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A new Riccati equation rational expansion method and its application to (2 + 1)-dimensional Burgers equation

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Abstract

In this paper, we present a new Riccati equation rational expansion method to uniformly construct a series of exact solutions for nonlinear evolution equations. Compared with most existing tanh methods and other sophisticated methods, the proposed method not only recover some known solutions, but also find some new and general solutions. The solutions obtained in this paper include rational triangular periodic wave solutions, rational solitary wave solutions and rational wave solutions. The efficiency of the method can be demonstrated on (2 + 1)-dimensional Burgers equation.
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1. Introduction

Since Gaedner, Greene, Kruskal and Miural solved the Korteweg–de Vries equation by means of the inverse scattering transformation approach in 1967, the modern theory of soliton has been widely applied in physics and deeply studied in mathematics. There has been a great amount of activities aiming to find methods for exact solution of nonlinear differential equations, such as Bäcklund transformation, Darboux transformation, Cole–Hopf transformation, various tanh methods, various Jacobi elliptic function methods, variable separation approach, Painlevé method, homogeneous balance method, similarity reduction method and so on [1–19]. In [17], Wang and Chen present a new elliptic function rational expansion method and is more powerful than exiting Jacobi elliptic function method [10–12] to uniformly construct more new exact doubly-periodic solutions in terms of rational formal Jacobi elliptic function of nonlinear evolution equations (NLEEs). The key ideas of generalization in elliptic function rational expansion method is that the solution sought is expressible as a finite series of rational formal Jacobi elliptic function. On the lines of the rational expansion thought, the present work is motivated by the desire to present a new Riccati equation rational expansion (RERE) method, in which the solution sought is expressible as a polynomial in a rational formal variable

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which satisfies Riccati equation, to obtained more types and general formal solutions which contain not only the results obtained by using the method [6–10,14–17] but also other types of solutions. For illustration, we apply the generalized method to solve and successfully construct new and more general solutions including rational solitary wave solutions, rational solutions, rational triangular periodic solutions for (2 + 1)-dimensional Burgers equation.

This paper is organized as follows. In Section 2, we summarize the Riccati equation rational expansion method. In Section 3, we apply the method to (2 + 1)-dimensional Burgers equation and bring out many solutions. Conclusions will be presented in finally.

2. Summary of the Riccati equation rational expansion method

In the following we would like to outline the main steps of our method:

Step 1. For a given NLEE system with some physical fields $u_i(x,y,t)$ in three variables x,y,t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0, \tag{2.1}$$

by using the wave transformation

$$u_i(x,y,t) = U_i(\xi), \quad \xi = k(x + ly + \lambda t), \tag{2.2}$$

where k, l and λ are constants to be determined later. Then the nonlinear partial differential Eq. (2.1) is reduced to a nonlinear ordinary differential equation (ODE):

$$G_i(U_i, U_i', U_i'', \ldots) = 0. \tag{2.3}$$

Step 2. We introduce a new ansätz in terms of finite rational formal expansion in the following forms:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}\phi^j(\xi) + b_{ij}\phi^{j-1}(\xi)\phi'(\xi)}{(\mu_1\phi(\xi) + \mu_2\phi'(\xi) + 1)^j} \tag{2.4}$$

and the new variable $\phi = \phi(\xi)$ satisfying

$$\phi' - (h_1 + h_2\phi^2) = \frac{d\phi}{d\xi} - (h_1 + h_2\phi^2) = 0, \tag{2.5}$$

where h_1, h_2, a_{i0}, a_{ij} and b_{ij} ($i = 1,2,\ldots$; $j = 1,2,\ldots,m_i$) are constants to be determined later.

Step 3. The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions to occur is that differ effects that act to change wave forms in many nonlinear equations, i.e. dispersion, dissipation and nonlinearity, either separately or various combination are able to balance out. We define the degree of $U_i(\xi)$ as $D[U_i(\xi)] = n_i$, which gives rise to the degrees of other expressions as

$$D[U_i^{(\alpha)}] = n_i + \alpha, \quad D[U_i^{\beta}(U_j^{(\alpha)})^s] = n_i\beta + (\alpha + n_j)s. \tag{2.6}$$

Therefore we can get the value of m_i in Eq. (2.4). If n_i is a nonnegative integer, then we first make the transformation $U_i = V_i^{n_i}$.

Step 4. Substitute Eq. (2.4) into Eq. (2.3) along with Eq. (2.5) and then set all coefficients of $\phi^i(\xi)$ ($i = 1,2,\ldots$) of the resulting system's numerator to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, μ_1, μ_2, a_{i0}, a_{ij} and b_{ij} ($i = 1,2,\ldots$; $j = 1,2,\ldots,m_i$).

Step 5. Solving the over-determined system of nonlinear algebraic equations by use of *Maple*, we would end up with the explicit expressions for k, μ_1, μ_2, a_{i0}, a_{ij} and b_{ij} ($i = 1,2,\ldots$; $j = 1,2,\ldots,m_i$).

