Available online at www.sciencedirect.com
SCIENCE DIRECT®
Chaos, Solitons and Fractals 23 (2005) 477–483
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

A new Jacobi elliptic function rational expansion method and its application to (1 + 1)-dimensional dispersive long wave equation

Qi Wang [a,d], Yong Chen [b,c,d,*], Zhang Hongqing [a,d]

[a] Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China
[b] Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, China
[c] Department of Mathematics, Ningbo University, Ningbo 315211, China
[d] MM Key Lab, Chinese Academy of Sciences, Beijing 100080, China

Received 16 April 2004

Communicated by Prof. M. Wadati

Abstract

With the aid of computerized symbolic computation, a new elliptic function rational expansion method is presented by means of a new general ansätz and is very powerful to uniformly construct more new exact doubly-periodic solutions in terms of rational formal Jacobi elliptic function of nonlinear evolution equations (NLEEs). As an application of the method, we choose a (1 + 1)-dimensional dispersive long wave equation to illustrate the method. As a result, we can successfully obtain the solutions found by most existing Jacobi elliptic function methods and find other new and more general solutions at the same time. Of course, more shock wave solutions or solitary wave solutions can be gotten at their limit condition.

1. Introduction

In recent years, the nonlinear partial differential equations (NPDEs) are widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, biology, etc. With the development of soliton theory, There has been a great amount of activities aiming to find methods for exact solution of nonlinear differential equations, such as Bäcklund transformation, Darboux transformation, Cole–Hopf transformation, varied tanh methods, varied Jacobi elliptic function methods, variable separation approach, Painlevé method, homogeneous balance method, similarity reduction method and so on [1–12].

Among those, the direct ansätz method [6–12] provides a straightforward and effective algorithm to obtain such particular solutions for a large number of nonlinear equations, in which the starting point is the ansätz that the solution sought is expressible as a finite series of special function, such as tanh function, sech function, tan function, sec function, sine–cosine function, Weierstrass elliptic function, theta function and Jacobian elliptic function [13,14].

In this paper a new Jacobi elliptic function rational expansion method is presented by means of a new general ansätz and is more powerful than above exiting Jacobi elliptic function methods [10–12] to uniformly construct more new exact

* Corresponding author. Address: Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, China.
E-mail address: chenyong@dlut.edu.cn (Y. Chen).

0960-0779/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.chaos.2004.04.029

doubly-periodic solutions in terms of rational formal elliptic function of nonlinear evolution equations (NLEEs). The algorithm and its applications are demonstrated later.

This paper is organized as follows. In Section 2, we summarize the elliptic function rational expansion method. In Section 3, we apply the generalized method to (1 + 1)-dimensional dispersive long wave equation and bring out many solutions. Conclusions will be presented in finally.

2. Summary of the Jacobi elliptic function rational expansion method

In the following we would like to outline the main steps of our general method:

Step 1. For a given nonlinear partial differential equation(NPDE) system with some physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0, \tag{2.1}$$

by using the wave transformation

$$u_i(x,y,t) = u_i(\xi), \quad \xi = x + ly - \lambda t, \tag{2.2}$$

where l and λ are constants to be determined later. Then the nonlinear partial differential Eq. (2.1) is reduced to a nonlinear ordinary differential equation(ODE):

$$G_i(u_i, u_i', u_i'', \ldots) = 0. \tag{2.3}$$

Step 2. We introduce a new ansätz in terms of finite rational formal elliptic function expansion in the following forms:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \left(a_{ij} \frac{\mathrm{sn}^j(\xi)}{(\mu \mathrm{sn}(\xi) + 1)^j} + b_{ij} \frac{\mathrm{sn}^{j-1}(\xi)\mathrm{cn}(\xi)}{(\mu \mathrm{sn}(\xi) + 1)^j} \right). \tag{2.4}$$

