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New exact solutions for some nonlinear differential equations using symbolic computation

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Abstract

Based on computerized symbolic computation and a new general ansätze, a generalized tanh-function method for constructing multiple travelling wave solutions of nonlinear evolution equations (NEEs) is presented and implemented in a computer algebraic system. Applying the generalized method, with the aid of *Maple*, we consider some NEEs with physics interests. As a result, we can successfully recover the previously known solitary wave solutions that had been found by the extended tanh-function method and other more sophisticated methods. More importantly, for some equations, we also obtain other new and more general solutions at same time. The results include kink-profile solitary-wave solutions, bell-profile solitary-wave solutions, periodic wave solutions, rational solutions, singular solutions and new formal solutions. The properties of the new soliton solutions for WBK equations are shown by some figures.

Keywords: Nonlinear evolution equations; Exact solutions; Symbolic computation; Riccati equation

1. Introduction

In recent years, nonlinear evolution equations (NEEs) have attracted considerable attention. Directly searching for solutions of NEEs has become more

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and more attractive [1–8]. This is due to their occurrence in many fields of science and the interesting features and rich variety of properties of their solutions, on the other hand, due to the availability of computer symbolic system like *Maple* or *Mathematica* which allows us to perform some complicated and tedious algebraic calculation on a computer and help us to find new exact solutions of NEEs. Various powerful methods have been presented, such as, Bäcklund transformation, Darboux transformation, Cole–Hopf transformation, tanh method, sine–cosine method, Painlevé method, Hirota method [1,2], rank analysis method [3], homogeneous balance method (HBM) [9,10]. One of most effectively straightforward methods to construct exact solutions of NEES is tanh method [11–15]. Recently, Fan [16,17] has proposed an extended tanh-function method. More recently Fan et al. [18] and Yan [19] further developed this idea and made it much more lucid and straightforward for a class of NEES. Most recently, Elwakil et al. [20] modified extended tanh-function method and obtained some new exact solutions. In this paper, based on the above work [16–20], by introducing a new more general ansätz than the ansätzs

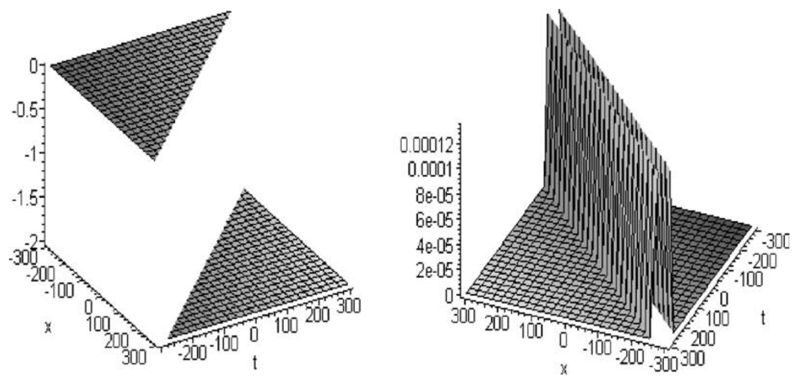

Fig. 1. The Case-1 solution u and v, where $d_{11} = i$, $\alpha = \beta = 2$.

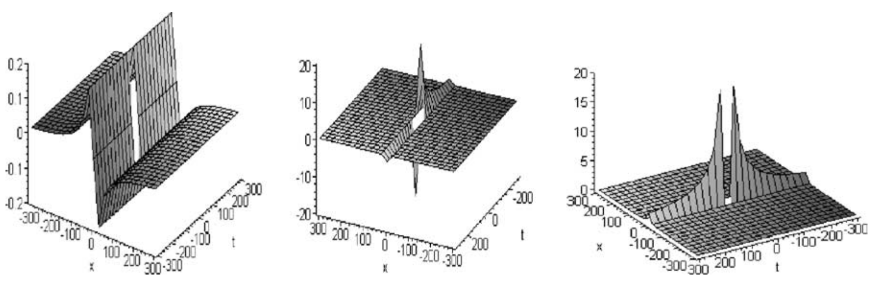

Fig. 2. The Case-1 solution u, the real part, imaginary part and the modulus, where $d_{11} = 0.01$, $\alpha = \beta = 2$.

in above methods, we present the generalized tanh-function method. Applying the method, with the help of *Maple*, we consider some NEEs: the generalized KP equation [21], the generalized WBK equations [22]. And the properties of the new soliton solutions for WBK equations, as some illustrative samples, are shown by some figures (Figs. 1–6).

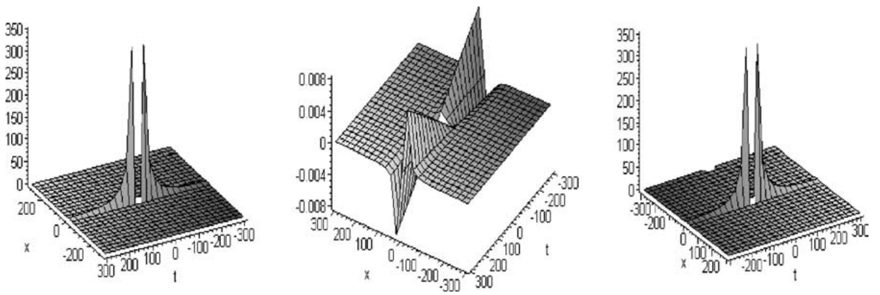

Fig. 3. The Case-1 solution v, the real part, imaginary part and the modulus, where $d_{11} = 0.01$, $\alpha = \beta = 2$.

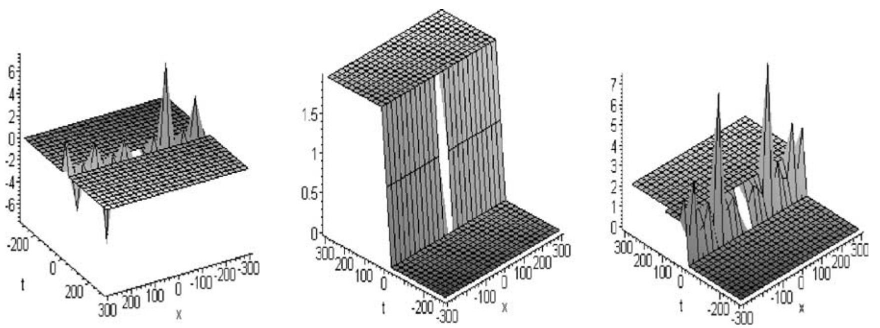

Fig. 4. The Case-4 solution u, the real part, imaginary part and the modulus, where $R = 0.01$, $\alpha = \beta = 2$.

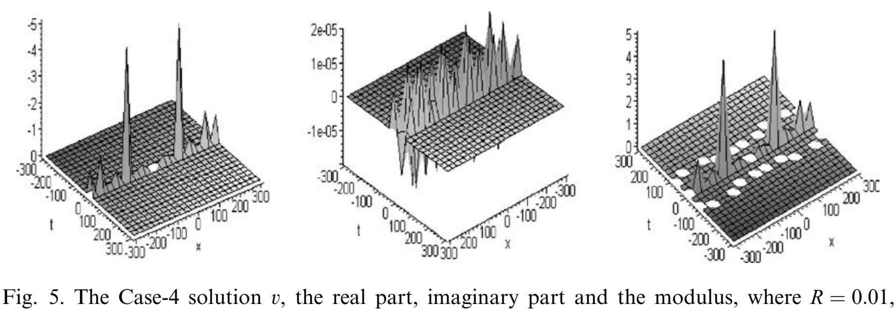

Fig. 5. The Case-4 solution v, the real part, imaginary part and the modulus, where $R = 0.01$, $\alpha = \beta = 2$.

