu_2^2 - \mu_2^2 - m^2},$$
$$a_0 = \frac{\pm(k\mu_2^3 - \mu_2^3 m^2 k - k\mu_2 + k\mu_2 m^2) - \lambda\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}}{\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}},$$
$$A_1 = \pm\frac{l\sqrt{-m^2\mu_2^2 + \mu_2^2 + m^2}\,k^2\mu_2(-\mu_2^2 + m^2\mu_2^2 + 1 - m^2)}{\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}}, \qquad B_1 = -lk^2\mu_2. \tag{3.9}$$

Case 6.

$$a_1 = \pm\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}\,k, \qquad a_0 = \frac{\pm(k\mu_1^3 - k\mu_1) - \lambda\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}}{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}},$$
$$B_1 = \pm\frac{l\sqrt{-m^2 + \mu_1^2}\,k^2\mu_1(\mu_1^2 - 1)}{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}}, \qquad A_0 = -\frac{-lk^2\mu_1^4 + 2lk^2\mu_1^2 m^2 - \mu_1^2 + m^2 - lk^2 m^2}{-\mu_1^2 + m^2},$$

$$b_1=\pm\sqrt{-m^2+\mu_1^2}k,\qquad B_2=\pm l\sqrt{-m^2+\mu_1^2}k^2\sqrt{\mu_1^4-\mu_1^2-\mu_1^2m^2+m^2},\qquad \mu_2=0,$$
$$A_1=2lk^2\mu_1^3-lk^2\mu_1-lk^2\mu_1m^2,\qquad A_2=-lk^2\mu_1^4+lk^2\mu_1^2+lk^2\mu_1^2m^2-lm^2k^2. \tag{3.10}$$

Case 7.

$$k=\pm\frac{1}{2}\frac{a_1}{\sqrt{1-m^2}},\qquad a_0=-\frac{1}{2}a_1-\lambda,\qquad b_1=0,\qquad \mu_1=\mu_2=\pm1,\qquad B_2=\frac{1}{2}la_1^2-A_1,$$
$$A_2=-\frac{1}{2}la_1^2,\qquad B_1=-\frac{1}{2}la_1^2+A_1,\qquad A_0=\frac{2A_1+la_1^2m^2+4-4m^2-2A_1m^2}{4(m^2-1)}. \tag{3.11}$$

From (3.2), (3.4) and Cases 1–7, we obtain the following solutions for Eqs. (3.1).

Family 1. From Eqs. (3.5), we obtain the following $\operatorname{sn}\xi$ and $\operatorname{cn}\xi$ rational formal periodic solutions for the DLWE, as follows:

$$u_1=-\lambda+a_1\operatorname{sn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2m}\right), \tag{3.12.1}$$
$$v_1=\frac{-4m^2+la_1^2+la_1^2m^2}{4m^2}-\frac{1}{2}la_1^2\operatorname{sn}^2\left(\pm\frac{a_1(x+ly+\lambda t)}{2m}\right), \tag{3.12.2}$$

where a_1, l and λ are arbitrary constants.

Family 2. From Eqs. (3.6), we obtain the following $\operatorname{sn}\xi$ and $\operatorname{cn}\xi$ rational formal periodic solutions for the DLWE, as follows:

$$u_2=-\lambda+b_1\operatorname{cn}\left(\pm\frac{ib_1(x+ly+\lambda t)}{2m}\right), \tag{3.13.1}$$
$$v_2=-\frac{lb_1^2+4m^2}{4m^2}+\frac{1}{2}lb_1^2\operatorname{sn}^2\left(\pm\frac{ib_1(x+ly+\lambda t)}{2m}\right), \tag{3.13.2}$$

where b_1, l and λ are arbitrary constants.

Family 3. From Eqs. (3.7), we obtain the following $\operatorname{sn}\xi$ and $\operatorname{cn}\xi$ rational formal periodic solutions for the DLWE, as follows:

$$u_3=-\lambda+a_1\operatorname{sn}\left(\frac{a_1(x+ly+\lambda t)}{m}\right)+ia_1\operatorname{cn}\left(\frac{a_1(x+ly+\lambda t)}{m}\right), \tag{3.14.1}$$
$$v_3=\frac{-m^2+la_1^2}{m^2}-la_1^2\operatorname{sn}^2\left(\pm\frac{a_1(x+ly+\lambda t)}{m}\right)-ila_1^2\operatorname{sn}\left(\pm\frac{a_1(x+ly+\lambda t)}{m}\right)\operatorname{cn}\left(\pm\frac{a_1(x+ly+\lambda t)}{m}\right), \tag{3.14.2}$$

where a_1, l and λ are arbitrary constants.

Family 4. From Eqs. (3.8), we obtain the following $\operatorname{sn}\xi$ and $\operatorname{cn}\xi$ rational formal periodic solutions for the DLWE, as follows:

$$u_4 = \frac{\pm(2k\mu_2^3 m^2 - 2k\mu_2^3 - 2k\mu_2 m^2 + k\mu_2) - \sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2m^2\mu_2^2 - \mu_2^2 - m^2}\lambda}{\sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2m^2\mu_2^2 - \mu_2^2 - m^2}} \pm 2\frac{\sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2m^2\mu_2^2 - \mu_2^2 - m^2}k\,\mathrm{cn}(k(x+ly+\lambda t))}{1+\mu_2\,\mathrm{cn}(k(x+ly+\lambda t))}, \tag{3.15.1}$$