Step 6. It is well known that the general solutions of Eq. (2.5) are

(1) when $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$,

$$\phi(\xi) = \tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi), \quad \phi(\xi) = \coth(\xi) \pm \mathrm{csch}(\xi), \tag{2.7}$$

(2) when $h_1 = h_2 = \pm\frac{1}{2}$,

$$\phi(\xi) = \sec(\xi) \pm \tan(\xi), \quad \phi(\xi) = \csc(\xi) \pm \cot(\xi), \tag{2.8}$$

(3) when $h_1 = 1$ and $h_2 = -1$,

$$\phi(\xi) = \tanh(\xi), \quad \phi(\xi) = \coth(\xi), \tag{2.9}$$

(4) when $h_1 = h_2 = 1$,

$$\phi(\xi) = \tan(\xi), \tag{2.10}$$

(5) when $h_1 = h_2 = -1$,

$$\phi(\xi) = \cot(\xi), \tag{2.11}$$

(6) when $h_1 = 0$ and $h_2 \neq 0$,

$$\phi(\xi) = -\frac{1}{h_2\xi + c_0}. \tag{2.12}$$

Thus according to Eqs. (2.2), (2.4), (2.7)–(2.12) and the conclusions in *Step 5*, we can obtain following rational formal travelling-wave solutions of Eq. (2.1).

(1) When $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^j + b_{ij}(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^{j-1}(\mathrm{sech}^2(\xi) \mp \mathrm{i}\,\mathrm{sech}(\xi)\tanh(\xi))}{\left(1 + \mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\mathrm{sech}^2(\xi) \mp \mathrm{i}\,\mathrm{sech}(\xi)\tanh(\xi))\right)^j}, \tag{2.13.1}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(\coth(\xi) \pm \mathrm{csch}(\xi))^j - b_{ij}(\coth(\xi) \pm \mathrm{csch}(\xi))^{j-1}(\mathrm{csch}^2(\xi) \pm \mathrm{csch}(\xi)\coth(\xi))}{\left(1 + \mu_1(\coth(\xi) \pm \mathrm{i}\,\mathrm{csch}(\xi)) - \mu_2(\mathrm{csch}^2(\xi) \pm \mathrm{csch}(\xi)\coth(\xi))\right)^j}, \tag{2.13.2}$$

(2) when $h_1 = h_2 = \pm\frac{1}{2}$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(\sec(\xi) \pm \tan(\xi))^j + b_{ij}(\sec(\xi) \pm \tan(\xi))^{j-1}(\sec(\xi)\tan(\xi) \pm \sec^2(\xi))}{(1 + \mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\sec(\xi)\tan(\xi) \pm \sec^2(\xi)))^j}, \tag{2.14.1}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(\csc(\xi) \pm \cot(\xi))^j - b_{ij}(\csc(\xi) \pm \cot(\xi))^{j-1}(\csc(\xi)\cot(\xi) \pm \csc^2(\xi))}{(1 + \mu_1(\csc(\xi) \pm \cot(\xi)) - \mu_2(\csc(\xi)\cot(\xi) \pm \csc^2(\xi)))^j}, \tag{2.14.2}$$

(3) when $h_1 = 1$ and $h_2 = -1$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}\tanh^j(\xi) + b_{ij}\tanh^{j-1}(\xi)\,\mathrm{sech}^2(\xi)}{(1 + \mu_1\tanh(\xi) + \mu_2\,\mathrm{sech}^2(\xi))^j}, \tag{2.15.1}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}\coth^j(\xi) - b_{ij}\coth^{j-1}(\xi)\,\mathrm{csch}^2(\xi)}{\left(1 + \mu_1\coth(\xi) - \mu_2\,\mathrm{csch}^2(\xi)\right)^j}, \tag{2.15.2}$$

(4) when $h_1 = h_2 = 1$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}\tan^j(\xi) + b_{ij}\tan^{j-1}(\xi)\sec^2(\xi)}{(1 + \mu_1\tan(\xi) + \mu_2\sec^2(\xi))^j}, \tag{2.16}$$

(5) when $h_1 = h_2 = -1$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}\cot^j(\xi) - b_{ij}\cot^{j-1}(\xi)\csc^2(\xi)}{(1 + \mu_1\cot(\xi) - \mu_2\csc^2(\xi))^j}, \tag{2.17}$$

(6) when $h_1 = 0$ and $h_2 \neq 0$,

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}(h_2\xi + c_0)^j - b_{ij}h_2(h_2\xi + c_0)^{j-1}}{\left((h_2\xi + c_0)^2 + \mu_1(h_2\xi + c_0) - \mu_2 h_2\right)^j}, \tag{2.18}$$

where $\xi = k(x + ly + \lambda t)$, $\mathrm{i} = \sqrt{-1}$ and c_0 is an arbitrary constants.

Remark. The ansatz proposed here is more general than the ansatz in the tanh function method [6,7], extended tanh function method [8], improved extended tanh function method [9], projective Riccati equations method [14] and general projective Riccati equations method [15,16]. If we set the parameters in (2.4) and (2.5) to different values, the above methods all can be recovered by the RERE method.