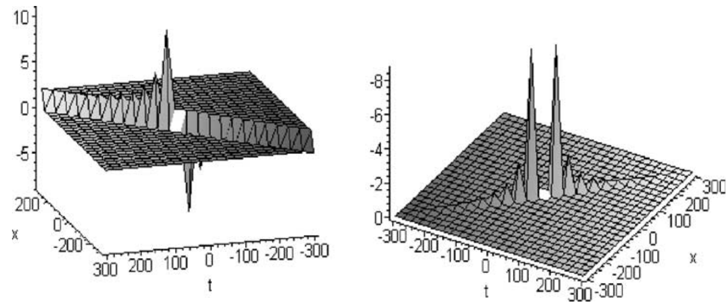

Fig. 6. The Case-4 solution u and v, the real part, imaginary part and the modulus, where $R = -0.01$, $\alpha = \beta = 2$.

2. Summary of method

Now, we simply describe the generalized tanh-function method. Consider a given system of NEEs, say, in two variables, x, t

$$H_i(u_i, u_{i,t}, u_{i,x}, u_{i,xt}, u_{i,tt}, \cdots) = 0, \quad (i = 1, 2, 3, \ldots, n) \tag{1}$$

where i means the ith equation, n denotes the number of the equations. We first consider the following formal travelling wave solutions $u_i(x,t) = \phi_i(\xi)$, $\xi = x - \lambda t$, where λ is a constant to be determined later. Then Eq. (1) becomes a nonlinear ordinary differential equation (ODE)

$$F_i(\phi_i, \phi_i', \phi_i'', \phi_i''' \ldots) = 0, \tag{2}$$

where "′" denotes $\mathrm{d}/\mathrm{d}\xi$. In order to seek the travelling wave solutions of Eq. (2), we introduce the following new ansätze

$$\phi_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \left\{ a_{ij}\omega^j + b_{ij}\omega^{-j} + c_{ij}\omega^{j-1}\sqrt{R + \omega^2} + d_{ij}\frac{\sqrt{R+\omega^2}}{\omega^j} \right\} \tag{3}$$

and the new variable $\omega = \omega(\xi)$ satisfying

$$\omega' = \frac{\mathrm{d}\omega}{\mathrm{d}\xi} = R + \omega^2. \tag{4}$$

where a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij} ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m_i$) and R are constants to be determined later. The value of m_i in Eq. (3) can be determined by balancing the highest-order derivative term with the nonlinear term [9,10] in Eqs. (1) or (2). After that, substitute Eq. (3) along with Eq. (4) into Eq. (2), the corresponding ODEs, and then let all coefficients of $\omega^p(\sqrt{(R + \omega^2)})^q$ ($q = 0, 1$; $p = 0, 1, 2, \ldots$) be zero to get an over-determined system of nonlinear algebraic equations with respect to a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}, R, λ, $(i = 1, 2, \dots, n;\ j = 1, 2, \dots, m_i)$. With the aid of Maple, we apply Wu-elimination method [25,26] to solve the above mentioned over-determined system of nonlinear algebraic equations to yield the values of a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}, R, λ, $(i = 1, 2, \dots, n;\ j = 1, 2, \dots, m_i)$. Because the Raccati equation Eq. (4) has the general solutions

(i) If $R < 0$,

$$\omega(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi), \tag{5}$$

$$\omega(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi), \tag{6}$$

(ii) If $R = 0$,

$$\omega(\xi) = -\frac{1}{\xi}, \tag{7}$$

(iii) If $R > 0$,

$$\omega(\xi) = \sqrt{R}\tan(\sqrt{R}\xi), \tag{8}$$

$$\omega(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi), \tag{9}$$

then combined the values of a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}, R, λ, $(i = 1, 2, \dots, n;\ j = 1, 2, \dots, m)$ with Eq. (3), and Eqs. (5)–(9), more travelling wave solutions of Eq. (1) are obtained.

The algorithm is more powerful than the typical tanh method [11–16], the extended tanh-function method [17–19] and the modified extended tanh-function method [20]. When $b_{ij} = 0$, $c_{ij} = 0$, $d_{ij} = 0$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$ in Eq. (3), Eq. (3) becomes the transformation proposed by Wang et al. [10]. But as $b_{ij} = 0$, $d_{ij} = 0$, $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$ in Eq. (3), Eq. (3) becomes the transformation proposed by Yan [19]. But as $c_{ij} = 0$, $d_{ij} = 0$, $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$ in Eq. (3), Eq. (3) becomes the transformation proposed by Elwakil et al. [20]. So we would find many new exact solutions of system (1) by our generalized tanh-function method.

In what follows we would like to apply our method to KP equation [21] and WBK equations [22–24] to illustrate our algorithm which is more powerful than the typical tanh method, the extended tanh-function method the modified tanh-function method.

Note: Since tan- and cot-type solutions appear in pairs with tanh- and coth-type solutions, respectively, we omit them in this letter. In addition, some rational solutions also are omitted.

3. Applications

Example 1. Consider the following KP equation [21]

$$u_{xt} + 6u_x^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0, \tag{10}$$

Firstly, we let

$$u(x,y,z,t) = \phi(\xi), \quad \xi = x + \beta y + rz - \lambda t, \tag{11}$$

where β, r, λ are constants to be determined later. Then Eq. (10) reduces to a system of nonlinear ODE

$$\phi'' - 3\phi^2 + \phi\lambda + \phi\beta^2 + \phi r^2 = 0, \tag{12}$$

According to the description in Section 2, By balancing the highest-order derivative term with the nonlinear term in Eq. (10) or (12), we support that Eq. (10) has the following formal solutions

$$\begin{aligned} \phi = {} & A_0 + A_1\omega + \frac{B_1}{\omega} + A_2\omega^2 + \frac{B_2}{\omega^2} + \frac{d_1\sqrt{R+\omega^2}}{\omega} + \frac{d_2\sqrt{R+\omega^2}}{\omega^2} \\ & + C_1\sqrt{R+\omega^2} + C_2\omega\sqrt{R+\omega^2} \end{aligned} \tag{13}$$

and $\omega = \omega(\xi)$ satisfying Eq. (4) where $R, A_0, A_1, A_2, B_1, B_2, C_1, C_2, d_1, d_2, r, \lambda, \beta$, are constants to be determined later. With the aid of *Maple*, substituting Eq. (13) into Eq. (12) along with Eq. (4), and let the coefficients be zero of $\omega^p(\sqrt{(R+\omega^2)})^q$ $(q = 0, 1;\ p = 0, 1, 2, \ldots)$ with the same power, we get the following over-determined algebraic equations system

$$-3d_2^2R - 3B_2^2 + 6B_2R^2 = 0, \tag{14}$$

$$2A_1 - 6A_1A_2 - 6C_1C_2 = 0, \tag{15}$$

$$\begin{aligned} & -6A_0A_1 - 6C_1C_2R + \lambda A_1 - 6B_1A_2 - 6d_1C_1 + \beta^2 A_1 \\ & \quad + r^2A_1 - 6d_2C_2 + 2A_1R = 0, \end{aligned} \tag{16}$$

$$8A_2R - 3C_2^2R - 3C_1^2 + r^2A_2 - 6A_0A_2 + \lambda A_2 - 6d_1C_2 - 3A_1^2 + \beta^2A_2 = 0, \tag{17}$$

$$\begin{aligned} & \lambda B_1 - 6A_0B_1 - 6A_1B_2 - 6d_2C_2R + 2B_1R + \beta^2B_1 \\ & \quad - 6d_1C_1R - 6d_1d_2 + r^2B_1 = 0, \end{aligned} \tag{18}$$

$$-6A_2C_1 - 6A_1C_2 + 2C_1 = 0, \tag{19}$$

$$6C_2 - 6A_2C_2 = 0, \tag{20}$$

$$-3A_2^2 - 3C_2^2 + 6A_2 = 0, \tag{21}$$

$$
\begin{aligned}
&-6A_2B_2+\beta^2A_0-6A_1B_1-3A_0^2+2B_2-3C_1^2R+\lambda A_0\\
&\quad -6d_1C_2R+2A_2R^2-3d_1^2+r^2A_0-6d_2C_1=0,
\end{aligned}\tag{22}
$$