$$v_4 = \frac{\mu_2^4 - \mu_2^2 - \mu_2^4 m^2 - m^2 - 2lk^2\mu_2^2 m^2 + lk^2 m^2 + 2m^2\mu_2^2 + \mu_2^4 lk^2 m^2 - \mu_2^4 lk^2}{\mu_2^4 m^2 - \mu_2^4 - 2m^2\mu_2^2 + \mu_2^2 + m^2} - 2\frac{lk^2\mu_2\,\mathrm{cn}(k(x+ly+\lambda t))}{1+\mu_2\,\mathrm{cn}(k(x+ly+\lambda t))} + \frac{(2lk^2\mu_2^2 m^2 - 2lk^2\mu_2^2 - 2lk^2 m^2)\,\mathrm{sn}^2(k(x+ly+\lambda t))}{(1+\mu_2\,\mathrm{cn}(k(x+ly+\lambda t)))^2}, \tag{3.15.2}$$

where μ_2, k, l and λ are arbitrary constants.

Family 5. From Eqs. (3.9), we obtain the following $\mathrm{sn}\,\xi$ and $\mathrm{cn}\,\xi$ rational formal periodic solutions for the DLWE, as follows:

$$u_5 = \frac{\pm(k\mu_2^3 - \mu_2^3 m^2 k - k\mu_2 + k\mu_2 m^2) - \lambda\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}}{\sqrt{{\mu_2}^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}} \pm \frac{\sqrt{-m^2\mu_2^2 + \mu_2^2 + m^2}k\,\mathrm{sn}(k(x+ly+\lambda t))}{1+\mu_2\,\mathrm{cn}(k(x+ly+\lambda t))} \pm \frac{\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}k\,\mathrm{cn}(k(x+ly+\lambda t))}{1+\mu_2\,\mathrm{cn}(k(x+ly+\lambda t))}, \tag{3.16.1}$$

$$\begin{aligned} v_5 = {} & -\frac{-\mu_2^2 + m^2\mu_2^2 - m^2 + lk^2 m^2}{m^2\mu_2^2 - \mu_2^2 - m^2} - \frac{lk^2\mu_2\,\mathrm{cn}(k(x+ly+\lambda t))}{1+\mu_2\,\mathrm{cn}(k(x+ly+\lambda t))} \\ & \pm \frac{l\sqrt{-m^2\mu_2^2 + \mu_2^2 + m^2}k^2\mu_2(-\mu_2^2 + m^2\mu_2^2 + 1 - m^2)\,\mathrm{sn}(k(x+ly+\lambda t))}{\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}(1+\mu_2\,\mathrm{cn}(k(x+ly+\lambda t)))} \\ & + \frac{(-lk^2\mu_2^2 + lk^2\mu_2^2 m^2 - lk^2 m^2)\,\mathrm{sn}^2(k(x+ly+\lambda t))}{(1+\mu_2\,\mathrm{cn}(k(x+ly+\lambda t)))^2} \\ & \pm l\sqrt{-m^2\mu_2^2 + \mu_2^2 + m^2}\,\frac{k^2\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}\,\mathrm{sn}(k(x+ly+\lambda t))\,\mathrm{cn}(k(x+ly+\lambda t))}{(1+\mu_2\,\mathrm{cn}(k(x+ly+\lambda t)))^2}, \end{aligned} \tag{3.16.2}$$

where μ_2, k, l and λ are arbitrary constants.

Family 6. From Eqs. (3.10), we obtain the following $\mathrm{sn}\,\xi$ and $\mathrm{cn}\,\xi$ rational formal periodic solutions for the DLWE, as follows:

$$u_6 = \frac{\pm(k\mu_1^3 - k\mu_1) - \lambda\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}}{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}} \pm \frac{\sqrt{\mu_1^2 - m^2}k\,\mathrm{cn}(k(x+ly+\lambda t))}{\mu_1\,\mathrm{sn}(k(x+ly+\lambda t))+1} \pm \frac{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}k\,\mathrm{sn}(k(x+ly+\lambda t))}{\mu_1\,\mathrm{sn}(k(x+ly+\lambda t))+1}, \tag{3.17.1}$$

$$v_6 = \frac{lk^2\mu_1^4 - 2lk^2\mu_1^2m^2 + \mu_1^2 - m^2 + lm^2k^2}{-\mu_1^2 + m^2} + \frac{(2lk^2\mu_1^3 - lk^2\mu_1 - lk^2\mu_1m^2)\,\mathrm{sn}(k(x+ly+\lambda t))}{\mu_1\,\mathrm{sn}(k(x+ly+\lambda t))+1}$$
$$\pm \frac{l\sqrt{-m^2+\mu_1^2}k^2\mu_1(\mu_1^2-1)\,\mathrm{cn}(k(x+ly+\lambda t))}{\sqrt{\mu_1^4-\mu_1^2-\mu_1^2m^2+m^2}(\mu_1\,\mathrm{sn}(k(x+ly+\lambda t))+1)}$$
$$+ \frac{(-lk^2\mu_1^4 + lk^2\mu_1^2 + lk^2\mu_1^2m^2 - lm^2k^2)\,\mathrm{sn}^2(k(x+ly+\lambda t))}{(\mu_1\,\mathrm{sn}(k(x+ly+\lambda t))+1)^2}$$
$$\pm \frac{l\sqrt{-m^2+\mu_1^2}k^2\sqrt{\mu_1^4-\mu_1^2-\mu_1^2m^2+m^2}\,\mathrm{sn}(k(x+ly+\lambda t))\,\mathrm{cn}(k(x+ly+\lambda t))}{(\mu_1\,\mathrm{sn}(k(x+ly+\lambda t))+1)^2}, \tag{3.17.2}$$

where μ_1, k, l and λ are arbitrary constants.