3. Exact solutions of the (2 + 1)-dimensional Burgers equation

Let us consider the (2 + 1)-dimensional Burgers equation, i.e.,

$$\begin{cases} -u_t + uu_y + \alpha v u_x + \beta u_{yy} + \alpha\beta u_{xx} = 0, \\ u_x - v_y = 0. \end{cases} \tag{3.1}$$

An equivalent form of the Burgers Eq. (3.1) is derived from the generalized Painlevé integrability classification in Ref. [19].

By considering the wave transformations $u(x,y,t) = U(\xi)$, $v(x,y,t) = V(\xi)$ and $\xi = k(x + ly + \lambda t)$, we change the Eq. (3.1) to the form

$$\begin{cases} -\lambda U' + lUU' + \alpha VU' + \beta k l^2 U'' + \alpha\beta k U'' = 0, \\ U' - lV' = 0. \end{cases} \tag{3.2}$$

According to the proposed method, we expand the solution of Eq. (3.2) in the form

$$\begin{cases} U(\xi) = a_0 + \sum_{j=1}^{m_u} \dfrac{a_j\phi^j(\xi) + b_j\phi^{j-1}(\xi)\phi'}{(\mu_1\phi(\xi) + \mu_2\phi' + 1)^j}, \\ V(\xi) = A_0 + \sum_{j=1}^{m_v} \dfrac{A_j\phi^j(\xi) + B_j\phi^{j-1}(\xi)\phi'}{(\mu_1\phi(\xi) + \mu_2\phi' + 1)^j}, \end{cases}$$

where $\phi(\xi)$ satisfies Eq. (2.5). Balancing the term U'' with term UU' and the term U'' with term VU' in Eq. (3.2) gives $m_u = 1$ and $m_v = 1$. So we have

$$\begin{cases} U(\xi) = a_0 + \dfrac{a_1\phi(\xi) + b_1\phi'}{\mu_1\phi(\xi) + \mu_2\phi' + 1}, \\ V(\xi) = A_0 + \dfrac{A_1\phi(\xi) + B_1\phi'}{\mu_1\phi(\xi) + \mu_2\phi' + 1}, \end{cases} \tag{3.3}$$

where $\phi(\xi)$ satisfies Eq. (2.5).

With the aid of *Maple*, substituting (3.3) along with (2.5) into (3.2), yields a set of algebraic equations for $\phi^i(\xi)$ $(i = 0, 1, \ldots)$. Setting the coefficients of these terms $\phi^i(\xi)$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1, μ_1, μ_2 and k.

By use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method [20], solving the over-determined algebraic equations, we get the following results.

Case 1

$$k = \mp\frac{\sqrt{-\mu_2(\mu_2 h_1 + 1)h_2}\,b_1}{2(\mu_2 h_1 + 1)lh_2\beta\mu_2}, \quad a_0 = a_0, \quad a_1 = \pm\frac{\sqrt{-\mu_2(\mu_2 h_1 + 1)h_2}\,b_1}{\mu_2(\mu_2 h_1 + 1)}, \quad b_1 = b_1,$$

$$\mu_1 = 0, \quad \mu_2 = \mu_2, \quad A_0 = -\frac{-\lambda + la_0}{\alpha}, \quad A_1 = \pm\frac{\sqrt{-\mu_2(\mu_2 h_1 + 1)h_2}\,b_1}{l\mu_2(\mu_2 h_1 + 1)}, \quad B_1 = \frac{b_1}{l}. \tag{3.4}$$

Case 2

$$k = -\frac{b_1}{2\mu_1 l\beta}, \quad a_0 = a_0, \quad a_1 = \frac{(h_2 + h_1\mu_1^2)b_1}{\mu_1}, \quad \mu_1 = \mu_1, \quad b_1 = b_1,$$

$$\mu_2 = 0, \quad A_0 = \frac{-l^2 a_0 + l\lambda - l^2 h_1 b_1 - \alpha h_1 b_1}{l\alpha}, \quad A_1 = \frac{(h_2 + h_1\mu_1^2)b_1}{l\mu_1}, \quad B_1 = \frac{b_1}{l}. \tag{3.5}$$

Case 3

$$k = -\frac{a_1}{2(h_2 + h_1\mu_1^2)l\beta}, \quad a_0 = a_0, \quad a_1 = a_1, \quad \mu_2 = b_1 = B_1 = 0, \quad \mu_1 = \mu_1,$$

$$A_0 = \frac{\mu_1^2 h_1 l\lambda + lh_2\lambda - l^2 h_2 a_0 - h_1\mu_1 l^2 a_1 - \mu_1^2 h_1 l^2 a_0 - \alpha h_1 a_1 \mu_1}{l\alpha(h_2 + h_1\mu_1^2)}, \quad A_1 = \frac{a_1}{l}. \tag{3.6}$$