$$
-6A_0B_2+8B_2R-3d_1^2R+\lambda B_2-3d_2^2-6d_2C_1R-3B_1^2+r^2B_2+\beta^2B_2=0,\tag{23}
$$

$$
-6A_0C_1-6A_2d_2-6A_1d_1+\beta^2C_1+C_1R+\lambda C_1+r^2C_1-6B_1C_2=0,\tag{24}
$$

$$
2d_1R^2-6B_2d_1-6B_1d_2=0,\tag{25}
$$

$$
-6A_0d_2+\lambda d_2-6B_1d_1+5d_2R+\beta^2d_2-6B_2C_1+r^2d_2=0,\tag{26}
$$

$$
\beta^2d_1-6B_2C_2+d_1R-6A_1d_2-6A_0d_1+r^2d_1+\lambda d_1-6B_1C_1=0,\tag{27}
$$

$$
\lambda C_2+r^2C_2+\beta^2C_2+5C_2R-6A_0C_2-6A_2d_1-6A_1C_1=0,\tag{28}
$$

$$
-6B_1B_2+2B_1R^2-6d_1d_2R=0,\tag{29}
$$

$$
-6B_2d_2+6d_2R^2=0,\tag{30}
$$

With the aid of *Maple* and applying Wu-elimination [25,26], we have

Case 1

$$
\begin{aligned}
&A_0=\frac{2}{3}R,\quad A_1=0,\quad B_1=0,\quad C_1=0,\quad d_1=0,\quad A_2=0,\quad B_2=R^2,\\
&C_2=0,\quad d_2=\mp R^{\frac{3}{2}},\quad \lambda=-R-r^2-\beta^2,
\end{aligned}
$$

Case 2

$$
\begin{aligned}
&A_0=R,\quad A_1=0,\quad B_1=0,\quad C_1=0,\quad d_1=0,\quad A_2=0,\quad B_2=R^2,\\
&C_2=0,\quad d_2=\mp R^{3/2},\quad \lambda=R-r^2-\beta^2,
\end{aligned}
$$

Case 3

$$
\begin{aligned}
&A_0=4R,\quad A_1=0,\quad B_1=0,\quad C_1=0,\quad d_1=0,\quad A_2=2,\quad B_2=2R^2,\\
&C_2=0,\quad d_2=0,\quad \lambda=-\beta^2-r^2+16R,
\end{aligned}
$$

Case 4

$$
\begin{aligned}
&A_0=-\frac{4}{3}R,\quad A_1=0,\quad B_1=0,\quad C_1=0,\quad d_1=0,\quad A_2=2,\quad B_2=2R^2,\\
&C_2=0,\quad d_2=0,\quad \lambda=-\beta^2-r^2-16R,
\end{aligned}
$$

Case 5

$$
\begin{aligned}
&A_0=R,\quad A_1=0,\quad B_1=0,\quad C_1=0,\quad d_1=0,\quad A_2=1, B_2=0,\\
&C_2=\pm 1,\quad d_2=0,\quad \lambda=R-r^2-\beta^2,
\end{aligned}
$$

Case 6

$$A_0 = \frac{2}{3}R, \quad A_1 = 0, \quad B_1 = 0, \quad C_1 = 0, \quad d_1 = 0, \quad A_2 = 0, \quad B_2 = 2R^2,$$
$$C_2 = 0, \quad d_2 = 0, \quad \lambda = -4R - r^2 - \beta^2,$$

Case 7

$$A_0 = \frac{2}{3}R, \quad A_1 = 0, \quad B_1 = 0, \quad C_1 = 0, \quad d_1 = 0, \quad A_2 = 1, \quad B_2 = 0,$$
$$C_2 = \pm 1, \quad d_2 = 0, \quad \lambda = -R - r^2 - \beta^2,$$

Case 8

$$A_0 = \frac{2}{3}R, \quad A_1 = 0, \quad B_1 = 0, \quad C_1 = 0, \quad d_1 = 0, \quad A_2 = 2, \quad B_2 = 0,$$
$$C_2 = 0, \quad d_2 = 0, \quad \lambda = -4R - r^2 - \beta^2,$$

Case 9

$$A_0 = 2R, \quad A_{91} = 0, \quad B_1 = 0, \quad C_1 = 0, \quad d_1 = 0, \quad A_2 = 2, \quad B_2 = 0,$$
$$C_2 = 0, \quad d_2 = 0, \quad \lambda = -\beta^2 - r^2 + 4R,$$

Case 10

$$A_0 = 2R, \quad A_1 = 0, \quad B_1 = 0, \quad C_1 = 0, \quad d_1 = 0, A_2 = 0, \quad B_2 = 2R^2,$$
$$C_2 = 0, d_2 = 0, \quad \lambda = -\beta^2 - r^2 + 4R,$$

Therefore, combining Eq. (13) along with Cases 1–10, we obtain the travelling wave solutions of KP equation as follows

Case 1

$$u = \frac{2}{3}R + R\cot\left(\sqrt{R}\xi\right)^2 \pm R\cos\left(\sqrt{R}\xi\right)\csc\left(\sqrt{R}\xi\right)^2, \tag{31}$$

$$u = \frac{2}{3}R + R\tan\left(\sqrt{R}\xi\right)^2 \pm R\sin\left(\sqrt{R}\xi\right)\sec\left(\sqrt{R}\xi\right)^2, \tag{32}$$

$$u = \frac{2}{3}R - R\coth\left(\sqrt{-R}\xi\right)^2 \pm i\sqrt{R}\sqrt{-R}\cosh\left(\sqrt{-R}\xi\right)\mathrm{csch}\left(\sqrt{-R}\xi\right)^2, \tag{33}$$

$$u = \frac{2}{3}R - R\tanh\left(\sqrt{-R}\xi\right)^2 \pm \sqrt{R}\sqrt{-R}\sinh\left(\sqrt{-R}\xi\right)\mathrm{sech}\left(\sqrt{-R}\xi\right)^2, \tag{34}$$

where $\xi = x + \beta y + rz - (-R - r^2 - \beta^2)t$.

Case 2

$$u = R + R\cot\left(\sqrt{R}\xi\right)^2 \mp R\cos\left(\sqrt{R}\xi\right)\csc\left(\sqrt{R}\xi\right)^2, \tag{35}$$

$$u = R - R\coth\left(\sqrt{-R}\xi\right)^2 \pm i\sqrt{R}\sqrt{-R}\cosh\left(\sqrt{-R}\xi\right)\operatorname{csch}\left(\sqrt{-R}\xi\right)^2, \tag{36}$$

$$u = R - R\tanh\left(\sqrt{-R}\xi\right)^2 \pm \sqrt{R}\sqrt{-R}\sinh\left(\sqrt{-R}\xi\right)\operatorname{sech}\left(\sqrt{-R}\xi\right)^2, \tag{37}$$

where $\xi = x + \beta y + rz - (R - r^2 - \beta^2)t$.

Case 3

$$u = 4R + 2R\tan\left(\sqrt{R}\xi\right)^2 + 2R\cot\left(\sqrt{R}\xi\right)^2, \tag{38}$$

$$u = 4R - 2R\tanh\left(\sqrt{-R}\xi\right)^2 - 2R\coth\left(\sqrt{-R}\xi\right)^2, \tag{39}$$

$$u = 4R - 2R\tanh\left(\sqrt{-R}\xi\right)^2 - 2R\coth\left(\sqrt{-R}\xi\right)^2, \tag{40}$$

where $\xi = x + \beta y + rz - (-\beta^2 - r^2 + 16R)t$.

