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Chaos, Solitons and Fractals 23 (2005) 1465–1473

CHAOS
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# The stochastic soliton-like solutions of stochastic KdV equations

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Accepted 21 June 2004

Communicated by Prof. M. Wadati

**Abstract**

By means of a generalized method and symbolic computation, we consider a stochastic KdV equation $U_t + f(t)U \Diamond U_x + g(t)U_{xxx} = W(t) \Diamond R^{\Diamond}(t, U, U_x, U_{xxx})$. We construct new and more general formal solutions. At the same time, we recover all the solutions found by Xie [Phys. Lett. A 310 (2003) 161]. The solutions obtained include the non-travelling wave and coefficient function's stochastic soliton-like solutions, singular stochastic soliton-like solutions, stochastic triangular functions solutions.


## 1. Introduction

As is well known, the motion of long, unidirectional, weakly nonlinear water waves on a channel can be described by the Korteweg–de Vries (KdV) equation. When the surface of the fluid is submitted to a nonconstant pressure, or when the bottom of the layer is not flat, a forcing term has to be added to the equation. This term is given by the gradient of the exterior pressure or of the function whose graph defines the bottom. We are interested in the case when the forcing term is random, which is a very natural approach if it is assumed that the exterior pressure is generated by a turbulent velocity field for instance. We also assume that this random force is of white noise type [1–5]. In 1983, for the first time M. Wadati answers an interesting question, ‘‘how does external noise affect the motion of solitons?’’. In [1], the KdV equation under Gaussian noise is studied and it is showed that a soliton under Gaussian noise satisfies a diffusion equation in transformed coordinates; the deformation of the soliton during the propagation is explicitly obtained; the phenomenon is designated as the diffusion of soliton. In 1990, a nonlinear partial differential equation which describes wave propagations in random media is presented by Wadati [2]. The stochastic equation (17) in [2] is useful for study of similar problem in hydrodynamics and plasmas physics. Recently, the stochastic KdV equation arises when modelling

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0960-0779/$ - see front matter 
doi:10.1016/j.chaos.2004.06.049

the propagation of weakly nonlinear waves in a noisy plasma [2–6]. The remarkable achievement (see [1–6] and references for detail) of the study of stochastic partial differential equation have been obtained, for example, de Bouard and Debussche [6] use function space similar to those introduced by Bourgain to prove well posedness results for the KdV equation in $L^2$-function and obtain the global existence of $L^2(R)$ solution when the covariance operator of the noise is Hilbert–Schmidt in $L^2(R)$. Holden et al. [7] gave while noise functional approach to research stochastic partial differential equations in Wick version. More recently, based on the theory in [7], using Hermite transform and the homogenous balance method, Xie studied Wick-type stochastic KdV equation and obtained stochastic soliton solution of this equation [8].

In this paper, we would like to further extended the method presented by Fan [9–11] and recently improved by Chen et al. [12–14], to find stochastic soliton-like solutions of a Wick-type stochastic KdV equation as the following form [8]:

$$U_t + f(t)U\diamondsuit U_x + g(t)U_{xxx} = W(t)\diamondsuit R^{\diamondsuit}(t, U, U_x, U_{xxx}) \tag{1.1}$$

which is the perturbation of the KdV equation with variable coefficients

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0 \tag{1.2}$$

by random force $W(t)\diamondsuit R^{\diamondsuit}(R, U, U_x, U_{xxx})$, where $f(t)$ and $g(t)$ are functions of $t$, $W(t)$ is Gaussian white noise, i.e., $W(t) = \dot{B}(t)$ and $B(t)$ is a Brownian motion, $R(u, u_x, u_{xxx}) = -\alpha uu_x - \beta u_{xxx}$ is a functional of $u$, $u_x$ and $u_{xxx}$ for some constants $\alpha$, $\beta$ and $R^{\diamondsuit}$ is the Wick version of the functional $R$. For more detail about the exchange between Wick-type stochastic equation and common partial differential equation, the reader is advised to see the remarkable achievement by Holden et al. [7] and the second section of Ref. [8] by Xie. As a result, we construct new and more general formal solutions for Eq. (1), which include the nontravelling wave and coefficient function's stochastic soliton-like solutions, singular stochastic soliton-like solutions, stochastic triangular functions solutions.

The rest of this paper is organized as follows. In Section 2, we establish a generalized method. In Section 3, we apply the generalized method to a stochastic KdV equation and obtain some exact analytical solutions for this model. A short summary and discussion are given in final.

