

# Extended Jacobi elliptic function method and its applications to (2+1)-dimensional dispersive long-wave equation\*

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An extended Jacobi elliptic function method is proposed for constructing the exact double periodic solutions of nonlinear partial differential equations (PDEs) in a unified way. It is shown that these solutions exactly degenerate to the many types of soliton solutions in a limited condition. The Wu-Zhang equation (which describes the (2+1)-dimensional dispersive long wave) is investigated by this means and more formal double periodic solutions are obtained.

**Keywords:** Jacobi elliptic function, double periodic solutions, solitary wave solutions

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## 1. Introduction

In recent years, nonlinear evolution equations have attracted considerable attention, partly due to their occurrence in many fields of science, in physics as well as in chemistry and biology, and partly due to the interesting feature and rich variety of properties of their solutions, in particular due to the availability of computer symbolic systems like *Maple* and *Mathematica*, which allow us to perform complicated and tedious algebraic calculation by computer, and help us to find new exact solutions of PDEs.<sup>[1–13]</sup>

Recently, Liu *et al*<sup>[14,15]</sup> and Fan,<sup>[16]</sup> based on different Jacobi elliptic functions, proposed the Jacobi elliptic function expansion method and applied it to some nonlinear wave equations. Many new periodic solutions were obtained by means of this method and it was shown that these solutions exactly degenerate to the well-known soliton solutions in a limited condition.

In this paper, a more powerful method—extended Jacobi elliptic function method is presented. This method is a natural generalization of the improved tanh method by Yan,<sup>[17,18]</sup> for finding solitary-wave

solutions. We present a more general Jacobi elliptic function combination expansion method. Many new formal double periodic solutions are obtained, and their corresponding solitons, singular solitary solutions and triangular function formal solutions are obtained under their limited condition. Our improved method is not only used to find the formal solution obtained by Liu *et al*<sup>[14,15]</sup> and Fan,<sup>[16]</sup> but is also used to seek new formal solutions. To illustrate the improved method, we consider the Wu-Zhang equation,<sup>[19,20]</sup> which describes the (2+1)-dimensional dispersive long wave

$$u_t + uu_x + vv_y + w_x = 0, \quad (1)$$

$$v_t + uv_x + vv_y + w_y = 0, \quad (2)$$

$$w_t + (uw)_x + (vw)_y + \frac{1}{3}(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}) = 0, \quad (3)$$

where  $w$  is the elevation of the water,  $u$  is the surface velocity of water along the  $x$ -direction and  $v$  is the surface velocity of water along the  $y$ -direction. Chen *et al*, using Painlevé truncation expansion, found many

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types of exact solutions of the Wu–Zhang equation (see Ref.[20] for details). By scaling transformation and symmetry reduction, Eqs.(1)–(3) can be reduced to the (1+1)-dimensional dispersive long-wave equation [19,20]

$$v_t + vv_y + w_y = 0, \quad (4)$$

$$w_t + (wv)_y + \frac{1}{3}v_{yyy} = 0. \quad (5)$$

We have found four new families of exact double periodic solutions and the corresponding solitary wave solutions. To our knowledge, these exact double periodic solutions have been, to date, unknown.

This paper is arranged as follows. In Section 2, we present the extended Jacobi elliptic function expansion method. In Section 3, the extended method is used to deal with the system of Wu–Zhang equations. Finally, the summary or conclusions are given.

## 2. Extended Jacobi elliptic function expansion method

In this section, we will improve the Jacobi elliptic function expansion method developed by some authors.[13–15] For a given nonlinear evolution equations, say, in two variables  $x$  and  $t$ ,

$$F(u, u_t, u_x, u_{xt}, u_{tt}, \dots) = 0, \quad (6)$$

we seek the following formal travelling wave solutions:

$$u(x, t) = u(\xi), \quad \xi = x - \lambda t, \quad (7)$$

where  $\lambda$  is a constant to be determined later. Then Eq.(6) reduces to a nonlinear ordinary differential equation under Eq.(7)