Family 7. From Eqs. (3.11), we obtain the following $\mathrm{sn}\,\xi$ and $\mathrm{cn}\,\xi$ rational formal periodic solutions for the DLWE, as follows:

$$u_7 = -\frac{1}{2}a_1 - \lambda + \frac{a_1\,\mathrm{sn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}})}{\pm\,\mathrm{sn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}) \pm \mathrm{cn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}) + 1}, \tag{3.18.1}$$

$$v_7 = \frac{2A_1 + la_1^2m^2 + 4 - 4m^2 - 2A_1m^2}{4(m^2-1)} - \frac{\frac{1}{2}la_1^2\,\mathrm{sn}^2(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}})}{(\pm\,\mathrm{sn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}) \pm \mathrm{cn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}) + 1)^2}$$
$$+ \frac{A_1\,\mathrm{sn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}})}{\pm\,\mathrm{sn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}) \pm \mathrm{cn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}) + 1} + \frac{(-\frac{1}{2}la_1^2 + A_1)\,\mathrm{cn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}})}{\pm\,\mathrm{sn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}) \pm \mathrm{cn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}) + 1}$$
$$+ \frac{(\frac{1}{2}la_1^2 - A_1)\,\mathrm{sn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}})\,\mathrm{cn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}})}{(\pm\,\mathrm{sn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}) \pm \mathrm{cn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}) + 1)^2}, \tag{3.18.2}$$

where a_1, A_1, l and λ are arbitrary constants.

3.2. $\mathrm{ns}\,\xi$ *and* $\mathrm{cs}\,\xi$ *rational expansion*

Now we consider the ansatz (2.4.2). For Eq. (3.3), the ansatz (2.4.2) becomes

$$\begin{cases} \phi(\xi) = a_0 + a_1\frac{\mathrm{ns}(\xi)}{\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1} + b_1\frac{\mathrm{cs}(\xi)}{\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1}, \\ \sigma(\xi) = A_0 + A_1\frac{\mathrm{ns}(\xi)}{\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1} + B_1\frac{\mathrm{cs}(\xi)}{\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1} \\ \qquad + A_2\frac{\mathrm{ns}^2(\xi)}{(\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1)^2} + B_2\frac{\mathrm{ns}(\xi)\,\mathrm{cs}(\xi)}{(\mu_1\,\mathrm{ns}(\xi)+\mu_2\,\mathrm{cs}(\xi)+1)^2}, \end{cases} \tag{3.19}$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 are constants to be determined later. Following the same steps in Section 3.1, we can obtain the following $\mathrm{ns}\,\xi$ and $\mathrm{cs}\,\xi$ rational formal periodic solution:

Family 8.

$$u_8 = -\lambda + b_1 \operatorname{cs}\left(\pm\frac{1}{2}b_1(x+ly+\lambda t)\right), \tag{3.20.1}$$

$$v_8 = -1 + \frac{1}{4}lb_1^2m^2 - \frac{1}{2}lb_1^2 \operatorname{ns}^2\left(\pm\frac{1}{2}b_1(x+ly+\lambda t)\right), \tag{3.20.2}$$

where b_1, l and λ are arbitrary constants.

Family 9.

$$u_9 = -\lambda + a_1 \operatorname{ns}\left(\pm\frac{1}{2}a_1(x+ly+\lambda t)\right), \tag{3.21.1}$$

$$v_6 = -1 + \frac{1}{4}la_1^2m^2 + \frac{1}{4}la_1^2 - \frac{1}{2}la_1^2 \operatorname{ns}^2\left(\pm\frac{1}{2}a_1(x+ly+\lambda t)\right), \tag{3.21.2}$$

where a_1, l and λ are arbitrary constants.

Family 10.

$$u_{10} = -\lambda + a_1 \operatorname{ns}\big(\pm a_1(x+ly+\lambda t)\big) - a_1 \operatorname{cs}\big(\pm a_1(x+ly+\lambda t)\big), \tag{3.22.1}$$

$$v_{10} = la_1^2m^2 - 1 - la_1^2 \operatorname{ns}^2\big(\pm a_1(x+ly+\lambda t)\big) + la_1^2 \operatorname{ns}\big(\pm a_1(x+ly+\lambda t)\big) \operatorname{cs}\big(\pm a_1(x+ly+\lambda t)\big), \tag{3.22.2}$$

where a_1, l and λ are arbitrary constants.

Family 11.

$$u_{11} = a_0 \pm \frac{(\mu_2^2m^2 - 1 - \mu_2^2)(a_0+\lambda)\operatorname{ns}(\xi)}{(m^2-1)\sqrt{1+\mu_2^2}\mu_2(\mu_2\operatorname{cs}(\xi)+1)} \mp \frac{(\mu_2^2m^2 - 1 - \mu_2^2)(a_0+\lambda)\operatorname{cs}(\xi)}{(m^2-1)\mu_2(\mu_2\operatorname{cs}(\xi)+1)}, \tag{3.23.1}$$