Case 4

$$k = \frac{b_1}{2\mu_1 l\beta}, \quad a_0 = a_0, \quad a_1 = -\frac{(h_2 + h_1\mu_1^2)b_1}{\mu_1}, \quad b_1 = b_1, \quad \mu_1 = \mu_1,$$

$$A_0 = \frac{l^2 h_1 b_1 + l\lambda - l^2 a_0 + \alpha h_1 b_1}{l\alpha}, \quad A_1 = -\frac{(h_2 + h_1\mu_1^2)b_1}{\mu_1 l}, \quad B_1 = \frac{b_1}{l}, \quad \mu_2 = -\frac{1}{h_1}. \tag{3.7}$$

Case 5

$$k = -\frac{a_1}{4(h_2 + h_1\mu_1^2)l\beta}, \quad a_0 = a_0, \quad a_1 = a_1, \quad b_1 = B_1 = 0, \quad \mu_1 = \mu_1, \quad \mu_2 = -\frac{1}{2h_1},$$

$$A_0 = -\frac{-\mu_1^2 h_1 l\lambda - lh_2\lambda + l^2 h_2 a_0 + h_1\mu_1 l^2 a_1 + \mu_1^2 h_1 l^2 a_0 + \alpha h_1 a_1 \mu_1}{l\alpha(h_2 + h_1\mu_1^2)}, \quad A_1 = \frac{a_1}{l}. \tag{3.8}$$

Case 6

$$k = -\frac{a_1}{2(h_2 + h_1\mu_1^2)l\beta}, \quad a_0 = a_0, \quad a_1 = a_1, \quad b_1 = B_1 = 0, \quad \mu_1 = \mu_1, \quad \mu_2 = -\frac{1}{h_1},$$

$$A_0 = -\frac{-\mu_1^2 h_1 l\lambda - lh_2\lambda + l^2 h_2 a_0 + h_1\mu_1 l^2 a_1 + \mu_1^2 h_1 l^2 a_0 + \alpha h_1 a_1 \mu_1}{l\alpha(h_2 + h_1\mu_1^2)}, \quad A_1 = \frac{a_1}{l}. \tag{3.9}$$

From (3.2) and (3.3) and Cases 1–6, we obtain the following solutions for Eq. (3.1). Note: Since tan- and cot-type solution appear in pairs with tanh- and coth-type solutions, respectively, we omit them in this paper. In addition, some rational solutions are also omitted.

Family 1. According to Eq. (3.4), when $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$, we obtain the following solutions for the (2 + 1)-dimensional Burgers equation, as follows.

$$u_{11} = a_0 \pm \frac{\sqrt{\mu_2(\mu_2+2)}b_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))}{(2\mu_2 + \mu_2^2)\left(1 + \mu_2\,\mathrm{sech}^2(\xi) \mp \mathrm{i}\mu_2\,\mathrm{sech}(\xi)\tanh(\xi)\right)} + \frac{b_1(\mathrm{sech}^2(\xi) \mp \mathrm{i}\,\mathrm{sech}(\xi)\tanh(\xi))}{1 + \mu_2\,\mathrm{sech}^2(\xi) \mp \mathrm{i}\mu_2\,\mathrm{sech}(\xi)\tanh(\xi)}, \tag{3.10.1}$$

$$v_{11} = -\frac{-\lambda + la_0}{\alpha} \pm \frac{\sqrt{\mu_2(\mu_2+2)}b_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))}{l(2\mu_2 + \mu_2^2)\left(1 + \mu_2\,\mathrm{sech}^2(\xi) \mp \mathrm{i}\mu_2\,\mathrm{sech}(\xi)\tanh(\xi)\right)} + \frac{b_1(\mathrm{sech}^2(\xi) \mp \mathrm{i}\,\mathrm{sech}(\xi)\tanh(\xi))}{l\left(1 + \mu_2\,\mathrm{sech}^2(\xi) \mp \mathrm{i}\mu_2\,\mathrm{sech}(\xi)\tanh(\xi)\right)}, \tag{3.10.2}$$

$$u_{12} = a_0 \pm \frac{\sqrt{\mu_2(\mu_2+2)}b_1(\coth(\xi) \pm \mathrm{csch}(\xi))}{(2\mu_2 + \mu_2^2)\left(1 - \mu_2\,\mathrm{csch}^2(\xi) \mp \mu_2\,\mathrm{csch}(\xi)\coth(\xi)\right)} - \frac{b_1(\mathrm{csch}^2(\xi) \pm \mathrm{csch}(\xi)\coth(\xi))}{1 - \mu_2\,\mathrm{csch}^2(\xi) \mp \mu_2\,\mathrm{csch}(\xi)\coth(\xi)}, \tag{3.11.1}$$

$$v_{12} = -\frac{-\lambda + la_0}{\alpha} \pm \frac{\sqrt{\mu_2(\mu_2+2)}b_1(\coth(\xi) \pm \mathrm{csch}(\xi))}{l(2\mu_2 + \mu_2^2)\left(1 - \mu_2\,\mathrm{csch}^2(\xi) \mp \mu_2\,\mathrm{csch}(\xi)\coth(\xi)\right)} - \frac{b_1(\mathrm{csch}^2(\xi) \pm \mathrm{csch}(\xi)\coth(\xi))}{l\left(1 - \mu_2\,\mathrm{csch}^2(\xi) \mp \mu_2\,\mathrm{csch}(\xi)\coth(\xi)\right)}, \tag{3.11.2}$$

where $\xi = k(x + ly + \lambda t)$, k is determined by (3.4), a_0, b_1, μ_2, l and λ are arbitrary constants.