$$G(u, u', u'', u''' \dots) = 0, \quad (8)$$

where “'” denotes  $d/d\xi$ . In order to seek the travelling wave solutions of Eq.(8), we take the following two ansatzes in the forms

$$u_1(\xi) = a_0 + \sum_{j=1}^n \text{sn}^{j-1} \xi [a_j \text{sn} \xi + b_j \text{cn} \xi], \quad (9)$$

$$u_2(\xi) = c_0 + \sum_{j=1}^n \text{ns}^{j-1} \xi [c_j \text{ns} \xi + d_j \text{cs} \xi], \quad (10)$$

where  $a_0, a_i, b_i (i = 1, 2, \dots, n)$  are constants to be determined later. The parameter  $n$  can be found by balancing the highest-order linear term with the nonlinear terms in Eq.(6) or Eq.(8). It is necessary to point out that  $n$  is a positive integer. However, when we balance the highest-order partial derivative term

and the nonlinear term in Eq.(6) or Eq.(8), we find that the constant  $n$  need not be a positive integer. In order to apply the above transformation when  $n$  is equal to a fraction or a negative integer, we make the following transformation:

(1) When  $n = q/p$  (where  $n = q/p$  is a fraction in lowest terms), we let

$$u(\xi) = \varphi^{q/p}(\xi), \quad (11)$$

then substitute Eq.(11) into Eq.(8) and return to determine the value of  $n$  by balancing the highest-order partial derivative term and the nonlinear term in the new Eq.(8).

(2) when  $n$  is a negative integer, we let

$$u(\xi) = \varphi^n(\xi), \quad (12)$$

then substitute Eq.(12) into Eq.(8) and return to determine the value of  $n$  once again.

Notice that

$$\begin{aligned} \frac{du_1}{d\xi} &= \sum_{j=1}^n (ja_j \text{sn}^{j-1} \xi \text{cn} \xi \text{dn} \xi + b_j(j-1) \\ &\quad \times \text{sn}^{j-2} \xi \text{cn}^2 \xi \text{dn} \xi + (-b_j) \text{sn}^j \xi \text{dn} \xi), \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{du_2}{d\xi} &= \sum_{j=1}^n (-jc_j \text{ns}^{j-1} \xi \text{ds} \xi \text{cs} \xi - d_j(j-1) \\ &\quad \times \text{ns}^{j-2} \xi \text{cs}^2 \xi \text{ds} \xi - d_j \text{ns}^j \xi \text{ds} \xi), \end{aligned} \quad (14)$$

where  $\text{sn} \xi, \text{cn} \xi, \text{dn} \xi, \text{ns} \xi, \text{cs} \xi$ , and  $\text{ds} \xi$ , etc., are Jacobi elliptic functions, which are double periodic and possess the following properties.

(1) Properties of triangular function

$$\text{cn}^2 \xi + \text{sn}^2 \xi = \text{dn}^2 \xi + m^2 \text{sn}^2 \xi = 1, \quad (15)$$

$$\text{ns}^2 \xi = 1 + \text{cs}^2 \xi, \quad \text{ns}^2 \xi = m^2 + \text{ds}^2 \xi; \quad (16)$$

(2) Derivatives of the Jacobi elliptic functions

$$\begin{aligned} \text{sn}' \xi &= \text{cn} \xi \text{dn} \xi, \\ \text{cn}' \xi &= -\text{sn} \xi \text{dn} \xi, \\ \text{dn}' \xi &= -m^2 \text{sn} \xi \text{cn} \xi, \end{aligned} \quad (17)$$

$$\begin{aligned} \text{ns}' \xi &= -\text{ds} \xi \text{cs} \xi, \\ \text{ds}' \xi &= -\text{cs} \xi \text{ns} \xi, \\ \text{cs}' \xi &= -\text{ns} \xi \text{ds} \xi, \end{aligned} \quad (18)$$

where  $m$  is a modulus. The Jacobi–Glaisher functions for elliptic function can be found in Refs.[21, 22]. Substituting Eqs.(9), (10) into Eq.(8) and using Eqs.(15),