Case 4

$$u = -\frac{4}{3}R + 2R\tan\left(\sqrt{R}\xi\right)^2 + 2R\cot\left(\sqrt{R}\xi\right)^2, \tag{41}$$

$$u = -\frac{4}{3}R - 2R\tanh\left(\sqrt{-R}\xi\right)^2 - 2R\coth\left(\sqrt{-R}\xi\right)^2, \tag{42}$$

$$u = -\frac{4}{3}R - 2R\tanh\left(\sqrt{-R}\xi\right)^2 - 2R\coth\left(\sqrt{-R}\xi\right)^2, \tag{43}$$

where $\xi = x + \beta y + rz - (-\beta^2 - r^2 - 16R)t$.

Case 5

$$u = R + R\tan\left(\sqrt{R}\xi\right)^2 \pm R\tan\left(\sqrt{R}\xi\right)\sec\left(\sqrt{R}\xi\right), \tag{44}$$

$$u = R - R\tanh\left(\sqrt{-R}\xi\right)^2 \pm iR\tanh\left(\sqrt{-R}\xi\right)\operatorname{sech}\left(\sqrt{-R}\xi\right), \tag{45}$$

$$u = R - R\coth\left(\sqrt{-R}\xi\right)^2 \pm R\coth\left(\sqrt{-R}\xi\right)\operatorname{csch}\left(\sqrt{-R}\xi\right), \tag{46}$$

where $\xi = x + \beta y + rz - (R - r^2 - \beta^2)t$.

Case 6

$$u = \frac{2}{3}R + 2R\cot\left(\sqrt{R}\xi\right)^2, \tag{47}$$

$$u=\frac{2}{3}R-2R\coth\left(\sqrt{-R}\xi\right)^{2}, \tag{48}$$

$$u=\frac{2}{3}R-2R\tanh\left(\sqrt{-R}\xi\right)^{2}, \tag{49}$$

where $\xi=x+\beta y+rz-(-4R-r^{2}-\beta^{2})t$.

Case 7

$$u=\frac{2}{3}R+R\tan\left(\sqrt{R}\xi\right)^{2}\pm R\tan\left(\sqrt{R}\xi\right)\sec\left(\sqrt{R}\xi\right), \tag{50}$$

$$u=\frac{2}{3}R-R\tanh\left(\sqrt{-R}\xi\right)^{2}\pm iR\tanh\left(\sqrt{-R}\xi\right)\mathrm{sech}\left(\sqrt{-R}\xi\right), \tag{51}$$

$$u=\frac{2}{3}R-R\coth\left(\sqrt{-R}\xi\right)^{2}\pm R\coth\left(\sqrt{-R}\xi\right)\mathrm{csch}\left(\sqrt{-R}\xi\right), \tag{52}$$

where $\xi=x+\beta y+rz-(-R-r^{2}-\beta^{2})t$.

Case 8

$$u=\frac{2}{3}R+2R\tan\left(\sqrt{R}\xi\right)^{2}, \tag{53}$$

$$u=\frac{2}{3}R-2R\tanh\left(\sqrt{-R}\xi\right)^{2}, \tag{54}$$

$$u=\frac{2}{3}R-2R\coth\left(\sqrt{-R}\xi\right)^{2}, \tag{55}$$

where $\xi=x+\beta y+rz-\left(-4R-r^{2}-\beta^{2}\right)t$.

Case 9

$$u=2R+2R\tan\left(\sqrt{R}\xi\right)^{2}, \tag{56}$$

$$u=2R-2R\tanh\left(\sqrt{-R}\xi\right)^{2}, \tag{57}$$

$$u=2R-2R\coth\left(\sqrt{-R}\xi\right)^{2}, \tag{58}$$

where $\xi=x+\beta y+rz-\left(-\beta^{2}-r^{2}+4R\right)t$.

Case 10

$$u=2R+2R\cot\left(\sqrt{R}\xi\right)^{2}, \tag{59}$$

$$u=2R-2R\coth\left(\sqrt{-R}\xi\right)^{2}, \tag{60}$$

$$u=2R-2R\tanh\left(\sqrt{-R}\xi\right)^{2}, \tag{61}$$

where $\xi=x+\beta y+rz-(-\beta^{2}-r^{2}+4R)t$.

Remark 1. From the families of obtained solutions, it is not difficult to see that we not only recover the previously known solitary solutions found by the extended tanh-function method [21] but also can obtain new formal solutions, such as the solutions in Cases 1, 2, which cannot be found by the known tanh-method.

Example 2. Consider WBK equations [22]

$$u_t + uu_x + v_x + \beta u_{xx} = 0, \tag{62}$$

$$v_t + u_x v + uv_x + \alpha u_{xxx} - \beta v_{xx} = 0. \tag{63}$$

Firstly, we let

$$u = \phi(\xi), \quad v = \theta(\xi), \quad \xi = x - \lambda t, \tag{64}$$

then Eqs. (62) and (63) reduce into the following nonlinear ODEs

$$-\lambda\phi + \frac{1}{2}\phi^2 + \theta + \beta\phi' = 0, \tag{65}$$

$$-\lambda\theta + \phi\theta + \alpha\phi'' - \beta\theta' = 0. \tag{66}$$

Like Example 1, we support Eqs. (62) and (63) have the following formal solutions

$$\phi = a_{10} + a_{11}\omega + \frac{b_{11}}{\omega} + c_{11}\sqrt{R+\omega^2} + \frac{d_{11}\sqrt{R+\omega^2}}{\omega}, \tag{67}$$

$$\theta = a_{20} + a_{21}\omega + \frac{b_{21}}{\omega} + c_{21}\sqrt{R+\omega^2} + \frac{d_{21}\sqrt{R+\omega^2}}{\omega} + a_{22}\omega^2 + \frac{b_{22}}{\omega^2} + c_{22}\omega\sqrt{R+\omega^2} + \frac{d_{22}\sqrt{R+\omega^2}}{\omega^2} \tag{68}$$

and $\omega = \omega(\xi)$ satisfying Eq. (4) where a_{10}, a_{11}, b_{11}, c_{11}, d_{11}, a_{20}, a_{21}, b_{21}, c_{21}, d_{21}, a_{22}, b_{22}, c_{22}, d_{22}, R, λ, are constants to be determined later. With the aid of *Maple*, substituting Eqs. (67) and (68) into Eqs. (65) and (66) along with Eq. (4), and let the coefficients be zero of $\omega^p(\sqrt{(R+\omega^2)})^q$ $(q = 0, 1;\ p = 0, 1, 2, \ldots)$ with the same power, we get the following over-determined algebraic equations system

$$b_{22} - \beta b_{11}R + \frac{1}{2}b_{11}^2 + \frac{1}{2}d_{11}^2R = 0, \tag{69}$$

$$-\lambda a_{11} + c_{11}d_{11} + a_{21} + a_{10}a_{11} = 0, \tag{70}$$

$$a_{11}c_{11} + \beta c_{11} + c_{22} = 0, \tag{71}$$

$$-\lambda b_{11} + a_{10}b_{11} + c_{11}d_{11}R + b_{21} = 0, \tag{72}$$

$$b_{11}c_{11} - \lambda d_{11} + d_{21} + a_{10}d_{11} = 0, \tag{73}$$

$$c_{21} + a_{11}d_{11} - \lambda c_{11} + a_{10}c_{11} = 0, \tag{74}$$

$$a_{22} + \frac{1}{2}a_{11}^2 + \frac{1}{2}c_{11}^2 + \beta a_{11} = 0, \tag{75}$$

$$-\beta d_{11}R + b_{11}d_{11} + d_{22} = 0, \tag{76}$$

$$-\beta b_{11} + \frac{1}{2}d_{11}^2 + a_{20} + a_{11}b_{11} + \frac{1}{2}a_{10}^2 + \frac{1}{2}c_{11}^2 R + \beta a_{11}R - \lambda a_{10} = 0, \tag{77}$$

$$d_{11}d_{22}R + 2\beta b_{22}R + b_{11}b_{22} + 2\alpha b_{11}R^2 = 0, \tag{78}$$

$$d_{11}c_{22}R - \beta a_{21}R + d_{11}d_{21} + \beta b_{21} + a_{11}b_{21} + c_{11}d_{22} - \lambda a_{20} + c_{11}c_{21}R + a_{10}a_{20} + b_{11}a_{21} = 0, \tag{79}$$