$$\begin{aligned} v_{11} = {} & \frac{-m^4\mu_2^4 - m^4\mu_2^2 + 2m^2\mu_2^4 + 2m^2\mu_2^2 + 2lm^2a_0\lambda + lm^2\lambda^2 - \mu_2^4 - \mu_2^2}{\mu_2^2(m^2-1)^2(1+\mu_2^2)} \\ & \mp \frac{l(\mu_2^2m^2 - 1 - \mu_2^2)(a_0+\lambda)^2\operatorname{ns}(\xi)}{(m^2-1)\sqrt{1+\mu_2^2}\mu_2(\mu_2\operatorname{cs}(\xi)+1)} + \frac{(\mu_2^2m^2 - 1 - \mu_2^2)^2(a_0+\lambda)^2lm^2\operatorname{cs}(\xi)}{(m^2-1)^2(1+\mu_2^2)\mu_2(\mu_2\operatorname{cs}(\xi)+1)} \\ & - \frac{l(\mu_2^2m^2 - 1 - \mu_2^2)^2(a_0+\lambda)^2\operatorname{ns}^2(\xi)}{(m^2-1)^2(1+\mu_2^2)\mu_2^2(\mu_2\operatorname{cs}(\xi)+1)^2} \pm \frac{l(\mu_2^2m^2 - 1 - \mu_2^2)^2(a_0+\lambda)^2\operatorname{ns}(\xi)\operatorname{cs}(\xi)}{(m^2-1)^2\sqrt{1+\mu_2^2}\mu_2^2(\mu_2\operatorname{cs}(\xi)+1)^2}, \end{aligned} \tag{3.23.2}$$

where $\xi = k(x+ly+\lambda t)$, $k = \mp\frac{\sqrt{(1-\mu_2^2m^2+\mu_2^2)(1+\mu_2^2)}(a_0+\lambda)}{(m^2-1)(1+\mu_2^2)\mu_2^2}$, a_0, μ_2, l and λ are arbitrary constants.

Family 12.

$$u_{12} = a_0 - \frac{(\mu_1^2m^2-1)(a_0+\lambda)\operatorname{ns}(\xi)}{\mu_1m^2(\mu_1\operatorname{ns}(\xi)+1)} \pm \frac{(\mu_1^2m^2-1)(a_0+\lambda)\operatorname{cs}(\xi)}{m^2\mu_1\sqrt{1-\mu_1^2}(\mu_1\operatorname{ns}(\xi)+1)}, \tag{3.24.1}$$

$$v_{12} = \frac{l\lambda^2\mu_1^4m^2 + 4l\mu_1^2a_0\lambda + \mu_1^4m^2 + l\mu_1^4m^2a_0^2 - \mu_1^2m^2 - 2l\lambda^2\mu_1^2 + l\lambda^2 - 4l\mu_1^2a_0\lambda + 2l\lambda a_0 - 2l\mu_1^2a_0^2 + la_0^2}{\mu_1^2m^2(1-\mu_1^2)}$$

$$- \frac{l(\mu_1^2 m^2 - 1)(a_0+\lambda)^2(1+m^2-2\mu_1^2 m^2)\,\mathrm{ns}(\xi)}{(\mu_1^2-1)m^4\mu_1(\mu_1\,\mathrm{ns}(\xi)+1)} \pm \frac{l(\mu_1^2 m^2-1)(a_0+\lambda)^2\,\mathrm{cs}(\xi)}{\sqrt{1-\mu_1^2}m^2\mu_1(\mu_1\,\mathrm{ns}(\xi)+1)}$$
$$- \frac{(\mu_1^2 m^2-1)^2(a_0+\lambda)^2\,\mathrm{ns}^2(\xi)}{m^4\mu_1^2(\mu_1\,\mathrm{ns}(\xi)+1)^2} \pm \frac{l(\mu_1^2 m^2-1)^2(a_0+\lambda)^2\,\mathrm{ns}(\xi)\,\mathrm{cs}(\xi)}{m^4\mu_1^2\sqrt{1-\mu_1^2}(\mu_1\,\mathrm{ns}(\xi)+1)^2}, \tag{3.24.2}$$

where $\xi = k(x+ly+\lambda t)$, $k = \mp\frac{\sqrt{(\mu_1^2 m^2-1)(\mu_1^2-1)}(a_0+\lambda)}{(\mu_1^2-1)(1+\mu_2^2)\mu_1 m^2}$, a_0, μ_1, l and λ are arbitrary constants.

Family 13.

$$u_{13} = a_0 - \frac{2(a_0+\lambda)\,\mathrm{cs}(\pm(a_0+\lambda)(x+ly+\lambda t))}{\pm\,\mathrm{ns}(\pm(a_0+\lambda)(x+ly+\lambda t)) \pm \mathrm{cs}(\pm(a_0+\lambda)(x+ly+\lambda t))+1}, \tag{3.25.1}$$

$$v_{13} = la_0^2 m^2 - 1 + 2la_0 m^2\lambda + lm^2\lambda^2 - \frac{2l(a_0+\lambda)^2\,\mathrm{cs}(\pm(a_0+\lambda)(x+ly+\lambda t))}{\pm\,\mathrm{ns}(\pm(a_0+\lambda)(x+ly+\lambda t)) \pm \mathrm{cs}(\pm(a_0+\lambda)(x+ly+\lambda t))+1}$$
$$+ \frac{2(la_0^2+4l\lambda a_0+2l\lambda^2)\,\mathrm{ns}(\pm(a_0+\lambda)(x+ly+\lambda t))\,\mathrm{cs}(\pm(a_0+\lambda)(x+ly+\lambda t))}{(\pm\,\mathrm{ns}(\pm(a_0+\lambda)(x+ly+\lambda t)) \pm \mathrm{cs}(\pm(a_0+\lambda)(x+ly+\lambda t))+1)^2}, \tag{3.25.2}$$

where a_0, l and λ are arbitrary constants.