## 2. Summary of the generalized method

In the following we would like to outline the main steps of our general method:

*Step 1.* For a given nonlinear partial differential equation (NPDE) system with some physical fields $u_i(x, y, t)$ $(i = 1, \ldots, n)$ in three variables $x, y, t$,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0. \tag{2.1}$$

We express the solutions of the NPDE by the new more general ansatz

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left\{a_{ij}\phi^j + b_{ij}\phi^{j-1}\sqrt{\sum_{\rho=0}^{4} h_\rho\phi^\rho}\right\}, \tag{2.2}$$

where $m_i$ is an integer to be determined by balance the highest-order derivative terms with the nonlinear terms in Eq. (2.1), the new variable $\phi = \phi(\xi)$ satisfying:

$$\phi' = \frac{d\phi}{d\xi} = \sqrt{\sum_{\rho=0}^{4} rh_\rho\phi^\rho}, \tag{2.3}$$

and $a_{i0} = a_{i0}(x, y, t)$, $a_{ij} = a_{ij}(x, y, t)$, $b_{ij} = b_{ij}(x, y, t)$, $(i = 1, 2, \ldots; j = 1, 2, \ldots, m_i)$ and $\xi = \xi(x, y, t)$ are all differentiable functions to be determined later. Here $h_0, h_1, h_2, h_3, h_4$ are constants.

*Step 2.* Substitute Eq. (2.2) into Eq. (2.1) along with Eq. (2.3) and then set all coefficients of $\phi^p(\sqrt{\sum_{\rho=0}^{4} h_\rho\phi^\rho})^q$ $(q = 0, 1;$ $p = 0, 1, 2, \ldots)$ to be zero to get an over-determined partial differential equations with respect to $a_{i0}, a_{ij}, b_{ij}$ $(i = 1, 2, \ldots; j = 1, 2, \ldots, m_i)$ and $\xi$.

*Step 3.* Solving the over-determined partial differential equations by use of *Maple*, we would end up with the explicit expressions for $a_{i0}, a_{ij}, b_{ij}$ $(i = 1, 2, \ldots; j = 1, 2, \ldots, m_i)$ and $\xi$ or the constrains among them.

*Step 4.* By using the results obtained in the above steps, we can derive a series of fundamental-like solutions such as polynomial-like, exponential-like, solitary-like wave, rational-like, triangular-like periodic, Jacobi and Weierstrass doubly-like periodic solutions. Because we are interested in solitary-like wave, Jacobi and Weierstrass doubly-like periodic solutions and tan-like and cot-like type solutions appearing in pairs with tanh-like and coth-like type solutions respectively, therefore polynomial-like, rational-like, triangular-like periodic solutions are omitted in this paper. By considering the different values of $h_0, h_1, h_2, h_3$ and $h_4$, Eq. (2.3) has many kinds of solitary-like wave, Jacobi and Weierstrass doubly-like periodic solutions which are listed as follows.

(i) Solitary-like wave solutions.

a. Bell shaped soliton-like solutions.

$$\phi = \sqrt{-\frac{h_2}{h_4}}\operatorname{sech}\left(\sqrt{h_2}\xi\right), \quad h_0 = h_1 = h_3 = 0,\ h_2 > 0,\ h_4 < 0, \tag{2.4}$$

$$\phi = -\frac{h_2}{h_3}\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right), \quad h_0 = h_1 = h_4 = 0,\ h_2 > 0. \tag{2.5}$$

b. Kink shaped soliton-like solutions.

$$\phi = \sqrt{-\frac{h_2}{2h_4}}\tanh\left(\sqrt{-\frac{h_2}{2}}\xi\right), \quad h_0 = \frac{h_2^2}{4h_4},\ h_1 = h_3 = 0,\ h_2 < 0,\ h_4 > 0. \tag{2.6}$$

c. Soliton-like solutions.

$$\phi = \frac{h_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{h_2}\xi\right)}{2\sqrt{h_2h_4}\tanh\left(\frac{1}{2}\sqrt{h_2}\xi\right) - h_3}, \quad h_0 = h_1 = 0, \quad h_2 > 0. \tag{2.7}$$

(ii) Jacobi and Weierstrass doubly-like periodic solutions.

$$\phi = \sqrt{\frac{-h_2m^2}{h_4(2m^2-1)}}\operatorname{cn}\left(\sqrt{\frac{h_2}{2m^2-1}}\xi\right), \quad h_4 < 0,\ h_2 > 0,\ h_0 = \frac{h_2^2m^2(1-m^2)}{h_4(2m^2-1)^2}, \tag{2.8}$$

$$\phi = \sqrt{\frac{-m^2}{h_4(2-m^2)}}\operatorname{dn}\left(\sqrt{\frac{h_2}{2-m^2}}\xi\right), \quad h_4 < 0,\ h_2 > 0,\ h_0 = \frac{h_2^2(1-m^2)}{h_4(2-m^2)^2}, \tag{2.9}$$

$$\phi = \sqrt{\frac{-h_2m^2}{h_4(m^2+1)}}\operatorname{sn}\left(\sqrt{-\frac{h_2}{m^2+1}}\xi\right), \quad h_4 > 0,\ h_2 < 0,\ h_0 = \frac{h_2^2m^2}{h_4(m^2+1)^2}, \tag{2.10}$$

where $m$ is a modulus.