Notice that

$$\begin{aligned}\frac{\mathrm{d}u_i}{\mathrm{d}\xi} &= \sum_{j=1}^{m_i} \frac{\mathrm{dn}(\xi)(a_{ij}j(\mathrm{sn}(\xi))^{j-1}\mathrm{cn}(\xi) - b_{ij}(\mathrm{sn}(\xi))^{j-2} - b_{ij}\mu(\mathrm{sn}(\xi))^{j-1})}{(\mu \mathrm{sn}(\xi) + 1)^{j+1}} \\ &\quad + \sum_{j=1}^{m_i} \frac{\mathrm{dn}(\xi)(b_{ij}j(\mathrm{sn}(\xi))^{j-2} - b_{ij}j(\mathrm{sn}(\xi))^{j})}{(\mu \mathrm{sn}(\xi) + 1)^{j+1}},\end{aligned} \tag{2.5}$$

where $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$, $\mathrm{dn}\,\xi$, $\mathrm{ns}\,\xi$, $\mathrm{cs}\xi$, and $\mathrm{ds}\,\xi$ etc. are Jacobi elliptic functions, which are double periodic and posses the following properties:

1. Properties of triangular function

$$\mathrm{cn}^2\xi + \mathrm{sn}^2\xi = \mathrm{dn}^2\xi + m^2\mathrm{sn}^2\xi = 1, \tag{2.6.1}$$

$$\mathrm{ns}^2\xi = 1 + \mathrm{cs}^2\xi, \quad \mathrm{ns}^2\xi = \mathrm{m}^2 + \mathrm{ds}^2\xi, \tag{2.6.2}$$

2. Derivatives of the Jacobi elliptic functions

$$\mathrm{sn}'\xi = \mathrm{cn}\,\xi\mathrm{dn}\,\xi, \quad \mathrm{cn}'\xi = -\mathrm{sn}\,\xi\mathrm{dn}\,\xi, \quad \mathrm{dn}'\xi = -m^2\mathrm{sn}\xi\mathrm{cn}\,\xi, \tag{2.7.1}$$

$$\mathrm{ns}'\xi = -\mathrm{ds}\,\xi\mathrm{cs}\,\xi, \quad \mathrm{ds}'\xi = -\mathrm{cs}\,\xi\mathrm{ns}\,\xi, \quad \mathrm{cs}'\xi = -\mathrm{ns}\xi\mathrm{ds}\,\xi, \tag{2.7.2}$$

where m is a modulus. The Jacobi–Glaisher functions for elliptic function can be found in Refs. [13,14].

Step 3. The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly-periodic solutions to occur is that differ effects that act to change wave forms in many nonlinear equations, i.e. dispersion, dissipation and nonlinearity, either separately or various combination are able to balance out. We define the degree of $u_i(\xi)$ as $D[u_i(\xi)] = n_i$, which gives rise to the degrees of other expressions as

$$D[u_i^{(\alpha)}] = n_i + \alpha, \quad D[u_i^{\beta}(u_j^{(\alpha)})^s] = n_i\beta + (\alpha + n_j)s. \tag{2.8}$$

Therefore we can get the value of m_i in Eq. (2.4). If n_i is a nonnegative integer, then we first make the transformation $u_i = \omega^{n_i}$.

Step 4. Substitute Eq. (2.4) into Eq. (2.3) along with Eqs. (2.6) and (2.7) and then set all coefficients of $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$ of the resulting system's numerator, $(i = 1, 2, \cdots; j = 0, 1)$ to be zero to get an over-determined system of nonlinear algebraic equations with respect to λ, l, μ, a_{i0}, a_{ij} and b_{ij} $(i = 1, 2, \ldots; j = 1, 2, \ldots, m_i)$.

Step 5. Solving the over-determined system of nonlinear algebraic equations by use of Maple, we would end up with the explicit expressions for λ, l, μ, a_{i0}, a_{ij} and b_{ij}, $(i = 1, 2, \ldots; j = 1, 2, \ldots, m_i)$.

From which λ, l, μ, a_{i0}, a_{ij} and b_{ij} $(i = 1, 2, \ldots; j = 1, 2, \ldots, m_i)$ can be obtained. In this way, we can get double periodic solutions with Jacobi elliptic function.