(16) and (17), (18) yields a set of algebraic equations, from which  $a_j, b_j, \lambda$  can be obtained. In this way, we can obtain double periodic solutions with the Jacobi elliptic function. Since

$$\begin{aligned} \lim_{m \rightarrow 1} \operatorname{sn} \xi &= \tanh \xi, \\ \lim_{m \rightarrow 1} \operatorname{cn} \xi &= \operatorname{sech} \xi, \\ \lim_{m \rightarrow 1} \operatorname{dn} \xi &= \operatorname{sech} \xi, \\ \lim_{m \rightarrow 1} \operatorname{ns} \xi &= \operatorname{coth} \xi, \\ \lim_{m \rightarrow 1} \operatorname{cs} \xi &= \operatorname{csch} \xi, \\ \lim_{m \rightarrow 1} \operatorname{ds} \xi &= \operatorname{csch} \xi, \\ \lim_{m \rightarrow 0} \operatorname{sn} \xi &= \sin \xi, \\ \lim_{m \rightarrow 0} \operatorname{cn} \xi &= \cos \xi, \\ \lim_{m \rightarrow 0} \operatorname{dn} \xi &= 1, \\ \lim_{m \rightarrow 0} \operatorname{ns} \xi &= \operatorname{csc} \xi, \\ \lim_{m \rightarrow 0} \operatorname{cs} \xi &= \operatorname{sec} \xi, \\ \lim_{m \rightarrow 0} \operatorname{ds} \xi &= \operatorname{sec} \xi, \end{aligned}$$

$u_1$  and  $u_2$  degenerate respectively as the following form.

(1) Solitary wave solutions

$$u_1 = a_0 + \left[ \sum_{i=1}^n \tanh^{(j-1)} \xi [a_j \tanh \xi + b_j \operatorname{sech} \xi] \right], \quad (19)$$

$$u_2 = c_0 + \left[ \sum_{i=1}^n \operatorname{coth}^{(j-1)} \xi [a_j \operatorname{coth} \xi + b_j \operatorname{csch} \xi] \right]; \quad (20)$$

(2) Triangular function formal solution

$$u_1 = a_0 + \left[ \sum_{i=1}^n \sin^{(j-1)} \xi [a_j \sin \xi + b_j \cos \xi] \right], \quad (21)$$

$$u_2 = c_0 + \left[ \sum_{i=1}^n \operatorname{csc}^{(j-1)} \xi [a_j \operatorname{csc} \xi + b_j \operatorname{sec} \xi] \right]. \quad (22)$$

So the extended Jacobi function expansion method is more powerful than the method by Liu *et al*.<sup>[14,15]</sup> and the method by Fan.<sup>[16]</sup> The solutions which contain solitary wave solutions, singular solitary solutions and triangular function formal solutions can be obtained by the extended method.

**Remark** If we replace the Jacobi elliptic functions in the two ansatzes (9), (10) by other Jacobi elliptic functions in Refs.[21, 22], other new periodic wave solutions can be obtained for some systems. For simplicity, we omit them here.

### 3. Exact solutions of the Wu–Zhang equation

In this section, we consider the Wu–Zhang equation, i.e. Eqs.(1)–(3). Firstly, we let

$$\begin{aligned} u(x, y, t) &= u(\xi), \\ v(x, y, t) &= v(\xi), \\ w(x, y, t) &= w(\xi), \\ \xi &= k(x + ly - \lambda t + \xi_0), \end{aligned} \quad (23)$$

then substituting (23) into Eqs.(1)–(3) yields

$$-k\lambda u' + kuu' + klvu' + kw' = 0, \quad (24)$$

$$-k\lambda v' + kuv' + klvv' + klvw' = 0, \quad (25)$$

$$\begin{aligned} &-k\lambda w' + k(u'w + uw') + k(lv'w + lvw') \\ &+ \frac{1}{3}((k^3 + k^3 l^2)u''' + (k^3 l + k^3 l^3)v''') = 0. \end{aligned} \quad (26)$$