3.3. snξ *and* dnξ *rational expansion*

Now we consider the ansatz (2.4.3). For Eq. (3.3), the ansatz (2.4.3) becomes

$$\begin{cases} \phi(\xi) = a_0 + a_1\frac{\mathrm{sn}(\xi)}{\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1} + b_1\frac{\mathrm{dn}(\xi)}{\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1}, \\ \sigma(\xi) = A_0 + A_1\frac{\mathrm{sn}(\xi)}{\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1} + B_1\frac{\mathrm{dn}(\xi)}{\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1} \\ \qquad + A_2\frac{\mathrm{sn}^2(\xi)}{(\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1)^2} + B_2\frac{\mathrm{sn}(\xi)\,\mathrm{dn}(\xi)}{(\mu_1\,\mathrm{sn}(\xi)+\mu_2\,\mathrm{dn}(\xi)+1)^2}, \end{cases} \tag{3.26}$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 are constants to be determined later. Following the same steps in Section 3.1, we can obtain the following snξ and dnξ rational formal periodic solution:

Family 14.

$$u_{14} = -\lambda + b_1\,\mathrm{dn}\left(\pm\frac{1}{2}ib_1(x+ly+\lambda t)\right), \tag{3.27.1}$$

$$v_{14} = -\frac{1}{4}lb_1^2 m^2 - 1 + \frac{1}{2}lb_1^2 m^2\,\mathrm{sn}^2\left(\pm\frac{1}{2}ib_1(x+ly+\lambda t)\right), \tag{3.27.2}$$

where b_1, l and λ are arbitrary constants.

Family 15.

$$u_{15} = -\lambda + a_1\,\mathrm{sn}\left(\pm\frac{a_1(x+ly+\lambda t)}{m}\right) \pm i\frac{a_1}{m}\,\mathrm{dn}\left(\pm\frac{a_1(x+ly+\lambda t)}{m}\right), \tag{3.28.1}$$

$$v_{15} = -1 + la_1^2 - la_1^2\,\mathrm{sn}^2\left(\pm\frac{a_1(x+ly+\lambda t)}{m}\right) \pm i\frac{la_1^2}{m}\,\mathrm{sn}\left(\pm\frac{a_1(x+ly+\lambda t)}{m}\right)\mathrm{dn}\left(\pm\frac{a_1(x+ly+\lambda t)}{m}\right), \tag{3.28.2}$$

where a_1, l and λ are arbitrary constants.

Family 16.

$$u_{16} = a_0 \pm \frac{(1-\mu_2^2+\mu_2^2m^2)(a_0+\lambda)m\,\mathrm{sn}(\xi)}{\sqrt{\mu_2^2-1}(m^2-1)\mu_2(1+\mu_2\,\mathrm{dn}(\xi))} - \frac{(a_0+\lambda)(1-\mu_2^2+\mu_2^2m^2)\,\mathrm{dn}(\xi)}{\mu_2(m^2-1)(1+\mu_2\,\mathrm{dn}(\xi))}, \tag{3.29.1}$$

$$\begin{aligned} v_{16} = {} & -\frac{\mu_2^4m^4-2\mu_2^4m^2+\mu_2^4-\mu_2^2m^4+2\mu_2^2m^2-\mu_2^2-lm^2\lambda^2-lm^2a_0^2-2lm^2a_0\lambda}{\mu_2^2(\mu_2^2-1)(m^2-1)^2} \\ & \pm \frac{lm(1-\mu_2^2+\mu_2^2m^2)(a_0+\lambda)^2\,\mathrm{sn}(\xi)}{\sqrt{\mu_2^2-1}(m^2-1)\mu_2(1+\mu_2\,\mathrm{dn}(\xi))} - \frac{l(1-\mu_2^2+\mu_2^2m^2)(a_0+\lambda)^2m^2\,\mathrm{dn}(\xi)}{\mu_2(m^2-1)^2(\mu_2^2-1)(1+\mu_2\,\mathrm{dn}(\xi))} \\ & - \frac{l(a_0+\lambda)^2(1-\mu_2^2+\mu_2^2m^2)^2m^2\,\mathrm{sn}^2(\xi)}{\mu_2^2(\mu_2^2-1)(m^2-1)^2(1+\mu_2\,\mathrm{dn}(\xi))^2} \pm \frac{(1-\mu_2^2+\mu_2^2m^2)^2(a_0+\lambda)^2lm\,\mathrm{sn}(\xi)\,\mathrm{dn}(\xi)}{(m^2-1)^2\sqrt{\mu_2^2-1}\mu_2^2(1+\mu_2\,\mathrm{dn}(\xi))^2}, \end{aligned} \tag{3.29.2}$$

where $\xi = k(x+ly+\lambda t)$, $k = \pm\frac{\sqrt{(\mu_2^2-1)(1-\mu_2^2+\mu_2^2m^2)}(a_0+\lambda)}{(\mu_2^2-1)(m^2-1)\mu_2}$, μ_2, a_0, l and λ are arbitrary constants.