Since

$$\lim_{m\to 1} \mathrm{sn}\,\xi = \tanh\xi, \quad \lim_{m\to 1} \mathrm{cn}\,\xi = \mathrm{sech}\,\xi, \quad \lim_{m\to 1} \mathrm{dn}\,\xi = \mathrm{sech}\,\xi, \tag{2.9.1}$$

$$\lim_{m\to 1} \mathrm{ns}\,\xi = \coth\xi, \quad \lim_{m\to 1} \mathrm{cs}\,\xi = \mathrm{csch}\,\xi, \quad \lim_{m\to 1} \mathrm{ds}\,\xi = \mathrm{csch}\,\xi, \tag{2.9.2}$$

$$\lim_{m\to 0} \mathrm{sn}\,\xi = \sin\xi, \quad \lim_{m\to 0} \mathrm{cn}\,\xi = \cos\xi, \quad \lim_{m\to 0} dn\xi = 1, \tag{2.9.3}$$

$$\lim_{m\to 0} \mathrm{ns}\,\xi = \csc\xi, \quad \lim_{m\to 0} \mathrm{cs}\xi = \cot\xi, \quad \lim_{m\to 0} \mathrm{ds}\,\xi = \sec\xi, \tag{2.9.4}$$

u_i degenerate respectively as the following form.

1. Solitary wave solutions:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \left(a_{ij} \frac{\tanh^j(\xi)}{(\mu\tanh(\xi) + 1)^j} + b_{ij} \frac{\tanh^{j-1}(\xi)\mathrm{sech}(\xi)}{(\mu\tanh(\xi) + 1)^j} \right). \tag{2.10}$$

2. Triangular function formal solution:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \left(a_{ij} \frac{\sin^j(\xi)}{(\mu\sin(\xi) + 1)^j} + b_{ij} \frac{\sin^{j-1}(\xi)\cos(\xi)}{(\mu\sin(\xi) + 1)^j} \right). \tag{2.11}$$

So the Jacobi function rational expansion method is more powerful than the method by Liu et al [10], the method by Fan [11] and the method by Yan [12]. The solutions which contain solitary wave solutions, singular solitary solutions and triangular function formal solutions can be gotten by the extended method.

Remark 1. If we replace the Jacobi elliptic functions $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ in the ansätzes (2.4) with other pairs of Jacobi elliptic functions such as $\mathrm{sn}(\xi)$ and $\mathrm{dn}(\xi)$; $\mathrm{ns}(\xi)$ and $\mathrm{cs}(\xi)$; $\mathrm{ns}(\xi)$ and $\mathrm{ds}(\xi)$; $\mathrm{sc}(\xi)$ and $\mathrm{nc}(\xi)$; $\mathrm{dc}(\xi)$ and $\mathrm{nc}(\xi)$; $\mathrm{sd}(\xi)$ and $\mathrm{nd}(\xi)$; $\mathrm{cd}(\xi)$ and $\mathrm{nd}(\xi)$ (Refs. [13,15]). It is necessary to point out that above combinations only require the recurrent coefficient relation or derivative relation for the terms of polynomial for computation closed. Therefore other new double periodic wave solutions, solitary wave solutions, and triangular functional solutions can be obtained for some system. For simplicity, we omit them here.

3. Exact solutions of the (1 + 1)-dimensional dispersive long wave equation (DLWE)

Let us consider the (1 + 1)-dimensional dispersive long wave equation (DLWE), i.e.,

$$\begin{cases} v_t + vv_x + w_x = 0, \\ w_t + (wv)_x + \frac{1}{3}v_{xxx} = 0. \end{cases} \tag{3.1}$$

where $w - 1$ is the elevation of the water wave, v is the surface velocity of water along x-direction. The equation system (3.1) can be traced back to the works of Broer [16], Kaup [17], Martinez [18], Kupershmidt [19] etc. A good understanding of all solutions of (3.1) is very helpful for coastal and civil engineers to apply the nonlinear water wave model in a harbor and coastal design. Therefore, finding more types of exact solutions of Eq. (3.1) is of fundamental interest in fluid dynamics. There is an amount of paper devoted to this equations [20–22]. According to the above method, to seek travelling wave solutions of Eq. (3.1), we make the transformation

$$w(x,t) = \sigma(\xi), \quad v(x,t) = \phi(\xi), \quad \xi = x - \lambda t, \tag{3.2}$$

where λ is a constant to be determined later, and thus Eq. (3.1) becomes

$$\begin{cases} -\lambda\phi' + \phi\phi' + \sigma' = 0, \\ -\lambda\sigma' + (\sigma\phi)' + \frac{1}{3}\phi''' = 0. \end{cases} \tag{3.3}$$