$$\phi = \wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right), \quad h_2 = 0,\ h_3 > 0, \tag{2.11}$$

where $g_2 = -4\frac{h_1}{h_3}$ and $g_3 = -4\frac{h_0}{h_3}$ are called invariants of Weierstrass elliptic function. The Jacobi elliptic functions are doubly periodic and possess properties of triangular functions:

$$\operatorname{sn}^2\xi + \operatorname{cn}^2\xi = 1, \quad \operatorname{dn}^2\xi = 1 - m^2\operatorname{sn}^2\xi,$$
$$(\operatorname{sn}\xi)' = \operatorname{cn}\xi\operatorname{dn}\xi, \quad (\operatorname{cn}\xi)' = -\operatorname{sn}\xi, \quad (\operatorname{dn}\xi)' = -m^2\operatorname{sn}\xi\operatorname{cn}\xi.$$

When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.

$$\operatorname{sn}\xi \to \tanh\xi, \quad \operatorname{cn}\xi \to \operatorname{sech}\xi,$$

when $m \to 0$, the Jacobi functions degenerate to the triangular functions, i.e.

$$\operatorname{sn}\xi \to \sin\xi, \quad \operatorname{cn}\xi \to \cos\xi.$$

The more detailed notations for the Weierstrass and Jacobi elliptic functions can be found in Refs. [19,20].

**Remark 1**

1. *Generalization*: The method proposed here is more general than the method [9–11] by Fan, the method [12–14] improved by Chen et al. and the method [15] improved by E. Yomba. Firstly, compared with the method [9–11] and the improved method [12–14] the restriction on $\xi(x,y,t)$ as merely a linear function $x,y,t$ and the restriction on the coefficients $a_{i0}, a_{ij}, b_{ij}$ $(i = 1,2,\ldots;\ j = 1,2,\ldots,m_i)$ as constants are removed. Secondly, compared with the improved method [15] by E. Yomba, the Eq. (2.3) that the new variable $\phi = \phi(\xi)$ satisfies is more general. More importantly, we add terms $b_{ij}\phi^{j-1}\sqrt{\sum_{\rho=0}^{4}h_\rho\phi^\rho}$ in new ansatz (2.2), so more types of solutions would be expected for some equations.
2. *Feasibility*: For the generalization of the ansatz, naturally more complicated computation is expected than ever before. Even if the availability of computer symbolic systems like *Maple* or *Mathematica* allows us to perform the complicated and tedious algebraic calculation and differentiation on a computer, in general it is very difficult, sometime impossible, to solve the set of over-determined partial differential equations in (step 3). As the calculation goes on, in order to drastically simplify the work or make the work feasible, we often choose special function forms for $a_{i0}, a_{ij}, b_{ij}$ $(i = 1,2,\ldots;\ j = 1,2,\ldots,m_i)$ and $\xi$, on a trial-and-error basis.
3. *Further extendable*: In fact, We naturally present a more general a ansatz, which reads,

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left\{a_{ij}\phi^j + b_{ij}\phi^{-j} + f_{ij}\phi^{j-1}\sqrt{\sum_{\rho=0}^{r}h_\rho\phi^\rho} + k_{ij}\frac{\sqrt{\sum\limits_{\rho=0}^{r}h_\rho\phi^\rho}}{\varphi^j}\right\}, \tag{2.12}$$

where $a_{i0}, a_{ij}, b_{ij}, f_{ij}, k_{ij}$ $(i = 1,2,\ldots;\ j = 1,2,\ldots,m_i)$ and $\xi$ are differentiable function to be determined later. When $a_{i0}, a_{ij}, b_{ij}, f_{ij}, k_{ij} (i = 1,2,\ldots; j = 1,2,\ldots,m_i)$ are constants and $\xi$ is linear function with respect to $x, y$ and $t$ in the above ansatz, we have studied in Refs. [16–18]. Therefore, for some nonlinear equations, more types of solutions would be expected.

## 3. The stochastic soliton solutions of stochastic KdV equations

In this section, we will give exact solutions of Eq. (1.1) by the generalized method. Taking the Hermite transform of (1.1), we get the equation

$$\widetilde{U_t}(t,x,z) + \theta(t,z)\widetilde{U}(t,x,z)\widetilde{U}_x(t,x,z) + \tau(t,z)\widetilde{U}_{xxx}(t,x,z) = 0, \tag{3.1}$$

where $\theta(t,z) = [f(t) + \alpha\widetilde{W}(t,z)]$, $\tau(t,z) = [g(t) + \beta\widetilde{W}(t,z)]$, the Hermite transformation of $W(t)$ is defined by $\widetilde{W}(t,z) = \sum_{k=1}^{\infty}\eta_k(t)z_k$ when $z = (z_1, z_2, \ldots,) \in (C^N)_c$ is parameter. We first solve Eq. (3.1).