Balancing the highest order of derivative term and nonlinear term in Eqs.(24)–(26), we can have  $n_u = 2$ ,  $n_v = 1$ ,  $n_w = 2$ . So the ansatz solutions of Eqs.(24)–(26) in terms of  $\operatorname{sn} \xi$  and  $\operatorname{cn} \xi$ , are as follows:

$$u(\xi) = a_0 + a_1 \operatorname{sn} \xi + a_2 \operatorname{cn} \xi, \quad (27)$$

$$v(\xi) = b_0 + b_1 \operatorname{sn} \xi + b_1 \operatorname{cn} \xi, \quad (28)$$

$$w(\xi) = c_0 + c_1 \operatorname{sn} \xi + c_2 \operatorname{sn}^2 \xi + c_3 \operatorname{cn} \xi + c_4 \operatorname{cn} \xi \operatorname{sn} \xi. \quad (29)$$

Substituting Eqs.(27)–(29) into Eqs.(24)–(26) and using *Mathematica* yields an algebraic system for  $a_i, b_i, c_j$  ( $i = 0, 1, 2; j = 0, \dots, 42$ ),  $k, l$  and  $\lambda$

$$k(c_4 + a_1(a_2 + b_2 l)) = 0, \quad (30)$$

$$-k(a_0 a_2 + c_3 + a_2 b_0 l - a_2 \lambda) = 0, \quad (31)$$

$$-k(2a_1 a_2 + 2c_4 + a_2 b_1 l + a_1 b_2 l) = 0, \quad (32)$$

$$k(a_0 a_1 + c_1 + a_1 b_0 l - a_1 \lambda) = 0, \quad (33)$$

$$k(a_1^2 - a_2^2 + 2c_2 + a_1 b_1 l - a_2 b_2 l) = 0, \quad (34)$$

$$k(a_2 b_1 + (b_1 b_2 + c_4)l) = 0, \quad (35)$$

$$-k(a_0 b_2 + b_0 b_2 l + c_3 l - b_2 \lambda) = 0, \quad (36)$$

$$-k(a_2 b_1 + a_1 b_2 + 2(b_1 b_2 + c_4)l) = 0, \quad (37)$$

$$k(a_0 b_1 + b_0 b_1 l + c_1 l - b_1 \lambda) = 0, \quad (38)$$

$$k(a_1 b_1 - a_2 b_2 + (b_1^2 - b_2^2 + 2c_2)l) = 0, \quad (39)$$

$$k(a_2 c_1 + a_1 c_3 + a_0 c_4 + b_2 c_1 l + b_1 c_3 l + b_0 c_4 l - c_4 \lambda) = 0, \quad (40)$$

$$\begin{aligned} & \frac{1}{3}k(-3a_0c_3 + 6a_1c_4 - 3b_2c_0l + 6b_2c_2l \\ & - 3b_0c_3l + 6b_1c_4l + b_2k^2l + 4b_2k^2m^2l \\ & + b_2k^2l^3 + 4b_2k^2m^2l^3 + a_2(-3c_0 + 6c_2 \\ & + k^2(1 + 4m^2)(1 + l^2)) + 3c_3\lambda) = 0, \end{aligned} \quad (41)$$

$$\begin{aligned} & - 2k(a_2c_1 + a_1c_3 + a_0c_4 + b_2c_1l \\ & + b_1c_3l + b_0c_4l - c_4\lambda) = 0, \end{aligned} \quad (42)$$

$$\begin{aligned} & - k(3a_1c_4 + a_2(3c_2 + 2k^2m^2(1 + l^2)) \\ & + l(3b_1c_4 + b_2(3c_2 + 2k^2m^2(1 + l^2)))) = 0, \end{aligned} \quad (43)$$