According to Step 1 in Section 2, by balancing $\phi'''(\xi)$ and $(\sigma(\xi)\phi(\xi))'$ in Eq. (3.3) and by balancing $\sigma'(\xi)$ and $\phi(\xi)\phi'(\xi)$ in Eq. (3.3), we suppose that Eq. (3.3) has the following formal solutions:

$$\begin{cases} \phi(\xi) = a_0 + a_1 \frac{\mathrm{sn}(\xi)}{\mu\,\mathrm{sn}(\xi)+1} + b_1 \frac{\mathrm{cn}(\xi)}{\mu\,\mathrm{sn}(\xi)+1}, \\ \sigma(\xi) = A_0 + A_1 \frac{\mathrm{sn}(\xi)}{\mu\,\mathrm{sn}(\xi)+1} + B_1 \frac{\mathrm{cn}(\xi)}{\mu\,\mathrm{sn}(\xi)+1} + A_2 \frac{\mathrm{sn}^2(\xi)}{(\mu\,\mathrm{sn}(\xi)+1)^2} + B_2 \frac{\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{(\mu\,\mathrm{sn}(\xi)+1)^2}, \end{cases} \tag{3.4}$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 are constants to be determined later.

With the aid of *Maple*, substituting (3.4) along with (2.6) and (2.7) into (3.3), yields a set of algebraic equations for $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$, $(i = 0, 1, \ldots; j = 0, 1)$. Setting the coefficients of these terms $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2 and λ.

By use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method [23], solving the over-determined algebraic equations, we get the following results:

Case 1.

$$a_0 = \lambda, \quad A_0 = \frac{1}{3}, \quad A_2 = \pm\frac{1}{3}m^2, \quad B_2 = \pm\frac{1}{3}m^2\mathrm{i}, \quad A_1 = B_1 = \mu = 0, \quad a_1 = \pm\frac{1}{3}\sqrt{3}m, \quad b_1 = \pm\frac{1}{3}\mathrm{i}\sqrt{3}m. \tag{3.5}$$

Case 2.

$$a_1 = A_1 = B_1 = B_2 = \mu = 0, \quad a_0 = \lambda, \quad A_0 = \frac{1}{3}, \quad A_2 = -\frac{2}{3}m^2, \quad b_1 = \pm\frac{2}{3}m^2\mathrm{i}. \tag{3.6}$$

Case 3.

$$b_1 = A_1 = B_1 = B_2 = \mu = 0, \quad a_0 = \lambda, \quad A_2 = -\frac{2}{3}m^2, \quad A_0 = \frac{1}{3} + \frac{1}{3}m^2, \quad a_1 = \pm\frac{2}{3}\sqrt{3}m. \tag{3.7}$$

Case 4.

$$\begin{aligned} &a_0 = \frac{\lambda\sqrt{15m^2+60} \pm 10\mathrm{i}}{\sqrt{15m^2+60}}, \quad a_1 = \pm\frac{1}{3}\sqrt{15m^2+60}, \quad b_1 = \pm\frac{1}{3}\sqrt{-3m^2-12}, \\ &A_0 = \frac{9m^2+16}{3(m^2+4)}, \quad A_1 = \pm\left(6\mathrm{i} + \frac{2}{3}\mathrm{i}m^2\right), \quad A_2 = -\frac{5}{3}m^2 - \frac{20}{3}, \quad \mu = \pm 2\mathrm{i}, \\ &B_1 = \pm\frac{-10\mathrm{i}\sqrt{-3m^2-12}}{3\sqrt{15m^2+60}}, \quad B_2 = \pm\frac{5\sqrt{-3m^2-12}(m^2+4)}{3\sqrt{15m^2+60}}. \end{aligned} \tag{3.8}$$

Case 5.

$$\begin{aligned} &b_1 = B_1 = B_2 = 0, \quad a_0 = \frac{\pm(\mu m^2 + \mu) + \lambda\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2m^2 + 3m^2} + 2\mu^3}{\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2m^2 + 3m^2}}, \\ &A_1 = -\frac{2}{3}(\mu m^2 + \mu - 2\mu^3), \quad A_0 = -\frac{-3\mu^4 + 6\mu^2m^2 - m^2 - 3m^2\mu^4 - m^4 + 2\mu^6}{3(m^2 + \mu^4 - \mu^2m^2 - \mu^2)}, \\ &a_1 = \pm\frac{2}{3}\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2m^2 + 3m^2}, \quad A_2 = \frac{2}{3}(\mu^2m^2 + \mu^2 - \mu^4 - m^2). \end{aligned} \tag{3.9}$$