According to the generalized method, we suppose that the solutions of (3.1) are the form

$$\begin{aligned}\widetilde{U}(t,x,z) = {} & a_0 + a_1\phi(\xi) + b_1\sqrt{h_0 + h_1\phi(\xi) + h_2\phi^2(\xi) + h_3\phi^3(\xi) + h_4\phi^4(\xi)} + a_2\phi^2(\xi) + b_2\phi(\xi)\\ & \times\sqrt{h_0 + h_1\phi(\xi) + h_2\phi^2(\xi) + h_3\phi^3(\xi) + h_4\phi^4(\xi)},\end{aligned} \tag{3.2}$$

where $a_0 = a_0(t,z), a_1 = a_1(t,z), a_2 = a_2(t,z), b_1 = b_1(t,z),\ b_2 = b_2(t,z), \xi = xp(t,z) + q(t,z)$ and $\phi(\xi)$ satisfy (2.3).

Then substituting (3.2) and (2.3) into (3.1), collecting coefficients of monomials of $\phi(\xi)$, $\sqrt{\sum_{\rho=0}^{4}h_\rho\phi^\rho}$ and $x$ of the resulting system's numerator (Notice that $\theta(t,z)$, $\tau(t,z)$, $a_0(t,z)$, $a_1(t,z)$, $a_2(t,z)$, $b_1(t,z)$, $b_2(t,z)$, $p(t,z)$, $q(t,z)$ are all independent of $x$), then setting each coefficients to zero, we obtain the following over-determined PDEs system.

$$\begin{aligned}& 8\theta(t,z)p(t,z)a_0(t,z)b_2(t,z)h_2 + 60h_4\tau(t,z)(p(t,z))^3b_1(t,z)h_1 + 10\theta(t,z)p(t,z)a_1(t,z)b_2(t,z)h_1\\ & \quad + 144h_4\tau(t,z)(p(t,z))^3b_2(t,z)h_0 + 6h_3b_1(t,z)\frac{\partial}{\partial t}q(t,z) + 8\theta(t,z)p(t,z)b_1(t,z)a_1(t,z)h_2 + 32\tau(t,z)(p(t,z))^3b_2(t,z)h_2^2\\ & \quad + 10\theta(t,z)p(t,z)b_1(t,z)a_2(t,z)h_1 + 30\tau(t,z)(p(t,z))^3b_1(t,z)h_2h_3 + 8b_2(t,z)\left(\frac{\partial}{\partial t}q(t,z)\right)h_2\\ & \quad + 12\theta(t,z)p(t,z)a_2(t,z)b_2(t,z)h_0 + 4\frac{\partial}{\partial t}a_2(t,z) + 84\tau(t,z)(p(t,z))^3b_2(t,z)h_1h_3 + 6h_3\theta(t,z)p(t,z)a_0(t,z)b_1(t,z) = 0,\end{aligned} \tag{3.3}$$

$$12h_4b_2(t,z)\frac{\partial}{\partial t}p(t,z)=0, \tag{3.4}$$

$$2\left(\frac{\partial}{\partial t}p(t,z)\right)(2h_0b_2(t,z)+b_1(t,z)h_1)=0, \tag{3.5}$$

$$\begin{aligned}&4\theta(t,z)p(t,z)b_1(t,z)b_2(t,z)h_0+4\theta(t,z)p(t,z)a_0(t,z)a_1(t,z)+4\tau(t,z)(p(t,z))^3a_1(t,z)h_2+2\theta(t,z)p(t,z)(b_1(t,z))^2h_1\\&\quad+12\tau(t,z)(p(t,z))^3a_2(t,z)h_1+4\frac{\partial}{\partial t}b_1(t,z)+4a_1(t,z)\frac{\partial}{\partial t}q(t,z)=0,\end{aligned} \tag{3.6}$$

$$\begin{aligned}&180h_4\tau(t,z)(p(t,z))^3b_2(t,z)h_1+8h_4\theta(t,z)p(t,z)a_0(t,z)b_1(t,z)+12\theta(t,z)p(t,z)b_1(t,z)a_2(t,z)h_2\\&\quad+14\theta(t,z)p(t,z)a_2(t,z)b_2(t,z)h_1+10h_3\theta(t,z)p(t,z)a_0(t,z)b_2(t,z)+10h_3\theta(t,z)p(t,z)b_1(t,z)a_1(t,z)\\&\quad+80h_4\tau(t,z)(p(t,z))^3b_1(t,z)h_2+8h_4b_1(t,z)\frac{\partial}{\partial t}q(t,z)+130\tau(t,z)(p(t,z))^3b_2(t,z)h_2h_3\\&\quad+12\theta(t,z)p(t,z)a_1(t,z)b_2(t,z)h_2+10h_3b_2(t,z)\frac{\partial}{\partial t}q(t,z)+30\tau(t,z)(p(t,z))^3b_1(t,z)h_3^2=0,\end{aligned} \tag{3.7}$$