Family 2. According to Eq. (3.4), when $h_1 = 1$ and $h_2 = -1$, we obtain the following solutions for the (2 + 1)-dimensional Burgers equation, as follows.

$$u_{21} = a_0 \pm \frac{\sqrt{\mu_2(\mu_2+1)}b_1\tanh(\xi)}{(\mu_2+\mu_2^2)(1+\mu_2\,\mathrm{sech}^2(\xi))} + \frac{b_1\,\mathrm{sech}^2(\xi)}{1+\mu_2\,\mathrm{sech}^2(\xi)}, \quad (3.12.1)$$

$$v_{21} = -\frac{-\lambda + la_0}{\alpha} \pm \frac{\sqrt{\mu_2(\mu_2+1)}b_1\tanh(\xi)}{l(\mu_2+\mu_2^2)(1+\mu_2\,\mathrm{sech}^2(\xi))} + \frac{b_1\,\mathrm{sech}^2(\xi)}{l(1+\mu_2\,\mathrm{sech}^2(\xi))}, \quad (3.12.2)$$

$$u_{22} = a_0 \pm \frac{\sqrt{\mu_2(\mu_2+1)}b_1\coth(\xi)}{(\mu_2+\mu_2^2)(1-\mu_2\,\mathrm{csch}^2(\xi))} - \frac{b_1\,\mathrm{csch}^2(\xi)}{1-\mu_2\,\mathrm{csch}^2(\xi)}, \quad (3.13.1)$$

$$v_{22} = -\frac{-\lambda + la_0}{\alpha} \pm \frac{\sqrt{\mu_2(\mu_2+1)}b_1\coth(\xi)}{l(\mu_2+\mu_2^2)(1-\mu_2\,\mathrm{csch}^2(\xi))} - \frac{b_1\,\mathrm{csch}^2(\xi)}{l(1-\mu_2\,\mathrm{csch}^2(\xi))}, \quad (3.13.2)$$

where $\xi = k(x + ly + \lambda t)$, k is determined by (3.4), a_0, b_1, μ_2, l and λ are arbitrary constants.

Family 3. According to Eq. (3.5), when $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$, we obtain the following solutions for the (2 + 1)-dimensional Burgers equation, as follows.

$$u_{31} = a_0 + \frac{(\mu_1^2-1)b_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))}{2\mu_1(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))+1)} + \frac{b_1(\mathrm{sech}^2(\xi) \mp \mathrm{i}\,\mathrm{sech}(\xi)\tanh(\xi))}{\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))+1}, \quad (3.14.1)$$

$$v_{31} = A_0 + \frac{(\mu_1^2-1)b_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))}{2l\mu_1(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))+1)} + \frac{b_1(\mathrm{sech}^2(\xi) \mp \mathrm{i}\,\mathrm{sech}(\xi)\tanh(\xi))}{\mu_1 l(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))+l}, \quad (3.14.2)$$

$$u_{32} = a_0 + \frac{(\mu_1^2-1)b_1(\coth(\xi) \pm \mathrm{csch}(\xi))}{2\mu_1(\mu_1(\coth(\xi) \pm \mathrm{csch}(\xi))+1)} - \frac{b_1(\mathrm{csch}^2(\xi) \pm \mathrm{csch}(\xi)\coth(\xi))}{\mu_1(\coth(\xi) \pm \mathrm{csch}(\xi))+1}, \quad (3.15.1)$$

$$v_{32} = A_0 + \frac{(\mu_1^2-1)b_1(\coth(\xi) \pm \mathrm{csch}(\xi))}{2l\mu_1(\mu_1(\coth(\xi) \pm \mathrm{csch}(\xi))+1)} - \frac{b_1(\mathrm{csch}^2(\xi) \pm \mathrm{csch}(\xi)\coth(\xi))}{\mu_1 l(\coth(\xi) \pm \mathrm{csch}(\xi))+l}, \quad (3.15.2)$$

where $\xi = k(x + ly + \lambda t)$, k and A_0 are determined by (3.5), a_0, b_1, μ_1, l and λ are arbitrary constants.