Case 6.

$$\begin{aligned}
a_0 &= \frac{-\mu \pm \lambda\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2 m^2 + 3m^2} + \mu^3}{\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2 m^2 + 3m^2}}, \quad a_1 = \pm\frac{1}{3}\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2 m^2 + 3m^2},\\
b_1 &= \pm\frac{1}{3}\sqrt{3\mu^2 - 3m^2}, \quad A_0 = -\frac{1}{3}\frac{\mu^4 - 2\mu^2 m^2 + m^2}{\mu^2 - m^2}, \quad A_1 = -\frac{1}{3}(\mu - 2\mu^3 + \mu m^2),\\
A_2 &= -\frac{1}{3}(\mu^4 - \mu^2 - \mu^2 m^2 + m^2), \quad B_1 = \pm\frac{\sqrt{\mu^2 - m^2}\,\mu(\mu^2 - 1)}{3\sqrt{\mu^4 - \mu^2 - \mu^2 m^2 + m^2}},\\
B_2 &= \pm\frac{1}{3}\sqrt{\mu^2 - m^2}\sqrt{m^2 + \mu^4 - \mu^2 m^2 - \mu^2}.
\end{aligned} \tag{3.10}$$

From (3.2), (3.4) and Case 1–6, we obtain the following solutions for Eq. (3.1):

Family 1. From Eq. (3.5), we obtain the following rational formal doubly-periodic solutions for the DLWE, as follows:

$$v_1(x,t) = \lambda \pm \frac{1}{3}\sqrt{3}m\,\mathrm{sn}(\xi) \pm \frac{1}{3}\mathrm{i}\sqrt{3}m\,\mathrm{cn}(\xi), \tag{3.11.1}$$

$$w_1(x,t) = \frac{1}{3} \pm \frac{1}{3}m^2\mathrm{sn}^2(\xi) \pm \frac{1}{3}\mathrm{i}m^2\mathrm{sn}(\xi)\mathrm{cn}(\xi), \tag{3.11.2}$$

where $\xi = x - \lambda t$, λ is an arbitrary constant.

Family 2. From Eq. (3.6), we obtain the following rational formal doubly-periodic solutions for the DLWE, as follows:

$$v_2(x,t) = \lambda \pm \frac{2}{3}\mathrm{i}\sqrt{3}m\,\mathrm{cn}(\xi), \tag{3.12.1}$$

$$w_2(x,t) = \frac{1}{3} - \frac{2}{3}m^2\mathrm{sn}^2(\xi), \tag{3.12.2}$$

where $\xi = x - \lambda t$, λ is an arbitrary constant.

Family 3. From Eq. (3.7), we obtain the following rational formal doubly-periodic solutions for the DLWE, as follows:

$$v_3(x,t) = \lambda \pm \frac{2}{3}\sqrt{3}m\,\mathrm{sn}(\xi), \tag{3.13.1}$$

$$w_3(x,t) = \frac{1}{3}(1 + m^2 - 2m^2\mathrm{sn}^2(\xi)), \tag{3.13.2}$$

where $\xi = x - \lambda t$, λ is an arbitrary constant.

Family 4. From Eq. (3.8), we obtain the following rational formal doubly-periodic solutions for the DLWE, as follows:

$$v_4(x,t) = \frac{\lambda\sqrt{15m^2 + 60} + 10\mathrm{i}}{\sqrt{15m^2 + 60}} \pm \frac{\sqrt{15m^2 + 60}\mathrm{sn}(\xi)}{3(\pm 2\mathrm{i}\,\mathrm{sn}(\xi) + 1)} \pm \frac{\sqrt{-3m^2 - 12}\mathrm{cn}(\xi)}{3(\pm 2\mathrm{i}\,\mathrm{sn}(\xi) + 1)}, \tag{3.14.1}$$

$$\begin{aligned}
w_4(x,t) = {} & \frac{9m^2 + 16}{3(m^2 + 4)} \pm \frac{(6\mathrm{i} + \frac{2}{3}\mathrm{i}m^2)\mathrm{sn}(\xi)}{\pm 2\mathrm{isn}(\xi) + 1} \pm \frac{10\mathrm{i}\sqrt{-m^2 - 4}\mathrm{cn}(\xi)}{3\sqrt{5m^2 + 20}(\pm 2\mathrm{isn}(\xi) + 1)} + \frac{(-5m^2 - 20)\mathrm{sn}^2(\xi)}{3(\pm 2\mathrm{isn}(\xi) + 1)^2}\\
& \pm \frac{5\sqrt{-m^2 - 4}(m^2 + 4)\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{3\sqrt{5m^2 + 20}(\pm 2\mathrm{isn}(\xi) + 1)^2},
\end{aligned} \tag{3.14.2}$$

where $\xi = x - \lambda t$, λ is an arbitrary constant.