$$2\left(\frac{\partial}{\partial t}p(t,z)\right)4h_2b_2(t,z)+3h_3b_1(t,z)=0, \tag{3.8}$$

$$2\left(\frac{\partial}{\partial t}p(t,z)\right)(2b_1(t,z)h_2+3h_1b_2(t,z))=0, \tag{3.9}$$

$$\begin{aligned}&30\tau(t,z)(p(t,z))^3b_2(t,z)h_1h_2+4\theta(t,z)p(t,z)a_0(t,z)b_1(t,z)h_2+48\tau(t,z)(p(t,z))^3b_1(t,z)h_0h_4\\&\quad+60\tau(t,z)(p(t,z))^3b_2(t,z)h_3h_0+6\theta(t,z)p(t,z)a_0(t,z)b_2(t,z)h_1+18\tau(t,z)(p(t,z))^3b_1(t,z)h_1h_3\\&\quad+8\theta(t,z)p(t,z)b_2(t,z)a_1(t,z)h_0+8\theta(t,z)p(t,z)b_1(t,z)a_2(t,z)h_0+4\frac{\partial}{\partial t}a_1(t,z)+4\tau(t,z)(p(t,z))^3b_1(t,z)h_2^2\\&\quad+4b_1(t,z)\left(\frac{\partial}{\partial t}q(t,z)\right)h_2+6\theta(t,z)p(t,z)b_1(t,z)a_1(t,z)h_1+6b_2(t,z)h_1\frac{\partial}{\partial t}q(t,z)=0,\end{aligned} \tag{3.10}$$

$$4a_1(t,z)\frac{\partial}{\partial t}p(t,z)=0, \tag{3.11}$$

$$\begin{aligned}&8p(t,z)(12h_4a_2(t,z)\tau(t,z)(p(t,z))^2+\theta(t,z)(a_2(t,z))^2+\theta(t,z)(b_1(t,z))^2h_4)+8p(t,z)(2h_3\theta(t,z)b_1(t,z)b_2(t,z)\\&\quad+\theta(t,z)(b_2(t,z))^2h_2)=0,\end{aligned} \tag{3.12}$$

$$\begin{aligned}&6p(t,z)(4(p(t,z))^2h_4\tau(t,z)a_1(t,z)+10(p(t,z))^2h_3\tau(t,z)a_2(t,z))+6p(t,z)(\theta(t,z)(b_2(t,z))^2h_1\\&\quad+2\theta(t,z)a_1(t,z)a_2(t,z))+6p(t,z)(2\theta(t,z)b_1(t,z)b_2(t,z)h_2+h_3\theta(t,z)(b_1(t,z))^2)=0,\end{aligned} \tag{3.13}$$

$$12\theta(t,z)p(t,z)(b_2(t,z))^2h_4=0, \tag{3.14}$$

$$20h_4p(t,z)b_2(t,z)(12(p(t,z))^2h_4\tau(t,z)+a_2(t,z)\theta(t,z))=0, \tag{3.15}$$

$$10\theta(t,z)p(t,z)b_2(t,z)(h_3b_2(t,z)+2h_4b_1(t,z))=0, \tag{3.16}$$

$$8a_2(t,z)\frac{\partial}{\partial t}p(t,z)=0, \tag{3.17}$$

$$\begin{aligned}&4\frac{\partial}{\partial t}a_0(t,z)+4\theta(t,z)p(t,z)a_1(t,z)b_1(t,z)h_0+12\tau(t,z)(p(t,z))^3b_1(t,z)h_3h_0+3\tau(t,z)(p(t,z))^3b_2(t,z)h_1^2\\&\quad+4\theta(t,z)p(t,z)a_0(t,z)b_2(t,z)h_0+2b_1(t,z)\left(\frac{\partial}{\partial t}q(t,z)\right)h_1+2\theta(t,z)p(t,z)a_0(t,z)b_1(t,z)h_1\\&\quad+2\tau(t,z)(p(t,z))^3b_1(t,z)h_1h_2+4b_2(t,z)\left(\frac{\partial}{\partial t}q(t,z)\right)h_0+16\tau(t,z)(p(t,z))^3b_2(t,z)h_2h_0=0,\end{aligned} \tag{3.18}$$

$$4\theta(t,z)p(t,z)(b_2(t,z))^2h_0 + 8\theta(t,z)p(t,z)b_1(t,z)b_2(t,z)h_1 + 4\theta(t,z)p(t,z)(a_1(t,z))^2 + 8a_2(t,z)\frac{\partial}{\partial t}q(t,z)$$
$$+ 8\theta(t,z)p(t,z)a_0(t,z)a_2(t,z) + 12h_3\tau(t,z)(p(t,z))^3a_1(t,z) + 4\frac{\partial}{\partial t}b_2(t,z) + 32\tau(t,z)(p(t,z))^3a_2(t,z)h_2$$
$$+ 4\theta(t,z)p(t,z)(b_1(t,z))^2h_2 = 0, \tag{3.19}$$