$$\begin{aligned} & - \frac{1}{3}k(-3a_0c_1 - 3a_2c_4 - 3b_1c_0l \\ & - 3b_0c_1l - 3b_2c_4l + b_1k^2l + b_1k^2m^2l \\ & + b_1k^2l^3 + b_1k^2m^2l^3 + a_1(-3c_0 \\ & + k^2(1 + m^2)(1 + l^2)) + 3c_1\lambda) = 0, \end{aligned} \quad (44)$$

$$\begin{aligned} & 2k(a_1c_1 + a_0c_2 - a_2c_3 + b_1c_1l \\ & + b_0c_2l - b_2c_3l - c_2\lambda) = 0, \end{aligned} \quad (45)$$

$$\begin{aligned} & k(-3a_2c_4 + a_1(3c_2 + 2k^2m^2(1 + l^2)) \\ & + l(-3b_2c_4 + b_1(3c_2 + 2k^2m^2(1 + l^2)))) = 0. \end{aligned} \quad (46)$$

To solve Eqs.(30)–(46), by use of the Wu elimination method, which is a sufficient method to solve the system of algebra polynomial equations with more unknowns,<sup>[23]</sup> we obtain the following sets of solutions.

#### Case 1

$$\begin{aligned} a_0 &= -b_0l + \lambda, \quad a_1 = 0, \quad a_2 = \pm i \frac{2km}{\sqrt{3}}, \\ b_0 &= \text{const.}, \quad b_1 = 0, \quad b_2 = \pm i \frac{2kml}{\sqrt{3}}, \\ c_0 &= \frac{k^2(1 + l^2)}{3}, \quad c_2 = -\frac{2k^2m^2(1 + l^2)}{3}, \\ c_1 &= c_3 = c_4 = 0. \end{aligned} \quad (47)$$

#### Case 2

$$\begin{aligned} a_0 &= a_2 = 0, \quad a_1 = \pm \frac{2km}{\sqrt{3}}, \\ b_0 &= \frac{\lambda}{l}, \quad b_1 = \pm \frac{2kml}{\sqrt{3}}, \quad b_2 = 0, \\ c_0 &= \frac{k^2(1 + m^2)(1 + l^2)}{3}, \quad c_2 = -\frac{2k^2m^2(1 + l^2)}{3}, \\ c_1 &= c_3 = c_4 = 0. \end{aligned} \quad (48)$$

#### Case 3

$$\begin{aligned} a_0 &= 0, \quad a_1 = \pm \frac{km}{\sqrt{3}}, \quad a_2 = \pm i \frac{km}{\sqrt{3}}, \\ b_0 &= \frac{\lambda}{l}, \quad b_1 = \pm \frac{kml}{\sqrt{3}}, \quad b_2 = \pm i \frac{kml}{\sqrt{3}}, \\ c_0 &= \frac{k^2(1 + l^2)}{3}, \quad c_2 = -\frac{k^2m^2(1 + l^2)}{3}, \\ c_4 &= \pm i \frac{k^2m^2(1 + l^2)}{3}, \quad c_1 = c_3 = 0. \end{aligned} \quad (49)$$

#### Case 4

$$\begin{aligned} a_0 &= -b_0l + \lambda, \quad a_1 = -b_1l, \quad a_2 = -b_2l, \\ b_0 &= b_0, \quad b_1 = b_1, \quad b_2 = b_2, \\ c_0 &= c_0, \quad c_1 = c_2 = c_3 = c_4 = 0. \end{aligned} \quad (50)$$

In this way, we gain the four types of periodic wave solutions for Eqs.(1)–(3).

#### Case 1

$$u = -b_0l + \lambda \pm i \frac{2km}{\sqrt{3}} \text{cn}\xi, \quad (51)$$

$$v = b_0 \pm i \frac{2kml}{\sqrt{3}} \text{cn}\xi, \quad (52)$$

$$w = \frac{k^2(1 + l^2)}{3} - \frac{2k^2m^2(1 + l^2)}{3} \text{sn}^2\xi. \quad (53)$$