Family 17.

$$u_{17} = a_0 - 2\frac{(-\mu_2^2+\mu_2^2m^2+1)(\mu_2^2-1)(a_0+\lambda)\,\mathrm{dn}(\xi)}{(2\mu_2^2m^2-2\mu_2^2-m^2+2)\mu_2(1+\mu_2\,\mathrm{dn}(\xi))}, \tag{3.30.1}$$

$$\begin{aligned} v_{17} = {} & A_0 - 2\frac{(\mu_2^2-1)(a_0+\lambda)^2m^2(-\mu_2^2+\mu_2^2m^2+1)l\,\mathrm{dn}(\xi)}{(2\mu_2^2m^2-2\mu_2^2-m^2+2)^2\mu_2(1+\mu_2\,\mathrm{dn}(\xi))} \\ & - 2\frac{lm^2(\mu_2^2-1)(-\mu_2^2+\mu_2^2m^2+1)^2(a_0+\lambda)^2\,\mathrm{sn}^2(\xi)}{\mu_2{}^2(2\mu_2^2m^2-2\mu_2^2-m^2+2)^2(1+\mu_2\,\mathrm{dn}(\xi))^2}, \end{aligned} \tag{3.30.2}$$

where $\xi = k(x+ly+\lambda t)$, $k = \pm\frac{\sqrt{(-\mu_2^2+\mu_2^2m^2+1)(\mu_2^2-1)}(a_0+\lambda)}{(2\mu_2^2m^2-2\mu_2^2-m^2+2)\mu_2}$, μ_2, l and λ are arbitrary constants.

Family 18.

$$u_{18} = a_0 - \frac{(\mu_1^2-1)(\lambda+a_0)\,\mathrm{sn}(\xi)}{\mu_1(\mu_1\,\mathrm{sn}(\xi)+1)} \pm \frac{(\mu_1^2-1)(\lambda+a_0)\,\mathrm{dn}(\xi)}{\sqrt{\mu_1^2-m^2}\mu_1(\mu_1\,\mathrm{sn}(\xi)+1)}, \tag{3.31.1}$$

$$\begin{aligned} v_{18} = {} & A_0 + \frac{l(\mu_1^2-1)(\lambda+a_0)^2(-2\mu_1^2+m^2+1)\,\mathrm{sn}(\xi)}{(m^2-\mu_1^2)\mu_1(\mu_1\,\mathrm{sn}(\xi)+1)} \pm \frac{l(\mu_1^2-1)(\lambda+a_0)^2\,\mathrm{dn}(\xi)}{\sqrt{\mu_1-m^2}\mu_1(\mu_1\,\mathrm{sn}(\xi)+1)} \\ & - \frac{(\mu_1^2-1)^2(\lambda+a_0)^2l\,\mathrm{sn}^2(\xi)}{\mu_1^2(\mu_1\,\mathrm{sn}(\xi)+1)^2} \pm \frac{(\mu_1^2-1)^2(\lambda+a_0)^2l\,\mathrm{sn}(\xi)\,\mathrm{dn}(\xi)}{\mu_1^2\sqrt{\mu_1^2-m^2}(\mu_1\,\mathrm{sn}(\xi)+1)^2}, \end{aligned} \tag{3.31.2}$$

where $\xi = k(x+ly+\lambda t)$, $k = \pm\frac{\sqrt{(\mu_1^2-m^2)(\mu_1^2-1)}(\lambda+a_0)}{(\mu_1^2-m^2)\mu_1}$, a_0, μ_1, l and λ are arbitrary constants.

Family 19.

$$u_{19} = a_0 - \frac{(m^2+3)(\lambda+a_0)\,\mathrm{sn}(\xi)}{\pm\mathrm{sn}(\xi)\pm\mathrm{dn}(\xi)+1}, \tag{3.32.1}$$

$$v_{19}=A_0-2\frac{(m^2A_0+m^2+8la_0\lambda-1+4la_0{}^2-A_0+4l\lambda^2)\,\mathrm{sn}(\xi)}{(m^2-1)(\pm\,\mathrm{sn}(\xi)\pm\mathrm{dn}(\xi)+1)}+\frac{B_1\,\mathrm{dn}(\xi)}{\pm\,\mathrm{sn}(\xi)\pm\mathrm{dn}(\xi)+1}$$
$$+\frac{A_2\,\mathrm{sn}^2(\xi)}{(\pm\,\mathrm{sn}(\xi)\pm\mathrm{dn}(\xi)+1)^2}+\frac{B_2\,\mathrm{sn}(\xi)\,\mathrm{dn}(\xi)}{(\pm\,\mathrm{sn}(\xi)\pm\mathrm{dn}(\xi)+1)^2}, \tag{3.32.2}$$

where $\xi=k(x+ly+\lambda t)$, $k=\pm\frac{\sqrt{(m^2-1)(m^2+3)(\lambda+a_0)}}{m^2-1}$, $B_1=\pm B_2$,

$$B_2=\pm\frac{m^4la_0^2+2m^4la_0\lambda+m^4l\lambda^2+4lm^2a_0\lambda+2lm^2a_0^2+2m^2+2m^2A_0+2l\lambda^2m^2+10la_0\lambda+5l\lambda^2-2+5la_0^2-2A_0}{(m-1)(m+1)},$$

$A_2=-2m^4la_0\lambda-m^4l\lambda^2-m^4la_0^2-8lm^2a_0\lambda-4lm^2a_0^2-m^2-m^2A_0-4l\lambda^2m^2-14la_0\lambda-7l\lambda^2+1-7la_0^2+A_0$, a_0, A_0, l and λ are arbitrary constants.