$$2\left(\frac{\partial}{\partial t}p(t,z)\right)(4h_4b_1(t,z) + 5h_3b_2(t,z)) = 0, \tag{3.20}$$

$$14h_3\theta(t,z)p(t,z)b_1(t,z)a_2(t,z) + 16\theta(t,z)p(t,z)a_2(t,z)b_2(t,z)h_2 + 12h_4\theta(t,z)p(t,z)b_1(t,z)a_1(t,z)$$
$$+ 12h_4\theta(t,z)p(t,z)a_0(t,z)b_2(t,z) + 105\tau(t,z)(p(t,z))^3b_2(t,z)h_3^2 + 120h_4\tau(t,z)(p(t,z))^3b_1(t,z)h_3$$
$$+ 12h_4b_2(t,z)\frac{\partial}{\partial t}q(t,z) + 240h_4\tau(t,z)(p(t,z))^3b_2(t,z)h_2 + 14h_3\theta(t,z)p(t,z)a_1(t,z)b_2(t,z) = 0, \tag{3.21}$$

$$2p(t,z)(48p^2(t,z)\tau(t,z)b_1(t,z)h_4^2 + 168p^2(t,z)h_4\tau(t,z)b_2(t,z)h_3) + 2p(t,z)\theta(t,z)(8h_4b_1(t,z)a_2(t,z)$$
$$+ a_1(t,z)b_2(t,z) + 9h_3a_2(t,z)b_2(t,z)) = 0, \tag{3.22}$$

Solving the system of PDEs. (3.3)–(3.22) by *Maple*, we obtain the following two solutions.

$$p = F_1(z), \quad \theta = \theta, \quad a_1 = F_2(z), \quad a_2 = 2F_2(z)\frac{h_4}{h3}, \quad b_2 = 0,$$
$$a_0 = F_3(z)q = -\frac{3F_1(z)\int\theta(t,z)\mathrm{d}th_3F_3(z) - F_1(z)\int\theta(t,z)\mathrm{d}th_2F_2(z) - 3F_5(z)h_3}{3h_3}, \quad \tau = -\frac{1}{3}\frac{\theta(t,z)F_2(z)}{h_3F_1^2(z)},$$
$$b_1 = \pm 2\frac{\sqrt{h_4}F_2(z)}{h_3}, \tag{3.23}$$

where $F_1(z), \ldots, F_3(z)$ and $\theta(t,z)$ are all arbitrary functions.

Thus from (3.2) and (3.23), we obtain two families of exact solutions of Eq. (3.1) as follows. For simplicity, we omit polynomial-like, rational-like, triangular-like periodic solutions in this paper.

*Family 1.* When $h_0 = h_1 = h_4 = 0,\ h_2 > 0$,

$$\widetilde{U}_1(t,x,z) = F_3(z) - \frac{F_2(z)h_2\,\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{h_3}, \tag{3.24}$$

where $\xi = xF_1(z) + q(t,z),\ q(t,z),\ \theta(t,z)$ and $\tau(t,z)$ are determined by (3.23).

*Family 2.* When $h_0 = h_1 = 0,\ h_2 > 0$

$$\widetilde{U}_2(x,t,z) = F_3(z) + \frac{F_2(z)h_2\,\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3} + 2\frac{F_2(z)h_4h_2^2\,\mathrm{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)}{h_3(2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3)^2} \pm 2$$
$$\times\frac{h_2^{\frac{5}{2}}\sqrt{h_4}}{h_3}F_2(z)\left(\frac{\mathrm{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)}{(2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3)^2} + \frac{\mathrm{sech}^6\left(\frac{\sqrt{h_2}}{2}\xi\right)\sqrt{h_2h_4}}{\left(2\sqrt{h_2h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)^3}\right), \tag{3.25}$$

where $\xi = xF_1(z) + q(t,z),\ q(t,z),\ \theta(t,z)$ and $\phi(t,z)$ determined by (3.23).

By (3.24) and (3.25), the definition of $\widetilde{W}$, Theorem 2.1 in Ref. [7] and $\exp^{\diamond}\{B(t)\} = \exp\{B(t) - \frac{1}{2}t^2\}$ (see Lemma 2.6.16 in [6]), we have the following stochastic solitary solutions:

*Family 1.*

$$U_1(t,x) = F_3 - 4\frac{F_2h_2\exp^{\diamond}(\sqrt{h_2}(xF_1 + Q(t)))}{(\exp^{\diamond}(\sqrt{h_2}(xF_1 + Q(t))) + 1)^{\diamond 2}h_3}, \tag{3.26}$$

where

$$Q(t) = -\frac{(3F_1h_3F_3 - F_1h_2F_2)\left\{\int f(t)\,\mathrm{d}t + \alpha[B(t) - \frac{1}{2}t^2]\right\} - 3F_4h_3}{3h_3},$$

$F_1, F_2, F_3$ and $F_4$ are all arbitrary constants.