Family 4. According to Eq. (3.5), when $h_1 = 1$ and $h_2 = -1$, we obtain the following solutions for the (2 + 1)-dimensional Burgers equation, as follows.

$$u_{41} = a_0 + \frac{(-1+\mu_1^2)b_1\tanh(\xi)}{\mu_1(\mu_1\tanh(\xi)+1)} + \frac{b_1\,\mathrm{sech}^2(\xi)}{\mu_1\tanh(\xi)+1}, \quad (3.16.1)$$

$$v_{41} = A_0 + \frac{(-1+\mu_1^2)b_1\tanh(\xi)}{\mu_1 l(\mu_1\tanh(\xi)+1)} + \frac{b_1\,\mathrm{sech}^2(\xi)}{l(\mu_1\tanh(\xi)+1)}, \quad (3.16.2)$$

$$u_{42} = a_0 + \frac{(-1+\mu_1^2)b_1\coth(\xi)}{\mu_1(\mu_1\coth(\xi)+1)} - \frac{b_1\,\mathrm{csch}^2(\xi)}{\mu_1\coth(\xi)+1}, \quad (3.17.1)$$

$$v_{42} = A_0 + \frac{(-1+\mu_1^2)b_1\coth(\xi)}{\mu_1 l(\mu_1\coth(\xi)+1)} - \frac{b_1\,\mathrm{csch}^2(\xi)}{l(\mu_1\coth(\xi)+1)}, \quad (3.17.2)$$

where $\xi = k(x + ly + \lambda t)$, k and A_0 are determined by (3.5), a_0, b_1, μ_1, l and λ are arbitrary constants.

Family 5. According to Eq. (3.6), when $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$, we obtain the following solutions for the (2 + 1)-dimensional Burgers equation, as follows.

$$u_{51} = a_0 + \frac{a_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))}{\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))+1}, \quad (3.18.1)$$

$$v_{51} = \frac{\mu_1^2 l\lambda - l\lambda + l^2a_0 - \mu_1 l^2 a_1 - \mu_1^2 l^2 a_0 - \alpha a_1\mu_1}{\alpha l(-1+\mu_1^2)} + \frac{a_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))}{l(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))+1)}, \quad (3.18.2)$$

$$u_{52} = a_0 + \frac{a_1(\coth(\xi) \pm \operatorname{csch}(\xi))}{\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + 1}, \tag{3.19.1}$$

$$v_{52} = \frac{\mu_1^2 l\lambda - l\lambda + l^2 a_0 - \mu_1 l^2 a_1 - \mu_1^2 l^2 a_0 - \alpha a_1 \mu_1}{\alpha l(-1+\mu_1^2)} + \frac{a_1(\coth(\xi) \pm \operatorname{csch}(\xi))}{l(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + 1)}, \tag{3.19.2}$$

where $\xi = k(x + ly + \lambda t)$, k is determined by (3.6), μ_1, a_0, a_1, A_0, l and λ are arbitrary constants.

Family 6. According to Eq. (3.6), when $h_1 = 1$ and $h_2 = -1$, we obtain the following solutions for the (2 + 1)-dimensional Burgers equation, as follows.

$$u_{61} = a_0 + \frac{a_1 \tanh(\xi)}{\mu_1 \tanh(\xi) + 1}, \tag{3.20.1}$$

$$v_{61} = \frac{\mu_1^2 l\lambda - l\lambda + l^2 a_0 - \mu_1 l^2 a_1 - \mu_1^2 l^2 a_0 - \alpha a_1 \mu_1}{\alpha l(-1+\mu_1^2)} + \frac{a_1 \tanh(\xi)}{l(\mu_1 \tanh(\xi) + 1)}, \tag{3.20.2}$$

$$u_{62} = u_{61} = a_0 + \frac{a_1 \coth(\xi)}{\mu_1 \coth(\xi) + 1}, \tag{3.21.1}$$

$$v_{62} = \frac{\mu_1^2 l\lambda - l\lambda + l^2 a_0 - \mu_1 l^2 a_1 - \mu_1^2 l^2 a_0 - \alpha a_1 \mu_1}{\alpha l(-1+\mu_1^2)} + \frac{a_1 \coth(\xi)}{l(\mu_1 \coth(\xi) + 1)}, \tag{3.21.2}$$

where $\xi = k(x + ly + \lambda t)$, k is determined by (3.6), μ_1, a_0, a_1, A_0, l and λ are arbitrary constants.

Family 7. According to Eq. (3.7), when $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$, we obtain the following solutions for the (2 +1)-dimensional Burgers equation, as follows.

$$\begin{aligned} u_{71} = a_0 &+ \frac{(1-\mu_1^2) b_1 (\tanh(\xi) \pm \mathrm{i}\,\operatorname{sech}(\xi))}{2\mu_1\big(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\operatorname{sech}(\xi)) - 2\operatorname{sech}^2(\xi) \pm 2\mathrm{i}\,\operatorname{sech}(\xi)\tanh(\xi) + 1\big)} \\ &+ \frac{b_1(\operatorname{sech}^2(\xi) \mp i\,\operatorname{sech}(\xi)\tanh(\xi))}{\mu_1(\tanh(\xi) \pm \mathrm{i}\,\operatorname{sech}(\xi)) - 2\operatorname{sech}^2(\xi) \pm 2\mathrm{i}\,\operatorname{sech}(\xi)\tanh(\xi) + 1}, \end{aligned} \tag{3.22.1}$$