Family 5. From Eq. (3.9), we obtain the following rational formal doubly-periodic solutions for the DLWE, as follows:

$$v_5(x,t) = \frac{\pm(\mu m^2 + \mu) + \lambda\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2 m^2 + 3m^2} + 2\mu^3}{\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2 m^2 + 3m^2}} \pm \frac{2}{3}\frac{\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2 m^2 + 3m^2}\mathrm{sn}(\xi)}{\mu\,\mathrm{sn}(\xi) + 1}, \tag{3.15.1}$$

$$w_5(x,t) = -\frac{-3\mu^4 + 6\mu^2 m^2 - m^2 - 3m^2\mu^4 - m^4 + 2\mu^6}{3(m^2 + \mu^4 - \mu^2 m^2 - \mu^2)} + \frac{(-2\mu m^2 - 2\mu + 4\mu^3)\mathrm{sn}(\xi)}{3(\mu\,\mathrm{sn}(\xi) + 1)} + \frac{(2\mu^2 m^2 + 2\mu^2 - 2\mu^4 - 2m^2)\mathrm{sn}^2(\xi)}{3(\mu\,\mathrm{sn}(\xi) + 1)^2}, \tag{3.15.2}$$

where $\xi = x - \lambda t$, λ and μ are arbitrary constants.

Family 6. From Eq. (3.10), we obtain the following rational formal doubly-periodic solutions for the DLWE, as follows:

$$v_6(x,t) = \pm\frac{\sqrt{3\mu^2 - 3m^2}\mathrm{cn}(\xi)}{\mu\mathrm{sn}(\xi) + 1} \pm \frac{\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2 m^2 + 3m^2}\mathrm{sn}(\xi)}{3(\mu\,\mathrm{sn}(\xi) + 1)} + \frac{-\mu \pm \lambda\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2 m^2 + 3m^2} + \mu^3}{\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2 m^2 + 3m^2}}, \tag{3.16.1}$$

$$w_6(x,t) = -\frac{\mu^4 - 2\mu^2 m^2 + m^2}{3(\mu^2 - m^2)} \pm \frac{\sqrt{3\mu^2 - 3m^2}\mu(\mu^2 - 1)\mathrm{cn}(\xi)}{3\sqrt{3\mu^4 - 3\mu^2 - 3\mu^2 m^2 + 3m^2}(\mu\,\mathrm{sn}(\xi) + 1)} + \frac{(-\mu + 2\mu^3 - \mu m^2)\mathrm{sn}(\xi)}{3(\mu\,\mathrm{sn}(\xi) + 1)} + \frac{(-\mu^4 + \mu^2 + \mu^2 m^2 - m^2)\mathrm{sn}^2(\xi)}{3(\mu\,\mathrm{sn}(\xi) + 1)^2} \pm \frac{\sqrt{\mu^2 - m^2}\sqrt{\mu^4 - \mu^2 - \mu^2 m^2 + m^2}\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{3(\mu\mathrm{sn}(\xi) + 1)^2}, \tag{3.16.2}$$

where $\xi = x - \lambda t$, λ and μ are arbitrary constants.

4. Summary and conclusions

Generally speaking, to construct the more general and more formal solutions of NPDEs, the various extensions and improvement of Jacobi function method have developed and can be classified into two class: One is called the direct method, which represents the solutions of given NPDEs as the sum of a polynomial in exponential solutions. It requires solving the recurrent coefficient relation or derivative relation for the terms of polynomial for computation closed. For example, Jacobi elliptic function expansion method [10] and extended Jacobi elliptic function expansion algorithm [11,12] et al. The second is called the indirect method, which consists of looking for the solutions of given NPDEs as a polynomial in a variable which satisfies a equation or equations (subequation). For example, a new algebraic method presented by Fan in [24] to seek more new solitary wave solutions of NPDEs that can be expressed as a polynomal in a elementary which satisfies a more general subequation than Riccati equation.