*Family 2.*

$$\begin{aligned} U_2(x,t) = F_3 &- \frac{4\exp^{\diamond}(\sqrt{h_2}\xi)F_2h_2}{(-2\sqrt{h_2h_4}\exp^{\diamond}(\sqrt{h_2}\xi) + 2\sqrt{h_2h_4} + h_3\exp^{\diamond}(\sqrt{h_2}\xi) + h_3)\diamond(\exp^{\diamond}(\sqrt{h_2}\xi) + 1)} \\ &+ \frac{32\exp^{\diamond}(2\sqrt{h_2}\xi)h_2^2h_4F_2}{(-2\sqrt{h_2h_4}\exp^{\diamond}(\sqrt{h_2}\xi) + 2\sqrt{h_2h_4} + h_3\exp^{\diamond}(\sqrt{h_2}\xi) + h_3)^{\diamond 2}\diamond(\exp^{\diamond}(\sqrt{h_2}\xi) + 1)^{\diamond 2}h_3} \\ &\pm \frac{32(\exp^{\diamond}(\sqrt{h_2}\xi) - 1)\diamond\exp^{\diamond}(2\sqrt{h_2}\xi)F_2\sqrt{h_4}h_2^{\frac{5}{2}}}{(-2\sqrt{h_2h_4}\exp^{\diamond}(\sqrt{h_2}\xi) + 2\sqrt{h_2h_4} + h_3\exp^{\diamond}(\sqrt{h_2}\xi) + h_3)^{\diamond 2}\diamond(\exp^{\diamond}(\sqrt{h_2}\xi) + 1)^{\diamond 3}h_3} \\ &\pm \frac{128\exp^{\diamond}(3\sqrt{h_2}\xi)F_2h_4h_2^3}{(-2\sqrt{h_2h_4}\exp^{\diamond}(\sqrt{h_2}\xi) + 2\sqrt{h_2h_4} + h_3\exp^{\diamond}(\sqrt{h_2}\xi) + h_3)^{\diamond 3}\diamond(\exp^{\diamond}(\sqrt{h_2}\xi) + 1)^{\diamond 3}h_3}, \end{aligned} \tag{3.27}$$

where $\xi = xF_1 + Q(t)$,

$$Q(t) = -\frac{(3F_1h_3F_3 - F_1h_2F_2)\left\{\int f(t)\,\mathrm{d}t + \alpha[B(t) - \frac{1}{2}t^2]\right\} - 3F_4h_3}{3h_3},$$

$F_1, F_2, F_3$ and $F_4$ are all arbitrary constants.

**Remark 2**

(1) Due to the arbitrariness of $F_1, F_2, F_3, F_4$ and $f(t)$, it is not difficult to verify that the solution 3.10,3.11 obtained in [8] can be reproduced by the solution (3.26) obtained by us. But, to our knowledge, the other solutions obtained were not reported before.

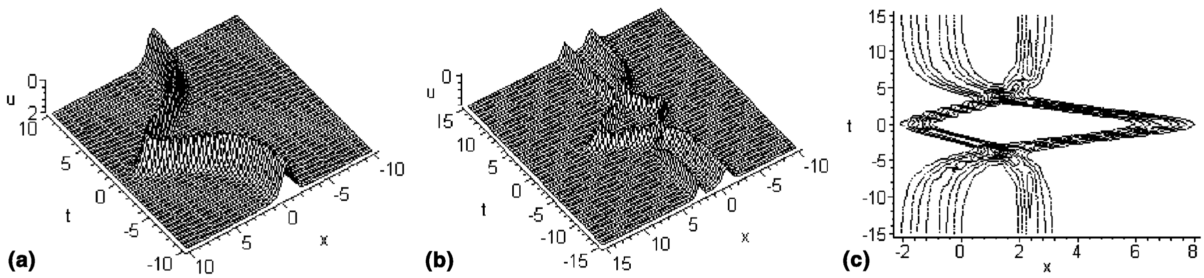


Fig. 1. (a) denotes a soliton-like solution scenario given by $U_1$, where $F_1 = F_2 = F_3 = F_4 = h_2 = h_3 = \alpha = 2, f(t) = -3\mathrm{sech}(\frac{1}{2}t)\tanh(\frac{1}{2}t), \beta(t) = \frac{1}{2}t^2$. (b) and (c) denote the interaction scenario of soliton-like solutions.

