Commun. Theor. Phys. (Beijing, China) **41** (2004) pp. 1–6
Vol. 41, No. 1, January 15, 2004

New Exact Travelling Wave Solutions for Generalized Zakharov–Kuzentsov Equations Using General Projective Riccati Equation Method*

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(Received May 6, 2003; Revised June 16, 2003)

Abstract *Applying the generalized method, which is a direct and unified algebraic method for constructing multiple travelling wave solutions of nonlinear partial differential equations (PDEs), and implementing in a computer algebraic system, we consider the generalized Zakharov–Kuzentsov equation with nonlinear terms of any order. As a result, we can not only successfully recover the previously known travelling wave solutions found by existing various tanh methods and other sophisticated methods, but also obtain some new formal solutions. The solutions obtained include kink-shaped solitons, bell-shaped solitons, singular solitons, and periodic solutions.*

PACS numbers: 02.30.Jr, 05.45.Yv
Key words: projective Riccati equation method, generalized Zakharov–Kuzentsov equation, exact solutions

1 Introduction

In recent years, there has been a great amount of activity aiming to find methods for the exact solutions of nonlinear differential equations. Many powerful methods have been presented, such as inverse scattering transform method,[1] Bäcklund transformation,[1,2] variable separation approach,[3] Darboux transformation,[4] various tanh methods,[5−8], homogeneous balance method,[9] and similarity reductions method,[10,11] and so on.

In Ref. [12], many solitary wave solutions of nonlinear partial differential equation can be written as a polynomial in two elementary functions that satisfy a projective (hence linearizable) Riccati system.[13] From the property, Conte and Musette deduced a method for building their solutions by determining only a finite number of coefficients. This method is much shorter and obtains more solutions than the one that consists of summing a perturbation series built from exponential solutions of the linearized equation. With the simplifying assumption of constant coefficients, one finds as solutions polynomials in two elementary bell-shaped and kink-shaped functions, and this covers the large majority of physically interesting solitary waves. Without this simplifying assumption, one finds more solutions, and one can even find the general solutions of some ODEs. Recently, Yan[14] developed Conte's method and presented the general projective Riccati equation method to find more exact solutions of nonlinear differential equations based upon a more general form of the projective Riccati equations,

$$\sigma'(\xi) = \epsilon\sigma(\xi)\tau(\xi)\,,$$
$$\tau'(\xi) = R + \epsilon\tau^2(\xi) - \mu\sigma(\xi)\,,$$
$$\epsilon = \pm 1, \quad R, \mu = \text{constants}\,, \tag{1}$$

which admits the first integral with $R \neq 0$,

$$\tau^2(\xi) = -\epsilon\Big[R - 2\mu\sigma(\xi) + \frac{\mu^2 - 1}{R}\sigma^2(\xi)\Big], \quad (R \neq 0)\,, \tag{2}$$

where $' = \mathrm{d}/\mathrm{d}\xi$. When $\epsilon = -1$, $R = 1$, $\mu \to \mu/K$, equation (1) becomes a projective Riccati equation.[13]

The present work is motivated by the desire to use the generally projective Riccati equation method to study the generalized Zakharov–Kuzentsov equations with nonlinear terms of any order,[15,16]

$$u_t + au^p u_x + bu^{2p} u_x + ru_{xy} + \delta u_{xxx} + su_{xyy} = 0\,, \tag{3}$$

where $a, b \neq 0$, r, $\delta \neq 0$, p and s are all constants. It is not difficult to see that when parameters a, b, r, δ, and s take different constants, equation (3) includes many important mathematical and physical equations, such as KdV equation, mKdV equation, ZK equation, and mZK equation, etc. (see, e.g., Refs. [15] ∼ [22] for detail).

This paper is organized as follows. In Sec. 2, we summarize the generalized method. In Sec. 3, we apply the improved method to generalized Zakharov–Kuzentsov equation and bring out many solutions. Conclusions will be presented in Sec. 4.

2 The Generalized Method

In this section, we describe the generalized method as follows.

*The project supported by National Natural Science Foundation of China under Grant No. 10072013 and the National Key Basic Research Development Program under Grant No. G1998030600

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Let us consider a given PDE, say, with three variable $\{x, y, t\}$,

$$p(u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots) = 0 \,. \tag{4}$$

Under the transformation $u(x,t) = u(\xi)$, $\xi = x - \lambda t$, equation (4) reduces to

$$G(u', u'', u''', \ldots) = 0 \,. \tag{5}$$

Step 1 Balancing the highest order derivative term and the nonlinear terms in Eq. (4) or (5), we get a balance constant m (m is usually a positive integer). If m is a fraction or a negative integer, we make the following transformation

$$u(\xi) = v^m(\xi) \,, \tag{6}$$

then return to determine balance constant m again.

Step 2 We express the solutions of Eq. (5) as the following forms.

Type 1 When $R \neq 0$ in Eqs. (1) and (2),

$$u(\xi) = A_0 + \sum_{i=1}^{m} \sigma(\xi)^{i-1}[A_i\sigma(\xi) + B_i\tau(\xi)] \,, \tag{7}$$

where $\sigma(\xi)$ and $\tau(\xi)$ satisfy Eqs. (1) and (2).

Type 2 When $R = \mu = 0$ in Eq. (1),

$$u(\xi) = \sum_{i=0}^{m} A_i\tau^i(\xi) \,, \tag{8}$$

where $\tau'(\xi) = \tau^2(\xi)$.