$$\begin{aligned} v_{71} = A_0 &+ \frac{(1-\mu_1^2) b_1 (\tanh(\xi) \pm \mathrm{i}\,\operatorname{sech}(\xi))}{2l\mu_1\big(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\operatorname{sech}(\xi)) - 2\operatorname{sech}^2(\xi) \pm 2\mathrm{i}\,\operatorname{sech}(\xi)\tanh(\xi) + 1\big)} \\ &+ \frac{b_1(\operatorname{sech}^2(\xi) \mp i\,\operatorname{sech}(\xi)\tanh(\xi))}{l\big(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\operatorname{sech}(\xi)) - 2\operatorname{sech}^2(\xi) \pm 2\mathrm{i}\,\operatorname{sech}(\xi)\tanh(\xi) + 1\big)}, \end{aligned} \tag{3.22.2}$$

$$\begin{aligned} u_{72} = a_0 &+ \frac{(1-\mu_1^2) b_1 (\coth(\xi) \pm \operatorname{csch}(\xi))}{2\mu_1\big(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + 2\operatorname{csch}^2(\xi) \pm 2\operatorname{csch}(\xi)\coth(\xi) + 1\big)} \\ &- \frac{b_1(\operatorname{csch}^2(\xi) \pm \operatorname{csch}(\xi)\coth(\xi))}{\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + 2\operatorname{csch}^2(\xi) \pm 2\operatorname{csch}(\xi)\coth(\xi) + 1}, \end{aligned} \tag{3.23.1}$$

$$\begin{aligned} v_{72} = A_0 &+ \frac{(1-\mu_1^2) b_1 (\coth(\xi) \pm \operatorname{csch}(\xi))}{2l\mu_1\big(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + 2\operatorname{csch}^2(\xi) \pm 2\operatorname{csch}(\xi)\coth(\xi) + 1\big)} \\ &- \frac{b_1(\operatorname{csch}^2(\xi) \pm \operatorname{csch}(\xi)\coth(\xi))}{l\big(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + 2\operatorname{csch}^2(\xi) \pm 2\operatorname{csch}(\xi)\coth(\xi) + 1\big)}, \end{aligned} \tag{3.23.2}$$

where $\xi = k(x + ly + \lambda t)$, k and A_0 determined by (3.7), μ_1, a_0, b_1, l and λ are arbitrary constants.

Family 8. According to Eq. (3.7), when $h_1 = 1$ and $h_2 = -1$, we obtain the following solutions for the (2 + 1)-dimensional Burgers equation, as follows.

$$u_{81} = a_0 + \frac{(\mu_1^2 - 1) b_1 \tanh(\xi) + b_1 \mu_1 \operatorname{sech}^2(\xi)}{\mu_1(\mu_1 \tanh(\xi) - \operatorname{sech}^2(\xi) + 1)}, \tag{3.24.1}$$

$$v_{81} = A_0 + \frac{(\mu_1^2 - 1)b_1 \tanh(\xi) + b_1\mu_1 \operatorname{sech}^2(\xi)}{l\mu_1(\mu_1 \tanh(\xi) - \operatorname{sech}^2(\xi) + 1)}, \tag{3.24.2}$$

$$u_{82} = a_0 + \frac{(\mu_1^2 - 1)b_1 \coth(\xi) - b_1\mu_1 \operatorname{csch}^2(\xi)}{\mu_1(\mu_1 \coth(\xi) + \operatorname{csch}^2(\xi) + 1)}, \tag{3.25.1}$$

$$v_{82} = A_0 + \frac{(\mu_1^2 - 1)b_1 \coth(\xi) - b_1\mu_1 \operatorname{csch}^2(\xi)}{l\mu_1(\mu_1 \coth(\xi) + \operatorname{csch}^2(\xi) + 1)}, \tag{3.25.2}$$

where $\xi = k(x + ly + \lambda t)$, k and A_0 determined by (3.7), μ_1, a_0, b_1, l and λ are arbitrary constants.

Family 9. According to Eq. (3.8), when $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$, we obtain the following solutions for the (2 + 1)-dimensional Burgers equation, as follows.

$$u_{91} = a_0 + \frac{a_1(\tanh(\xi) \pm \mathrm{i}\operatorname{sech}(\xi))}{\mu_1(\tanh(\xi) \pm \mathrm{i}\operatorname{sech}(\xi)) - \operatorname{sech}^2(\xi) \pm \mathrm{i}\operatorname{sech}(\xi)\tanh(\xi) + 1}, \tag{3.26.1}$$

$$v_{91} = A_0 + \frac{a_1(\tanh(\xi) \pm \mathrm{i}\operatorname{sech}(\xi))}{l\big(\mu_1(\tanh(\xi) \pm \mathrm{i}\operatorname{sech}(\xi)) - \operatorname{sech}^2(\xi) \pm \mathrm{i}\operatorname{sech}(\xi)\tanh(\xi) + 1\big)}, \tag{3.26.2}$$

$$u_{92} = a_0 + \frac{a_1(\coth(\xi) \pm \operatorname{csch}(\xi))}{\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \operatorname{csch}^2(\xi) \pm \operatorname{csch}(\xi)\coth(\xi) + 1}, \tag{3.27.1}$$

$$v_{92} = A_0 + \frac{a_1(\coth(\xi) \pm \operatorname{csch}(\xi))}{l\big(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \operatorname{csch}^2(\xi) \pm \operatorname{csch}(\xi)\coth(\xi) + 1\big)}, \tag{3.27.2}$$

where $\xi = k(x + ly + \lambda t)$, k and A_0 determined by (3.8), μ_1, a_0, a_1, l and λ are arbitrary constants.