$$\alpha c_{11}R + a_{11}d_{21} + d_{11}a_{21} - \beta c_{22}R + c_{11}a_{20} + a_{10}c_{21} - \lambda c_{21} + b_{11}c_{22} = 0, \tag{80}$$

$$a_{10}b_{22} + c_{11}d_{22}R + d_{11}d_{21}R - \lambda b_{22} + b_{11}b_{21} + \beta b_{21}R = 0, \tag{81}$$

$$\beta d_{21}R + b_{11}d_{21} + a_{10}d_{22} + d_{11}b_{21} - \lambda d_{22} + c_{11}b_{22} = 0, \tag{82}$$

$$\alpha d_{11}R + b_{11}c_{21} + a_{10}d_{21} + c_{11}b_{21} + a_{11}d_{22} + d_{11}a_{20} + \beta d_{22} - \lambda d_{21} = 0, \tag{83}$$

$$c_{11}c_{22}R + b_{11}a_{22} + 2\alpha a_{11}R + d_{11}c_{21} + c_{11}d_{21} - \lambda a_{21} - 2\beta a_{22}R + a_{11}a_{20} + a_{10}a_{21} = 0, \tag{84}$$

$$2\beta d_{22}R + b_{11}d_{22} + d_{11}b_{22} + 2\alpha d_{11}R^2 = 0, \tag{85}$$

$$-2\beta c_{22} + c_{11}a_{22} + a_{11}c_{22} + 2\alpha c_{11} = 0, \tag{86}$$

$$d_{11}a_{22} + a_{10}c_{22} - \beta c_{21} - \lambda c_{22} + a_{11}c_{21} + c_{11}a_{21} = 0, \tag{87}$$

$$2\alpha b_{11}R + d_{11}d_{22} + d_{11}c_{21}R + a_{11}b_{22} - \lambda b_{21} + b_{11}a_{20} + c_{11}d_{21}R + 2\beta b_{22} + a_{10}b_{21} = 0, \tag{88}$$

$$-\lambda a_{22} - \beta a_{21} + a_{10}a_{22} + c_{11}c_{21} + d_{11}c_{22} + a_{11}a_{21} = 0, \tag{89}$$

$$2\alpha a_{11} - 2\beta a_{22} + c_{11}c_{22} + a_{11}a_{22} = 0, \tag{90}$$

With the aid of Maple and applying Wu-elimination [25,26], we have

Case 1

$$a_{10} = \pm i d_{11}, \quad a_{11} = 0, \quad b_{11} = -\frac{d_{11}^2}{\sqrt{\alpha+\beta^2}}, \quad c_{11} = 0,$$

$$a_{20} = -\frac{d_{11}^2\left(\sqrt{\alpha+\beta^2}+\beta\right)}{\sqrt{\alpha+\beta^2}}, \quad a_{21} = 0, \quad b_{21} = 0 \quad c_{21} = 0, \quad d_{21} = 0,$$

$$a_{22} = 0, b_{22} = -\frac{d_{11}^4((\alpha+\beta^2)^{\frac{3}{2}} + \beta\alpha + \beta^3)}{(\alpha+\beta^2)^{\frac{5}{2}}}, \quad c_{22} = 0,$$

$$d_{22} = \frac{d_{11}^3\left(\beta\sqrt{\alpha+\beta^2}+\alpha+\beta^2\right)}{(\alpha+\beta^2)^{\frac{3}{2}}}, \quad R = \frac{d_{11}^2}{\alpha+\beta^2}, \quad \lambda = \pm i d_{11},$$

Case 2

$$a_{10}=\mp\frac{1}{2}\sqrt{2}d_{11},\quad a_{11}=0,\quad b_{11}=0,\quad c_{11}=0,\quad a_{20}=-\frac{1}{4}d_{11}^2,$$
$$a_{21}=0,\quad b_{21}=0,\quad c_{21}=0,\quad d_{21}=0,\quad a_{22}=0,\quad b_{22}=-\frac{1}{8}\frac{d_{11}^4}{\alpha+\beta^2},$$
$$c_{22}=0,\quad d_{22}=\frac{1}{4}\frac{\beta d_{11}^3}{\alpha+\beta^2},\quad R=\frac{1}{4}\frac{d_{11}^2}{\alpha+\beta^2},\quad \lambda=\mp\frac{1}{2}\sqrt{2}d_{11},$$

Case 3

$$a_{10}=\pm id_{11},\quad a_{11}=0,\quad b_{11}=\frac{d_{11}^2}{\sqrt{\alpha+\beta^2}},\quad c_{11}=0,$$
$$a_{20}=-\frac{d_{11}^2\left(\sqrt{\alpha+\beta^2}-\beta\right)}{\sqrt{\alpha+\beta^2}},\quad a_{21}=0,\quad b_{21}=0,\quad c_{21}=0,\quad d_{21}=0,$$
$$a_{22}=0,\quad b_{22}=-\frac{d_{11}^4((\alpha+\beta^2)^{\frac{3}{2}}-\beta\alpha-\beta^3)}{(\alpha+\beta^2)^{\frac{5}{2}}},\quad c_{22}=0,$$
$$d_{22}=\frac{d_{11}^3\left(\beta\sqrt{\alpha+\beta^2}-\alpha-\beta^2\right)}{(\alpha+\beta^2)^{\frac{3}{2}}},\quad R=\frac{d_{11}^2}{\alpha+\beta^2},\quad \lambda=\pm id_{11},$$

Case 4

$$a_{10}=4\sqrt{-R\alpha-\beta^2R},\quad a_{11}=\pm2\sqrt{\alpha+\beta^2},\quad b_{11}=\mp2\sqrt{\alpha+\beta^2}R,$$
$$c_{11}=0,\quad d_{11}=0,\quad a_{20}=\mp4\beta\sqrt{\alpha+\beta^2}R-4R\alpha-4\beta^2R,\quad a_{21}=0,$$
$$b_{21}=0,\quad c_{21}=0,\quad d_{21}=0,\quad a_{22}=-2\alpha-2\beta^2\mp2\beta\sqrt{\alpha+\beta^2},$$
$$b_{22}=-2\alpha R^2-2\beta^2R^2\mp2\beta\sqrt{\alpha+\beta^2}R^2,\quad c_{22}=0,\quad d_{22}=0,$$
$$\lambda=4\sqrt{-R\alpha-\beta^2R},$$

Case 5

$$a_{10}=\mp2\sqrt{2\beta^2R+2R\alpha},\quad a_{11}=\pm2\sqrt{\alpha+\beta^2},\quad b_{11}=\pm2\sqrt{\alpha+\beta^2}R,$$
$$c_{11}=d_{11}=0,\quad a_{20}=0,\quad a_{21}=0,\quad b_{21}=0,\quad c_{21}=0,\quad d_{21}=0,$$
$$a_{22}=-2\alpha-2\beta^2\mp2\beta\sqrt{\alpha+\beta^2},\quad b_{22}=-2\alpha R^2-2\beta^2R^2\pm2\beta\sqrt{\alpha+\beta^2}R^2,$$
$$c_{22}=0,\quad d_{22}=0,\quad \lambda=\mp2\sqrt{2\beta^2R+2R\alpha},$$

Case 6

$$a_{10} = -4\sqrt{-R\alpha - \beta^2 R}, \quad a_{11} = \pm 2\sqrt{\alpha + \beta^2}, \quad b_{11} = \mp 2\sqrt{\alpha + \beta^2}R,$$
$$c_{11} = 0, \quad d_{11} = 0, \quad a_{20} = \mp 4\beta\sqrt{\alpha + \beta^2}R - 4R\alpha - 4\beta^2 R, \quad a_{21} = 0,$$
$$b_{21} = 0, \quad c_{21} = 0, \quad d_{21} = 0, \quad a_{22} = -2\alpha - 2\beta^2 \mp 2\beta\sqrt{\alpha + \beta^2},$$
$$b_{22} = -2\alpha R^2 - 2\beta^2 R^2 \mp 2\beta\sqrt{\alpha + \beta^2}R^2, \quad c_{22} = 0, \quad d_{22} = 0,$$
$$\lambda = -4\sqrt{-R\alpha - \beta^2 R},$$