#### Case 2

$$u = \pm \frac{2km}{\sqrt{3}} \text{sn}\xi, \quad (54)$$

$$v = \frac{\lambda}{l} \pm \frac{2kml}{\sqrt{3}} \text{sn}\xi, \quad (55)$$

$$w = \frac{k^2(1 + m^2)(1 + l^2)}{3} - \frac{2k^2m^2(1 + l^2)}{3} \text{sn}^2\xi. \quad (56)$$

#### Case 3

$$u = \pm \frac{km}{\sqrt{3}} \text{sn}\xi \pm i \frac{km}{\sqrt{3}} \text{cn}\xi, \quad (57)$$

$$v = \frac{\lambda}{l} \pm \frac{kml}{\sqrt{3}} \text{sn}\xi \pm i \frac{kml}{\sqrt{3}} \text{cn}\xi, \quad (58)$$

$$\begin{aligned} w &= \frac{k^2(1 + l^2)}{3} - \frac{k^2m^2(1 + l^2)}{3} \text{sn}^2\xi \\ & \pm i \frac{k^2m^2(1 + l^2)}{3} \text{sn}\xi \text{cn}\xi. \end{aligned} \quad (59)$$

#### Case 4

$$u = -b_0l + \lambda - b_1l \text{sn}\xi - b_2l \text{cn}\xi, \quad (60)$$

$$v = b_0 + b_1 \text{sn}\xi + b_2 \text{cn}\xi, \quad (61)$$

$$w = c_0. \quad (62)$$

Their corresponding solitary wave solutions are as follows.

#### Case 1

$$u = -b_0l + \lambda \pm i \frac{2k}{\sqrt{3}} \text{sech}\xi, \quad (63)$$

$$v = b_0 \pm i \frac{2kl}{\sqrt{3}} \operatorname{sech} \xi, \quad (64)$$

$$w = \frac{k^2(1+l^2)}{3} - \frac{2k^2(1+l^2)}{3} \tanh^2 \xi. \quad (65)$$

**Case 2**

$$u = \pm \frac{2k}{\sqrt{3}} \tanh \xi, \quad (66)$$

$$v = \frac{\lambda}{l} \pm \frac{2kl}{\sqrt{3}} \tanh \xi, \quad (67)$$

$$w = \frac{2k^2(1+l^2)}{3} - \frac{2k^2(1+l^2)}{3} \tanh^2 \xi. \quad (68)$$

**Case 3**

$$u = \pm \frac{k}{\sqrt{3}} \tanh \xi \pm i \frac{k}{\sqrt{3}} \operatorname{sech} \xi, \quad (69)$$

$$v = \frac{\lambda}{l} \pm \frac{kl}{\sqrt{3}} \tanh \xi \pm i \frac{kl}{\sqrt{3}} \operatorname{sech} \xi, \quad (70)$$

$$w = \frac{k^2(1+l^2)}{3} - \frac{k^2(1+l^2)}{3} \tanh^2 \xi \pm i \frac{k^2(1+l^2)}{3} \tanh \xi \operatorname{sech} \xi. \quad (71)$$

**Case 4**

$$u = -b_0 l + \lambda - b_1 l \tanh \xi - b_2 l \operatorname{sech} \xi, \quad (72)$$

$$v = b_0 + b_1 \tanh \xi + b_2 \operatorname{sech} \xi, \quad (73)$$

$$w = c_0. \quad (74)$$

In a similar way, we also can obtain the ansatz solutions of Eqs.(1)–(3) in terms of  $\operatorname{ns} \xi$  and  $\operatorname{cs} \xi$ .