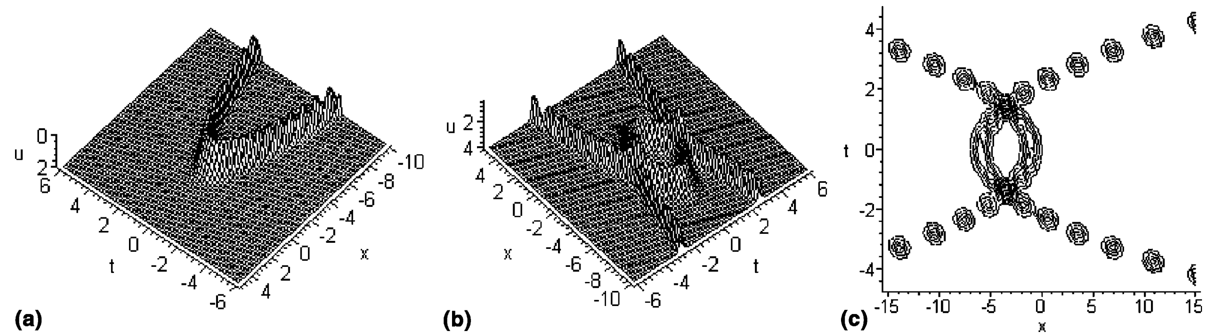


Fig. 2. (a) denotes the boomerang-like soliton solution scenario given by $U_1$, where $F_1 = F_2 = F_3 = F_4 = h_2 = h_3 = \alpha = 2$, $f(t) = 0$, $\beta(t) = \frac{1}{20}t^2$. (b) and (c) denote the interaction scenario of boomerang-like soliton solutions.

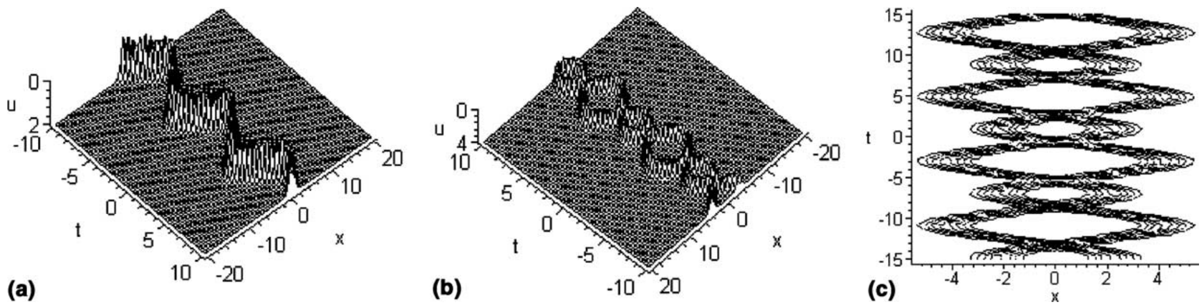


Fig. 3. (a) denotes the snake-like soliton solution scenario given by $U_1$, where $F_1 = F_2 = F_3 = F_4 = h_2 = h_3 = \alpha = 2, f(t) = 2\sin(\frac{4}{5}(t+3))$, $\beta(t) = \frac{1}{2}t^2$. (b) and (c) denote the interaction scenario of snake-like soliton solutions.

(2) The more general soliton-like solutions obtained by the generalized method contain the some arbitrary differentiable functions and some arbitrary constants, which may enable one to discuss the behavior of solutions as a function of these arbitrary differentiable functions and some arbitrary constants and this also provide enough freedom to build up solutions that may correspond to a particular physical situation, or initial condition have some desired features, which means a great variation in the solutions. In order to understand the significance of these soliton-like solutions obtained in the paper, by choosing the special functions, we find some very interesting special solutions (3.26), which including soliton-like solutions, snake-like soliton and Boomerang-like soliton, and choose the solutions (3.26) to be figured. Their interaction scenario also were shown in fellow figures (Figs. 1–3).

## 4. Summary and discussion

In summary, based on the generalized method and symbolic computation, by means of the theory of stochastic partial differential equations in Wick version [7], we study stochastic KdV equation $U_t + f(t)U\diamondsuit U_x + g(t)U_{xxx} = W(t)\diamondsuit R^{\diamondsuit}(t, U, U_x, U_{xxx})$. Some new and more general formal solutions for Eq. (1.1) are constructed, which include the nontravelling wave and coefficient function's stochastic soliton-like solutions, singular stochastic soliton-like solutions, stochastic triangular functions solutions, at the same time, the results in [7] are recovered. By choosing the special functions in the new and more general formal solution obtained, we would find some very interesting special solutions, for instance, from solution (3.26), soliton-like soliton solutions, snake-like soliton solutions, boomerage-like soliton solutions are fund, and their scenario and their interaction scenario are shown by the figures.

## Acknowledgments

We would express our sincere thanks for Prof. Wadati for his valuable suggestion and kind help. The work was supported by the National Outstanding Youth Foundation of China (no. 19925522).

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