3.4. sdξ *and* ndξ *rational expansion*

Now we consider the ansatz (2.4.4). For Eq. (3.3), the ansatz (2.4.4) becomes

$$\begin{cases}\phi(\xi)=a_0+a_1\frac{\mathrm{sd}(\xi)}{\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1}+b_1\frac{\mathrm{nd}(\xi)}{\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1},\\ \sigma(\xi)=A_0+A_1\frac{\mathrm{sd}(\xi)}{\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1}+B_1\frac{\mathrm{nd}(\xi)}{\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1}\\ \qquad+A_2\frac{\mathrm{sd}^2(\xi)}{(\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1)^2}+B_2\frac{\mathrm{sd}(\xi)\,\mathrm{nd}(\xi)}{(\mu_1\,\mathrm{sd}(\xi)+\mu_2\,\mathrm{nd}(\xi)+1)^2},\end{cases} \tag{3.33}$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 are constants to be determined later. Following the same steps in Section 3.1, we can obtain the following sdξ and ndξ rational formal periodic solution:

Family 20.

$$u_{20}=-\lambda+b_1\,\mathrm{nd}\left(\pm\frac{b_1(x+ly+\lambda t)}{2\sqrt{m^2-1}}\right), \tag{3.34.1}$$

$$v_{20}=-\frac{4m^2+lb_1^2m^2-4}{4(m^2-1)}-\frac{1}{2}lb_1^2m^2\,\mathrm{sd}^2\left(\pm\frac{b_1(x+ly+\lambda t)}{2\sqrt{m^2-1}}\right), \tag{3.34.2}$$

where b_1, l and λ are arbitrary constants.

Family 21.

$$u_{21}=-\lambda+a_1\,\mathrm{sd}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{m^2-1}m}\right), \tag{3.35.1}$$

$$v_{21}=-\frac{4m^4+2la_1^2m^2-4m^2-la_1^2}{4m^2(m^2-1)}-\frac{1}{2}la_1^2\,\mathrm{sd}^2\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{m^2-1}m}\right), \tag{3.35.2}$$

where a_1, l and λ are arbitrary constants.

Family 22.

$$u_{22}=-\lambda+a_1\,\mathrm{sd}\left(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^2-1}m}\right)-\frac{a_1}{m}\pm\mathrm{nd}\left(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^2-1}m}\right), \tag{3.36.1}$$

$$v_{22}=-\frac{m^2+la_1^2-1}{m^2-1}-la_1^2\,\mathrm{sd}^2\left(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^2-1}m}\right)+\frac{la_1^2}{m}\,\mathrm{nd}\left(\frac{a_1(x+ly+\lambda t)}{\sqrt{m^2-1}m}\right)\mathrm{sd}\left(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^2-1}m}\right), \tag{3.36.2}$$

where a_1, l and λ are arbitrary constants.

Family 23.

$$v_{23}=\frac{\pm(k\mu_2m^2-2k\mu_2+2\mu_2^3k)-\lambda\sqrt{-\mu_2^4-\mu_2^2m^2+2\mu_2^2+m^2-1}}{\sqrt{-\mu_2^4-\mu_2^2m^2+2\mu_2^2+m^2-1}}\pm 2\frac{\sqrt{-\mu_2^4-\mu_2^2m^2+2\mu_2^2+m^2-1}k\,\mathrm{nd}(k(x+ly+\lambda t))}{1+\mu_2\,\mathrm{nd}(k(x+ly+\lambda t))}, \tag{3.37.1}$$

$$w_{23}=\frac{\mu_2^4+\mu_2^4lk^2m^2-2lk^2\mu_2^2m^2-2\mu_2^2+\mu_2^2m^2+2lk^2\mu_2^2m^4+1-lk^2m^4+lk^2m^2-m^2}{m^2-\mu_2^4-\mu_2^2m^2+2\mu_2^2-1}+2\frac{lk^2\mu_2m^2\,\mathrm{nd}(k(x+ly+\lambda t))}{1+\mu_2\,\mathrm{nd}(k(x+ly+\lambda t))}+\frac{(2lk^2m^2-2lk^2\mu_2^2m^2-2lk^2m^4)\,\mathrm{sd}^2(k(x+ly+\lambda t))}{(1+\mu_2\,\mathrm{nd}(k(x+ly+\lambda t)))^2}, \tag{3.37.2}$$

where k, μ_2, l and λ are arbitrary constants.

Family 24.

$$u_{24}=\frac{\pm(k\mu_1m^2+\mu_1^3k)-\lambda\sqrt{m^4+2m^2\mu_1^2-m^2+\mu_1^4-\mu_1^2}}{\sqrt{m^4+2m^2\mu_1^2-m^2+\mu_1^4-\mu_1^2}}\pm\frac{\sqrt{m^4+2m^2\mu_1^2-m^2+\mu_1^4-\mu_1^2}k\,\mathrm{sd}(k(x+ly+\lambda t))}{\mu_1\,\mathrm{sd}(k(x+ly+\lambda t))+1}\pm\frac{\sqrt{m^2+\mu_1^2-1}k\,\mathrm{nd}(k(x+ly+\lambda t))}{\mu_1\,\mathrm{sd}(k(x+ly+\lambda t))+1}, \tag{3.38.1}$$