**Case 1**

$$u = -b_0 l + \lambda \pm \frac{2k}{\sqrt{3}} \operatorname{cs} \xi, \quad (75)$$

$$v = b_0 \pm \frac{2kl}{\sqrt{3}} \operatorname{cs} \xi, \quad (76)$$

$$w = \frac{k^2 m^2 (1+l^2)}{3} - \frac{2k^2 (1+l^2)}{3} \operatorname{ns}^2 \xi. \quad (77)$$

**Case 2**

$$u = \pm \frac{2k}{\sqrt{3}} \operatorname{ns} \xi, \quad (78)$$

$$v = \frac{\lambda}{l} \pm \frac{2kl}{\sqrt{3}} \operatorname{ns} \xi, \quad (79)$$

$$w = \frac{k^2 (1+m^2)(1+l^2)}{3} - \frac{2k^2 (1+l^2)}{3} \operatorname{ns}^2 \xi. \quad (80)$$

**Case 3**

$$u = \pm \frac{k}{\sqrt{3}} \operatorname{ns} \xi \pm \frac{k}{\sqrt{3}} \operatorname{cs} \xi, \quad (81)$$

$$v = \frac{\lambda}{l} \pm \frac{kl}{\sqrt{3}} \operatorname{ns} \xi \pm \frac{kl}{\sqrt{3}} \operatorname{cs} \xi, \quad (82)$$

$$w = \frac{k^2 m^2 (1+l^2)}{3} - \frac{k^2 (1+l^2)}{3} \operatorname{ns}^2 \xi - \frac{k^2 (1+l^2)}{3} \operatorname{ns} \xi \operatorname{cs} \xi. \quad (83)$$

**Case 4**

$$u = -b_0 l + \lambda - b_1 l \operatorname{ns} \xi - b_2 l \operatorname{cs} \xi, \quad (84)$$

$$v = b_0 + b_1 \operatorname{ns} \xi + b_2 \operatorname{cs} \xi, \quad (85)$$

$$w = c_0, \quad (86)$$

where  $b_0, b_1, b_2, c_0$  are arbitrary constants.

Their corresponding solitary wave solutions are as follows.

**Case 1**

$$u = -b_0 l + \lambda \pm \frac{2k}{\sqrt{3}} \operatorname{csch} \xi, \quad (87)$$

$$v = b_0 \pm \frac{2kl}{\sqrt{3}} \operatorname{csch} \xi, \quad (88)$$

$$w = \frac{k^2 (1+l^2)}{3} - \frac{2k^2 (1+l^2)}{3} \operatorname{coth}^2 \xi. \quad (89)$$

**Case 2**

$$u = \pm \frac{2k}{\sqrt{3}} \operatorname{coth} \xi, \quad (90)$$

$$v = \frac{\lambda}{l} \pm \frac{2kl}{\sqrt{3}} \operatorname{coth} \xi, \quad (91)$$

$$w = \frac{k^2 (1+m^2)(1+l^2)}{3} - \frac{2k^2 (1+l^2)}{3} \operatorname{coth}^2 \xi. \quad (92)$$

**Case 3**

$$u = \pm \frac{k}{\sqrt{3}} \operatorname{coth} \xi \pm \frac{k}{\sqrt{3}} \operatorname{csch} \xi, \quad (93)$$

$$v = \frac{\lambda}{l} \pm \frac{kl}{\sqrt{3}} \operatorname{coth} \xi \pm \frac{kl}{\sqrt{3}} \operatorname{csch} \xi, \quad (94)$$

$$w = \frac{k^2 m^2 (1+l^2)}{3} - \frac{k^2 (1+l^2)}{3} \operatorname{coth}^2 \xi - \frac{k^2 (1+l^2)}{3} \operatorname{coth} \xi \operatorname{csch} \xi. \quad (95)$$

**Case 4**

$$u = -b_0 l + \lambda - b_1 l \operatorname{coth} \xi - b_2 l \operatorname{csch} \xi, \quad (96)$$

$$v = b_0 + b_1 \operatorname{coth} \xi + b_2 \operatorname{csch} \xi, \quad (97)$$

$$w = c_0. \quad (98)$$

## 4. Conclusions

In this paper, based on Jacobi elliptic function combination expansion, the extended Jacobi function expansion method is presented and is applied to the Wu–Zhang equation (which describes the (2+1)-dimensional dispersive long wave). It is shown that more formal double periodic solutions are obtained and that solitary wave solutions, singular solitary wave solutions and triangular function form solutions

are obtained at their limit condition. The method used in this paper also can be applied to many other partial differential equations.

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