In this paper, we have presented the new Jacobi elliptic function rational expansion method. The method is more powerful than the method proposed recently by Liu [10], Fan [11] and Yan [12]. The (1 + 1)-dimensional dispersive long wave equation (DLWE) is chosen to illustrate the method such that three families of new Jacobi elliptic function solutions are obtained. When the modulus $m \to 1$, some of these obtained solutions degenerate as solitary wave solutions. The algorithm can be also applied to many nonlinear differential equations in mathematical physics. Further work about various extensions and improvement of Jacobi function method need us to find the more general ansätzes or the more general subequation.

Acknowledgement

The National Key Basic Research Development Project Program under the Grant No. G1998030600.

References

[1] Ablowitz MJ, Clarkson PA. Soliton, nonlinear evolution equations and inverse scatting. New York: Cambridge University Press; 1991.
[2] Wadati M. J Phys Soc Jpn 1975;38:673–80;
Wadati M. J Phys Soc Jpn 1975;38:681–6;
Wadati M, Sanuki H, Konno K. Prog Theor Phys 1975;53:419;
Konno K, Wadati M. Prog Theor Phys 1975;53:1652.

[3] Chen Y, Li B, Zhang HQ. Chaos, Solitons & Fractals 2003;17:693–8;
Chen Y, Yan ZY, Zhang HQ. Theor Math Phys 2002;132(1):970–5;
Chen Y, Li B, Zhang HQ. Commun Theor Phys (Beijing) 2003;39:135–40;
Li B, Chen Y, Zhang HQ. Phys Lett A 2002;305:377–82.
[4] Matveev VA, Salle MA. Darboux transformations and solitons. Berlin, Heidelberg: Springer-Verlag; 1991.
[5] Lou SY, Lu JZ. J Phys A 1996;29:4029;
Lou SY, Ruan HY. J Phys A 2001;34:305;
Lou SY. Phys Scr 2000;65:7;
Tang XY, Lou SY, Zhang Y. Phys Rev E 2002;66:046601.
[6] Fan E. Phys Lett A 2000;277:212;
Fan E. Phys Lett A 2002;294:26;
Fan E. Phys Lett A 2001;285:373;
Fan E, Hon YC. Z Naturforsch A 2002;57:692.
[7] Parkes EJ, Duffy BR. Comput Phys Commun 1996;98:288;
Parkes EJ, Duffy BR. Phys Lett A 1997;229:217.
[8] Yan ZY. Phys Lett A 2001;292:100–6;
Yan ZY, Zhang HQ. Appl Math Mech 2000;21:382.
[9] Wang ML. Phys Lett A 1996;216:67;
Wang ML, Wang YM, Zhou YB. Phys Lett A 2002;303:45–51.
[10] Liu SK et al. Phys Lett A 2001;290:72;
Liu SK et al. Phys Lett A 2001;289:69.
[11] Fan E, Hon BYC. Phys Lett A 2002;292:335;
Fan E, Zhang J. Phys Lett A 2002;305:383.
[12] Yan ZY. Chaos, Solitons & Fractals 2003;18:299;
Yan ZY. Chaos, Solitons & Fractals 2003;15:575;
Yan ZY. Comput Phys Commun 2002;148:30.
[13] Chandrasekharan K. Elliptic function. Berlin: Springer; 1978.
[14] Patrick DV. Elliptic function and elliptic curves. Cambridge: University Press; 1973.
[15] Wang ZX, Xia XJ. Special functions. Singapore: World Scientific; 1989.
[16] Broer LJF. Appl Sci Res 1975;31:377.
[17] Kaup DJ. Prog Theor Phys 1975;54:72.
[18] Martinez L. J MAth Phys 1980;21:2342.
[19] Kupershmidt BA. Commun Math Phys 1985;99:51.
[20] Chen CL, Lou SY. Chaos, Solitons & Fractals 2003;16:27.
[21] Wang ML. Phys Lett A 1995;199:169.
[22] Zheng XD, Chen Y, Zheng HQ. Phys Lett A 2003;311:145.
[23] Wu W. Algorithms and computation. Berlin: Springer; 1994. p. 1.
[24] Fan E. Comput Phys Commun 2003;153:17.