Case 7

$$a_{10} = -\sqrt{-R\alpha - \beta^2 R}, \quad a_{11} = \pm\sqrt{\alpha + \beta^2}, \quad b_{11} = 0, \quad c_{11} = \mp\sqrt{\alpha + \beta^2},$$
$$d_{11} = 0, \quad a_{20} = -R\alpha - \beta^2 R \mp \beta\sqrt{\alpha + \beta^2}R, \quad a_{21} = 0, \quad b_{21} = 0,$$
$$c_{21} = 0, \quad d_{21} = 0, \quad a_{22} = \mp\beta\sqrt{\alpha + \beta^2} - \alpha - \beta^2, \quad b_{22} = 0,$$
$$c_{22} = \pm\beta\sqrt{\alpha + \beta^2} + \alpha + \beta^2, \quad d_{22} = 0, \quad \lambda = -\sqrt{-R\alpha - \beta^2 R},$$

Case 8

$$a_{10} = \pm\sqrt{-R\alpha - \beta^2 R}, \quad a_{11} = \mp\sqrt{\alpha + \beta^2}, \quad c_{11} = \sqrt{\alpha + \beta^2}, \quad b_{11} = 0,$$
$$d_{11} = 0, \quad a_{20} = -R\alpha - \beta^2 R \pm \beta\sqrt{\alpha + \beta^2}R, \quad a_{21} = 0, \quad b_{21} = 0,$$
$$c_{21} = 0, \quad d_{21} = 0, \quad a_{22} = \pm\beta\sqrt{\alpha + \beta^2} - \alpha - \beta^2, \quad b_{22} = 0,$$
$$c_{22} = -\beta\sqrt{\alpha + \beta^2} \pm \alpha \pm \beta^2, \quad d_{22} = 0, \quad \lambda = \pm\sqrt{-R\alpha - \beta^2 R},$$

Case 9

$$a_{10} = \pm\sqrt{-R\alpha - \beta^2 R}, \quad a_{11} = \pm\sqrt{\alpha + \beta^2}, \quad b_{11} = 0, \quad c_{11} = \pm\sqrt{\alpha + \beta^2},$$
$$d_{11} = 0, \quad a_{20} = -R\alpha - \beta^2 R \mp \beta\sqrt{\alpha + \beta^2}R, \quad a_{21} = 0, \quad b_{21} = 0,$$
$$c_{21} = 0, \quad d_{21} = 0, \quad a_{22} = \mp\beta\sqrt{\alpha + \beta^2} - \alpha - \beta^2, \quad b_{22} = 0,$$
$$c_{22} = \mp\beta\sqrt{\alpha + \beta^2} - \alpha - \beta^2, \quad d_{22} = 0, \quad \lambda = \pm\sqrt{-R\alpha - \beta^2 R},$$

Case 10

$$a_{10} = \mp\sqrt{2\beta^2R + 2R\alpha}, \quad a_{11} = 0, \quad b_{11} = 0, \quad c_{11} = -2\sqrt{\alpha + \beta^2},$$
$$d_{11} = 0, \quad a_{20} = -R\alpha - \beta^2R, \quad a_{21} = 0, \quad b_{21} = 0, \quad c_{21} = 0, \quad d_{21} = 0,$$
$$a_{22} = -2\alpha - 2\beta^2, \quad b_{22} = 0, \quad c_{22} = 2\beta\sqrt{\alpha + \beta^2}, \quad d_{22} = 0,$$
$$\lambda = \mp\sqrt{2\beta^2R + 2R\alpha},$$

Case 11

$$a_{10} = \pm\sqrt{2\beta^2R + 2R\alpha}, \quad a_{11} = 0, b_{11} = 0, \quad c_{11} = 2\sqrt{\alpha + \beta^2},$$
$$d_{11} = 0, \quad a_{20} = -R\alpha - \beta^2R, \quad a_{21} = 0, \quad b_{21} = 0, \quad c_{21} = 0,$$
$$d_{21} = 0, \quad a_{22} = -2\alpha - 2\beta^2, \quad b_{22} = 0, \quad c_{22} = -2\beta\sqrt{\alpha + \beta^2},$$
$$d_{22} = 0, \quad \lambda = \pm\sqrt{2\beta^2R + 2R\alpha},$$

Case 12

$$a_{10} = \pm 2\sqrt{-R\alpha - \beta^2R}, \quad a_{11} = 2\sqrt{\alpha + \beta^2}, \quad b_{11} = 0, \quad c_{11} = 0,$$
$$d_{11} = 0, \quad a_{20} = -2\beta\sqrt{\alpha + \beta^2}R - 2R\alpha - 2\beta^2R, \quad a_{21} = 0, \quad b_{21} = 0,$$
$$c_{21} = 0, \quad d_{21} = 0, \quad a_{22} = -2\alpha - 2\beta^2 - 2\beta\sqrt{\alpha + \beta^2}, \quad b_{22} = 0,$$
$$c_{22} = 0, \quad d_{22} = 0, \quad \lambda = \pm 2\sqrt{-R\alpha - \beta^2R},$$

Case 13

$$a_{10} = \sqrt{-R\alpha - \beta^2R}, \quad a_{11} = \pm\sqrt{\alpha + \beta^2}, \quad b_{11} = 0, \quad c_{11} = -\sqrt{\alpha + \beta^2},$$
$$d_{11} = 0, \quad a_{20} = -R\alpha - \beta^2R \mp \beta\sqrt{\alpha + \beta^2}R, \quad a_{21} = 0, \quad b_{21} = 0,$$
$$c_{21} = 0, \quad d_{21} = 0, \quad d_{22} = 0, \quad a_{22} = \mp\beta\sqrt{\alpha + \beta^2} - \alpha - \beta^2,$$
$$b_{22} = 0, \quad c_{22} = \beta\sqrt{\alpha + \beta^2} \pm \alpha \pm \beta^2, \quad \lambda = \sqrt{-R\alpha - \beta^2R},$$

Case 14

$$a_{10} = 2\sqrt{-R\alpha - \beta^2R}, \quad a_{11} = 0, \quad b_{11} = \pm 2\sqrt{\alpha + \beta^2}R, \quad c_{11} = 0,$$
$$d_{11} = 0, \quad a_{20} = \pm 2\beta\sqrt{\alpha + \beta^2}R - 2R\alpha - 2\beta^2R, \quad a_{21} = 0, \quad b_{21} = 0,$$
$$c_{21} = 0, \quad d_{21} = 0, \quad a_{22} = 0, \quad b_{22} = -2\alpha R^2 - 2\beta^2R^2 \pm 2\beta\sqrt{\alpha + \beta^2}R^2,$$
$$c_{22} = 0, \quad d_{22} = 0, \quad \lambda = 2\sqrt{-R\alpha - \beta^2R},$$

Case 15

$$a_{10} = \pm 2\sqrt{-R\alpha - \beta^2 R}, \quad a_{11} = -2\sqrt{\alpha + \beta^2}, \quad b_{11} = 0, \quad c_{11} = 0,$$
$$d_{11} = 0, \quad a_{20} = 2\beta\sqrt{\alpha + \beta^2}R - 2R\alpha - 2\beta^2 R, \quad a_{21} = 0, \quad b_{21} = 0,$$
$$c_{21} = 0, \quad d_{21} = 0, \quad a_{22} = -2\alpha - 2\beta^2 + 2\beta\sqrt{\alpha + \beta^2}, \quad b_{22} = 0,$$
$$c_{22} = 0, \quad d_{22} = 0, \quad \lambda = \pm 2\sqrt{-R\alpha - \beta^2 R},$$

Case 16

$$a_{10} = -2\sqrt{-R\alpha - \beta^2 R}, \quad a_{11} = 0, \quad b_{11} = \pm 2\sqrt{\alpha + \beta^2}R, \quad c_{11} = 0,$$
$$d_{11} = 0, \quad a_{20} = \pm 2\beta\sqrt{\alpha + \beta^2}R - 2R\alpha - 2\beta^2 R, \quad a_{21} = 0, \quad b_{21} = 0,$$
$$c_{21} = 0, \quad d_{21} = 0, \quad a_{22} = 0, \quad b_{22} = -2\alpha R^2 - 2\beta^2 R^2 \pm 2\beta\sqrt{\alpha + \beta^2}R^2,$$
$$c_{22} = 0, \quad d_{22} = 0, \quad \lambda = -2\sqrt{-R\alpha - \beta^2 R},$$

Therefore, combining Eqs. (67) and (68) along with Cases 1–17, we obtain the travelling wave solutions of WBK equation as follows.