Family 10. According to Eq. (3.8), when $h_1 = 1$ and $h_2 = -1$, we obtain the following solutions for the (2 + 1)-dimensional Burgers equation, as follows.

$$u_{101} = a_0 + \frac{2a_1 \tanh(\xi)}{2\mu_1 \tanh(\xi) - \operatorname{sech}^2(\xi) + 2}, \tag{3.28.1}$$

$$v_{101} = A_0 + \frac{2a_1 \tanh(\xi)}{l(2\mu_1 \tanh(\xi) - \operatorname{sech}^2(\xi) + 2)}, \tag{3.28.2}$$

$$u_{102} = a_0 + \frac{2a_1 \coth(\xi)}{2\mu_1 \coth(\xi) + \operatorname{csch}^2(\xi) + 2}, \tag{3.29.1}$$

$$v_{102} = A_0 + \frac{2a_1 \coth(\xi)}{l(2\mu_1 \coth(\xi) + \operatorname{csch}^2(\xi) + 2)}, \tag{3.29.2}$$

where $\xi = k(x + ly + \lambda t)$, k and A_0 determined by (3.8), μ_1, a_0, a_1, l and λ are arbitrary constants.

Family 11. According to Eq. (3.9), when $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$, we obtain the following solutions for the (2 + 1)-dimensional Burgers equation, as follows.

$$u_{111} = a_0 + \frac{a_1(\tanh(\xi) \pm \mathrm{i}\operatorname{sech}(\xi))}{\mu_1(\tanh(\xi) \pm \mathrm{i}\operatorname{sech}(\xi)) - 2\operatorname{sech}^2(\xi) \pm 2\mathrm{i}\operatorname{sech}(\xi)\tanh(\xi) + 1}, \tag{3.30.1}$$

$$v_{111} = A_0 + \frac{a_1(\tanh(\xi) \pm \mathrm{i}\operatorname{sech}(\xi))}{l\big(\mu_1(\tanh(\xi) \pm \mathrm{i}\operatorname{sech}(\xi)) - 2\operatorname{sech}^2(\xi) \pm 2\mathrm{i}\operatorname{sech}(\xi)\tanh(\xi) + 1\big)}, \tag{3.30.2}$$

$$u_{112} = a_0 + \frac{a_1(\coth(\xi) \pm \operatorname{csch}(\xi))}{\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + 2\operatorname{csch}^2(\xi) \pm 2\operatorname{csch}(\xi)\coth(\xi) + 1}, \tag{3.31.1}$$

$$v_{112} = a_0 + \frac{a_1(\coth(\xi) \pm \operatorname{csch}(\xi))}{l\big(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + 2\operatorname{csch}^2(\xi) \pm 2\operatorname{csch}(\xi)\coth(\xi) + 1\big)}, \tag{3.31.2}$$

where $\xi = k(x + ly + \lambda t)$, k and A_0 determined by (3.9), μ_1, a_0, a_1, l and λ are arbitrary constants.

Family 12. According to Eq. (3.9), when $h_1 = 1$ and $h_2 = -1$, we obtain the following solutions for the (2 + 1)-dimensional Burgers equation, as follows.

$$u_{121} = a_0 + \frac{a_1 \tanh(\xi)}{\mu_1 \tanh(\xi) - \mathrm{sech}^2(\xi) + 1}, \tag{3.32.1}$$

$$v_{121} = A_0 + \frac{a_1 \tanh(\xi)}{l(\mu_1 \tanh(\xi) - \mathrm{sech}^2(\xi) + 1)}, \tag{3.32.2}$$

$$u_{122} = a_0 + \frac{a_1 \coth(\xi)}{\mu_1 \coth(\xi) + \mathrm{csch}^2(\xi) + 1}, \tag{3.33.1}$$

$$v_{122} = a_0 + \frac{a_1 \coth(\xi)}{l(\mu_1 \coth(\xi) + \mathrm{csch}^2(\xi) + 1)}, \tag{3.33.2}$$

where $\xi = k(x + ly + \lambda t)$, k and A_0 determined by (3.9), μ_1, a_0, a_1, l and λ are arbitrary constants.

4. Summary and conclusions

We present a new Riccati equation rational expansion method to uniformly construct a series of exact solutions for nonlinear evolution equations. By the use of the method, we obtain some rational solitary wave solutions, rational wave solutions and rational triangular periodic wave solutions of (2 + 1)-dimensional Burgers equation.

Further generalization about the Riccati equation rational expansion method need us to find the more general ansätzes or the more general subequation.

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