$$\begin{aligned}v_{24}={}&\frac{lk^2m^4+m^2+2lk^2\mu_1^2m^2-lk^2m^2+\mu_1^2-1-2lk^2\mu_1^2+lk^2\mu_1^4}{1-m^2-\mu_1^2}\\&+\frac{(2lk^2\mu_1m^2+2lk^2\mu_1^3-lk^2\mu_1)\,\mathrm{sd}(k(x+ly+\lambda t))}{\mu_1\,\mathrm{sd}(k(x+ly+\lambda t))+1}\\&\pm\frac{\sqrt{m^2+\mu_1^2-1}k^2l\mu_1(\mu_1^2+m^2)\,\mathrm{nd}(k(x+ly+\lambda t))}{\sqrt{m^4+2m^2\mu_1^2-m^2+\mu_1^4-\mu_1^2}(\mu_1\,\mathrm{sd}(k(x+ly+\lambda t))+1)}\\&+\frac{(-lk^2m^4-2lk^2\mu_1^2m^2+lk^2m^2-lk^2\mu_1^4+lk^2\mu_1^2)(\mathrm{sd}(k(x+ly+\lambda t)))^2}{(\mu_1\,\mathrm{sd}(k(x+ly+\lambda t))+1)^2}\\&\pm\frac{l\sqrt{m^4+2m^2\mu_1^2-m^2+\mu_1^4-\mu_1^2}k^2\sqrt{m^2+\mu_1^2-1}\,\mathrm{nd}(k(x+ly+\lambda t))\,\mathrm{sd}(k(x+ly+\lambda t))}{(\mu_1\,\mathrm{sd}(k(x+ly+\lambda t))+1)^2},\end{aligned} \tag{3.38.2}$$

where k, μ_1, l and λ are arbitrary constants.

Family 25.

$$v_{25}=-\frac{\lambda m^2+a_1+3\lambda}{m^2+3}+\frac{a_1\,\mathrm{sd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}})}{\pm\,\mathrm{sd}(-\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}})\pm\mathrm{nd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}})+1}, \tag{3.39.1}$$

$$w_{25} = \frac{2m^6 + B_1m^6 + 5B_1m^4 + la_1^2m^4 + 10m^4 + 6m^2 + 3B_1m^2 + 2la_1^2m^2 + 5la_1^2 - 18 - 9B_1}{2(3 - m^4 - 2m^2)(m^2 + 3)}$$
$$+ \frac{(B_1m^2 + 3B_1 + la_1^2)\,\mathrm{sd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}})}{(m^2+3)(\pm\,\mathrm{sd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}}) \pm \mathrm{nd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}}) + 1)}$$
$$+ \frac{B_1\,\mathrm{nd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}})}{\pm\,\mathrm{sd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}}) \pm \mathrm{nd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}}) + 1}$$
$$- \frac{1}{2}\frac{(B_1m^2 + B_1 + la_1^2)\,\mathrm{sd}^2(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}})}{(\pm\,\mathrm{sd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}}) \pm \mathrm{nd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}}) + 1)^2}$$
$$- \frac{B_1\,\mathrm{nd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}})\,\mathrm{sd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}})}{(\pm\,\mathrm{sd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}}) \pm \mathrm{nd}(\pm\frac{a_1(x+ly+\lambda t)}{\sqrt{m^4+2m^2-3}}) + 1)^2}, \tag{3.39.2}$$

where k, a_1, B_1, l and λ are arbitrary constants.

Remark.

(1) Family 1 reproduce the solutions (15) in [25], when $a_1 = \pm 2m\sqrt{\frac{2C_1+2+l\lambda^2}{2l(1+m^2)}}$ and $\lambda = -\lambda^*$. Family 2 reproduce the solutions (16) in [25], when $b_1 = \pm 2mi\sqrt{\frac{2C_1+2+l\lambda^2}{2l(1-2m^2)}}$ and $\lambda = -\lambda^*$. Family 3 reproduce the solutions (15) in [25], when $a_1 = \pm mk$ and $\lambda = -\lambda^*$.

(2) Family 8 reproduce the solutions (21) in [25], when $b_1 = \pm 2\sqrt{\frac{2C_1+2+l\lambda^2}{2l(-2+m^2)}}$ and $\lambda = -\lambda^*$. Family 9 reproduce the solutions (20) in [25], when $a_1 = \pm 2\sqrt{\frac{2C_1+2+l\lambda^2}{2l(1+m^2)}}$ and $\lambda = -\lambda^*$. Family 10 reproduce the solutions (20) in [25], when $a_1 = \pm k$ and $\lambda = -\lambda^*$.

(3) Family 14 reproduce the solutions (18) in [25], when $b_1 = \pm 2i\sqrt{\frac{2C_1+2+l\lambda^2}{2l(-2+m^2)}}$ and $\lambda = -\lambda^*$. Family 15 reproduce the solutions (19) in [25], when $a_1 = \pm k^*m$ and $\lambda = -\lambda^*$.

(4) Family 20 reproduce the solutions (29) in [25], when $b_1 = \pm 2\sqrt{m^2-1}\sqrt{\frac{2C_1+2+l\lambda^2}{2l(-2+m^2)}}$ and $\lambda = -\lambda^*$. Family 21 reproduce the solutions (28) in [25], when $a_1 = \pm 2m\sqrt{m^2-1}\sqrt{\frac{2C_1+2+l\lambda^2}{2l(1-2m^2)}}$ and $\lambda = -\lambda^*$. Family 22 reproduce the solutions (28) in [25], when $a_1 = \pm km\sqrt{m^2-1}$ and $\lambda = -\lambda^*$.

(5) The other solutions obtained here, to our knowledge, are all new families of rational formal periodic solution of the DLWE equation.

4. Summary and conclusions

In this Letter, we have presented the new extended Jacobi elliptic function rational expansion method. The method is more powerful than the method proposed recently by Liu et al. [10], Fan et al. [11], Yan [12] and Wang et al. [13]. The $(2+1)$-dimensional dispersive long wave equation (DLWE) is chosen to illustrate the method such that new rational formal Jacobi elliptic function solutions are obtained. When the modulus $m \to 1$, some of these obtained solutions degenerate as solitary wave solutions. The algorithm can be also applied to many nonlinear differential equations in mathematical physics. Further work about various extensions and improvement of Jacobi function method need us to find the more general ansätze or the more general subequations.

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