Case 1

$$u = \pm i d_{11} + d_{11}^2 \coth\left(\sqrt{-\frac{d_{11}^2}{\alpha + \beta^2}}\xi\right) \frac{1}{\sqrt{\alpha + \beta^2}} \frac{1}{\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}}}$$
$$- i d_{11}\mathrm{csch}\left(\sqrt{-\frac{d_{11}^2}{\alpha + \beta^2}}\xi\right), \tag{91}$$

$$v = -\frac{d_{11}^2\left(\sqrt{\alpha + \beta^2} + \beta\right)}{\sqrt{\alpha + \beta^2}} + d_{11}^2\Big(\left(\alpha + \beta^2\right)^{\frac{3}{2}}$$
$$+ \beta\alpha + \beta^3\Big) \coth\left(\sqrt{-\frac{d_{11}^2}{\alpha + \beta^2}}\xi\right)^2 (\alpha + \beta^2)^{-\frac{3}{2}}$$
$$- i d_{11}\left(\beta\sqrt{\alpha + \beta^2} + \alpha + \beta^2\right)\sqrt{-\frac{d_{11}^2}{\alpha + \beta^2}} \cosh\left(\sqrt{-\frac{d_{11}^2}{\alpha + \beta^2}}\xi\right)$$
$$\times \mathrm{csch}\left(\sqrt{-\frac{d_{11}^2}{\alpha + \beta^2}}\xi\right)^2 \frac{1}{\sqrt{\alpha + \beta^2}}, \tag{92}$$

where $\xi = x \mp i d_{11} t$.

Case 2

$$u = \mp \frac{1}{2}\sqrt{2} d_{11} - i d_{11} \operatorname{csch}\left(\frac{1}{2}\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}}\xi\right), \tag{93}$$

$$v = -\frac{1}{4} d_{11}^2 + \frac{1}{2} d_{11}^2 \coth\left(\frac{1}{2}\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}}\xi\right)^2 - \frac{1}{2} i\beta d_{11}\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}} \cosh\left(\frac{1}{2}\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}}\xi\right) \operatorname{csch}\left(\frac{1}{2}\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}}\xi\right)^2, \tag{94}$$

where $\xi = x \pm \frac{1}{2}\sqrt{2} d_{11} t$.

Case 3

$$u = \mp i d_{11} - d_{11}^2 \coth\left(\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}}\xi\right) \frac{1}{\sqrt{\alpha+\beta^2}} \frac{1}{\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}}} - i d_{11} \operatorname{csch}\left(\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}}\xi\right), \tag{95}$$

$$v = -\frac{d_{11}^2\left(\sqrt{\alpha+\beta^2}-\beta\right)}{\sqrt{\alpha+\beta^2}} + d_{11}^2\left(\left(\alpha+\beta^2\right)^{\frac{3}{2}} - \beta\alpha - \beta^3\right) \coth\left(\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}}\xi\right)^2 \left(\alpha+\beta^2\right)^{-\frac{3}{2}} - i d_{11}\left(\beta\sqrt{\alpha+\beta^2} - \alpha - \beta^2\right)\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}} \cosh\left(\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}}\xi\right) \times \operatorname{csch}\left(\sqrt{-\frac{d_{11}^2}{\alpha+\beta^2}}\xi\right)^2 \frac{1}{\sqrt{\alpha+\beta^2}}, \tag{96}$$

where $\xi = x \pm i d_{11} t$.

Case 4

$$u = 4\sqrt{-R\alpha - \beta^2 R} \mp 2\sqrt{\alpha + \beta^2}\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) \pm 2\frac{\sqrt{\alpha + \beta^2}R\coth\left(\sqrt{-R}\xi\right)}{\sqrt{-R}}, \tag{97}$$

$$v = \mp 4\beta\sqrt{\alpha + \beta^2}R - 4R\alpha - 4\beta^2 R - \left(-2\alpha - 2\beta^2 \mp 2\beta\sqrt{\alpha + \beta^2}\right)R \times \tanh\left(\sqrt{-R}\xi\right)^2 - \frac{\left(-2\alpha R^2 - 2\beta^2 R^2 \mp 2\beta\sqrt{\alpha + \beta^2}R^2\right)\coth\left(\sqrt{-R}\xi\right)^2}{R}, \tag{98}$$

where $\xi = x - 4\sqrt{-R\alpha - \beta^2 Rt}$.

Case 5

$$u = \mp 2\sqrt{2\beta^2 R + 2R\alpha} \mp 2\sqrt{\alpha + \beta^2}\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) \mp 2\frac{\sqrt{\alpha + \beta^2}R\coth\left(\sqrt{-R}\xi\right)}{\sqrt{-R}}, \tag{99}$$

$$v = -\left(-2\alpha - 2\beta^2 \mp 2\beta\sqrt{\alpha + \beta^2}\right)R\tanh\left(\sqrt{-R}\xi\right)^2 - \frac{\left(-2\alpha R^2 - 2\beta^2 R^2 \pm 2\beta\sqrt{\alpha + \beta^2}R^2\right)\coth\left(\sqrt{-R}\xi\right)^2}{R}, \tag{100}$$

where $\xi = x \pm 2\sqrt{2\beta^2 R + 2R\alpha t}$.

Case 6

$$u = -4\sqrt{-R\alpha - \beta^2 R} \mp 2\sqrt{\alpha + \beta^2}\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) \pm 2\frac{\sqrt{\alpha + \beta^2}R\coth\left(\sqrt{-R}\xi\right)}{\sqrt{-R}}, \tag{101}$$

$$v = \mp 4\beta\sqrt{\alpha + \beta^2}R - 4R\alpha - 4\beta^2 R - \left(-2\alpha - 2\beta^2 \mp 2\beta\sqrt{\alpha + \beta^2}\right)R\tanh\left(\sqrt{-R}\xi\right)^2 - \frac{\left(-2\alpha R^2 - 2\beta^2 R^2 \mp 2\beta\sqrt{\alpha + \beta^2}R^2\right)\coth\left(\sqrt{-R}\xi\right)^2}{R}, \tag{102}$$

where $\xi = x + 4\sqrt{-R\alpha - \beta^2 Rt}$.

Case 7

$$u = -\sqrt{-R\alpha - \beta^2 R} \mp \sqrt{\alpha + \beta^2}\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) \mp i\sqrt{\alpha + \beta^2}\sqrt{-R}\mathrm{sech}\left(\sqrt{-R}\xi\right), \quad (103)$$

$$v = -R\alpha - \beta^2 R \mp \beta\sqrt{\alpha + \beta^2}R - \left(\mp\beta\sqrt{\alpha + \beta^2} - \alpha - \beta^2\right)R\tanh\left(\sqrt{-R}\xi\right)^2 + i\left(\mp\beta\sqrt{\alpha + \beta^2} + \alpha + \beta^2\right)R\tanh\left(\sqrt{-R}\xi\right)\mathrm{sech}\left(\sqrt{-R}\xi\right), \quad (104)$$

where $\xi = x + \sqrt{-R\alpha - \beta^2 R}t$.

Case 8

$$u = \pm\sqrt{-R\alpha - \beta^2 R} \pm \sqrt{\alpha + \beta^2}\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) + i\sqrt{\alpha + \beta^2}\sqrt{-R}\mathrm{sech}\left(\sqrt{-R}\xi\right), \quad (105)$$

$$v = -R\alpha - \beta^2 R \pm \beta\sqrt{\alpha + \beta^2}R - \left(\pm\beta\sqrt{\alpha + \beta^2} - \alpha - \beta^2\right)R\tanh\left(\sqrt{-R}\xi\right)^2 + i\left(-\beta\sqrt{\alpha + \beta^2} \pm \alpha \pm \beta^2\right)R\tanh\left(\sqrt{-R}\xi\right)\mathrm{sech}\left(\sqrt{-R}\xi\right), \quad (106)$$

where $\xi = x \mp \sqrt{-R\alpha - \beta^2 R}t$.

Case 9

$$u = \pm\sqrt{-R\alpha - \beta^2 R} \mp \sqrt{\alpha + \beta^2}\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) \pm i\sqrt{\alpha + \beta^2}\sqrt{-R}\mathrm{sech}\left(\sqrt{-R}\xi\right), \quad (107)$$

$$v = -R\alpha - \beta^2 R \mp \beta\sqrt{\alpha + \beta^2}R - \left(\mp\beta\sqrt{\alpha + \beta^2} - \alpha - \beta^2\right)R\tanh\left(\sqrt{-R}\xi\right)^2 + i\left(\mp\beta\sqrt{\alpha + \beta^2} - \alpha - \beta^2\right)R\tanh\left(\sqrt{-R}\xi\right)\mathrm{sech}\left(\sqrt{-R}\xi\right), \quad (108)$$

where $\xi = x \mp \sqrt{-R\alpha - \beta^2 R}t$.

Case 10

$$u = \mp\sqrt{2\beta^2 R + 2R\alpha} - 2i\sqrt{\alpha+\beta^2}\sqrt{-R}\,\mathrm{sech}\left(\sqrt{-R}\xi\right), \tag{109}$$

$$\begin{aligned} v = {} & -R\alpha - \beta^2 R - (-2\alpha - 2\beta^2)R\tanh\left(\sqrt{-R}\xi\right)^2 \\ & + 2i\beta\sqrt{\alpha+\beta^2}R\tanh\left(\sqrt{-R}\xi\right)\mathrm{sech}\left(\sqrt{-R}\xi\right), \end{aligned} \tag{110}$$

where $\xi = x \pm \sqrt{2\beta^2 R + 2R\alpha t}$.

Case 11

$$u = \pm\sqrt{2\beta^2 R + 2R\alpha} + 2i\sqrt{\alpha+\beta^2}\sqrt{-R}\,\mathrm{sech}\left(\sqrt{-R}\xi\right), \tag{111}$$

$$\begin{aligned} v = {} & -R\alpha - \beta^2 R - (-2\alpha - 2\beta^2)R\left(\tanh\left(\sqrt{-R}\xi\right)\right)^2 \\ & - 2i\beta\sqrt{\alpha+\beta^2}R\tanh\left(\sqrt{-R}\xi\right)\mathrm{sech}\left(\sqrt{-R}\xi\right), \end{aligned} \tag{112}$$

where $\xi = x \mp \sqrt{2\beta^2 R + 2R\alpha t}$.

Case 12

$$u = \pm 2\sqrt{-R\alpha - \beta^2 R} - 2\sqrt{\alpha+\beta^2}\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right), \tag{113}$$

$$\begin{aligned} v = {} & -2\beta\sqrt{\alpha+\beta^2}R - 2R\alpha - 2\beta^2 R \\ & - \left(-2\alpha - 2\beta^2 - 2\beta\sqrt{\alpha+\beta^2}\right)R\tanh\left(\sqrt{-R}\xi\right)^2, \end{aligned} \tag{114}$$

where $\xi = x \mp 2\sqrt{-R\alpha - \beta^2 R t}$.

Case 13

$$\begin{aligned} u = {} & \sqrt{-R\alpha - \beta^2 R} \mp \sqrt{\alpha+\beta^2}\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) \\ & - i\sqrt{\alpha+\beta^2}\sqrt{-R}\,\mathrm{sech}\left(\sqrt{-R}\xi\right), \end{aligned} \tag{115}$$

$$\begin{aligned} v = {} & -R\alpha - \beta^2 R \mp \beta\sqrt{\alpha+\beta^2}R \\ & - \left(\mp\beta\sqrt{\alpha+\beta^2} - \alpha - \beta^2\right)R\tanh\left(\sqrt{-R}\xi\right)^2 \\ & + i\left(\beta\sqrt{\alpha+\beta^2} \pm \alpha \pm \beta^2\right)R\tanh\left(\sqrt{-R}\xi\right)\mathrm{sech}\left(\sqrt{-R}\xi\right), \end{aligned} \tag{116}$$

where $\xi = x - \sqrt{-R\alpha - \beta^2 R t}$.

Case 14

$$u = 2\sqrt{-R\alpha - \beta^2 R} \mp 2\frac{\sqrt{\alpha + \beta^2}R\coth\left(\sqrt{-R}\xi\right)}{\sqrt{-R}}, \tag{117}$$

$$v = \pm 2\beta\sqrt{\alpha + \beta^2}R - 2R\alpha - 2\beta^2 R - \frac{\left(-2\alpha R^2 - 2\beta^2 R^2 \pm 2\beta\sqrt{\alpha + \beta^2}R^2\right)\coth\left(\sqrt{-R}\xi\right)^2}{R}, \tag{118}$$

where $\xi = x - 2\sqrt{-R\alpha - \beta^2 R}t$.

Case 15

$$u = \pm 2\sqrt{-R\alpha - \beta^2 R} + 2\sqrt{\alpha + \beta^2}\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right), \tag{119}$$

$$v = 2\beta\sqrt{\alpha + \beta^2}R - 2R\alpha - 2\beta^2 R - \left(-2\alpha - 2\beta^2 + 2\beta\sqrt{\alpha + \beta^2}\right)R\tanh\left(\sqrt{-R}\xi\right)^2, \tag{120}$$

where $\xi = x \mp 2\sqrt{-R\alpha - \beta^2 R}t$.

Case 16

$$u = -2\sqrt{-R\alpha - \beta^2 R} \mp 2\frac{\sqrt{\alpha + \beta^2}R\coth\left(\sqrt{-R}\xi\right)}{\sqrt{-R}}, \tag{121}$$

$$v = \pm 2\beta\sqrt{\alpha + \beta^2}R - 2R\alpha - 2\beta^2 R - \frac{\left(-2\alpha R^2 - 2\beta^2 R^2 \pm 2\beta\sqrt{\alpha + \beta^2}R^2\right)\coth\left(\sqrt{-R}\xi\right)^2}{R}, \tag{122}$$

where $\xi = x + 2\sqrt{-R\alpha - \beta^2 R}t$.

Remark 2. Just like Remark 1, we also found new formal solutions that cannot be found by the known tanh-methods. As some illustrative samples, the properties of the solutions are shown by following some figures.

4. Conclusions

In this paper, we present the generalized tanh method by introducing a more general ansätz (3) and, with aid of *maple*, implement it in a computer algebraic system. The validity of the method is tested by applying it to KP equation and WBK equations with physics interests. By applying the improved method, we drive many travelling wave solutions for this two system. We can successfully recover the previously known solitary wave solutions that had been found by the extended tanh-function method and other more sophisticated methods. More importantly, for KP equation and WBK equations, we also obtain other new and more general solutions at same time. In addition, as some illustrative samples, the properties of new formal the soliton solutions for WBK equations are shown by some figures. The method can be used to many other nonlinear equations or coupled ones. In addition, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculation on a computer.

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