Step 3 When $R \neq 0$, substituting (7) along with the conditions (1) and (2) into Eq. (5) (When $R = \mu = 0$, substituting Eq. (8) along with $\tau'(\xi) = \tau^2(\xi)$ into Eq. (5)) yields a set of algebraic equations for $\sigma^j(\xi)\tau^i(\xi)$, $j = 0, 1, \ldots$; $i = 0, 1$ ($\tau^l(\xi), l = 0, 1, \ldots$). Setting the coefficients of these terms $\sigma^j(\xi)\tau^i(\xi)$ (or $\tau^l(\xi)$) to zero yields a set of over-determined algebraic equations in λ, A_i, B_i, R, and μ.

Step 4 With the aid of Maple, solving the above set of equations, yields the values of A_i, B_i, R, and λ.

Step 5 We know that Eq. (1) admits the following solutions.

Case 1 When $\epsilon = -1, R \neq 0$,

$$\begin{cases} \sigma_1(\xi) = \dfrac{R\,\mathrm{sech}(\sqrt{R}\,\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\,\xi) + 1}, & \tau_1(\xi) = \dfrac{\sqrt{R}\tanh(\sqrt{R}\,\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\,\xi) + 1}\,, \\ \sigma_2(\xi) = \dfrac{R\,\mathrm{csch}(\sqrt{R}\,\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\,\xi) + 1}, & \tau_2(\xi) = \dfrac{\sqrt{R}\coth(\sqrt{R}\,\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\,\xi) + 1}\,. \end{cases} \tag{9}$$

Case 2 When $\epsilon = 1, R \neq 0$,

$$\begin{cases} \sigma_3(\xi) = \dfrac{R\sec(\sqrt{R}\,\xi)}{\mu\sec(\sqrt{R}\,\xi) + 1}, & \tau_3(\xi) = \dfrac{\sqrt{R}\,\tan(\sqrt{R}\,\xi)}{\mu\sec(\sqrt{R}\,\xi) + 1}\,, \\ \sigma_4(\xi) = \dfrac{R\csc(\sqrt{R}\,\xi)}{\mu\csc(\sqrt{R}\,\xi) + 1}, & \tau_4(\xi) = \dfrac{-\sqrt{R}\cot(\sqrt{R}\,\xi)}{\mu\csc(\sqrt{R}\,\xi) + 1}\,. \end{cases} \tag{10}$$

Case 3 When $R = \mu = 0$,

$$\sigma_5(\xi) = \frac{C}{\xi} = C\epsilon\tau_5(\xi), \qquad \tau_5(\xi) = \frac{1}{\epsilon\xi}\,, \tag{11}$$

where C is a constant.

Thus according to Eqs. (7) $\sim$ (11) and the conclusions in Step 4, we can obtain many solutions for Eq. (4).

3 Exact Solutions of GZK Equation

Let us consider the GZK equation, i.e., Eq. (3). According to the above method, to seek travelling wave solutions of Eq. (3), we make the transformation

$$u(x, y, t) = v(\xi), \qquad \xi = x + ly + \lambda t\,, \tag{12}$$

where l and λ are constants to be determined later. Equation (3) becomes

$$\lambda v_\xi + av^p v_\xi + bv^{2p}v_\xi + rlv_{\xi\xi} + (\delta + sl^2)v_{\xi\xi\xi} = 0\,, \tag{13}$$

Integrating the above equation twice with regard to ξ, we obtain

$$(\delta + sl^2)v''(\xi) + rlv'(\xi) + \lambda v(\xi) + \frac{a}{p+1}v^{p+1}(\xi) + \frac{b}{2p+1}v^{2p+1}(\xi) = 0 \tag{14}$$

with the integration constants taken as zero. According to ***Step 1*** in Sec. 2, if $\delta \neq 0$, $b \neq 0$, and $p \neq 0, 1/2$, by balancing $v''(\xi)$ and $v^{(2p+1)}(\xi)$ in Eq. (14), we get $m = 1/p$. Therefore we make the following transformation,

$$v(\xi) = \varphi^{1/p}(\xi)\,, \tag{15}$$

then substituting Eq. (15) into Eq. (14) yields

$$a_0[p\varphi(\xi)\varphi''(\xi) + (1-p){\varphi'}^2(\xi)] + a_1\varphi(\xi)\varphi'(\xi) + p^2[a_2\varphi^2(\xi) + a_3\varphi^3(\xi) + a_4\varphi^4(\xi)] = 0\,, \tag{16}$$

where

$$a_0=(1+p)(1+2p)(\delta+sl^2),\qquad a_1=p(1+p)(1+2p)rl\,,$$
$$a_2=(1+p)(1+2p)\lambda,\qquad a_3=(1+2p)a,\qquad a_4=(1+p)b\,. \tag{17}$$

According to ***step 1*** in Sec. 2, by balancing $\varphi(\xi)\varphi''(\xi)$ (or $\varphi'^2(\xi)$) and $\varphi^4(\xi)$ in Eq. (16), we get $m=1$. Therefore we suppose that equation (16) has the following formal solutions:

$$\varphi(\xi)=A_0+A_1\sigma(\xi)+B_1\tau(\xi)\,, \tag{18}$$

where $\sigma(\xi)$ and $\tau(\xi)$ satisfies Eqs. (1) and (2), and A_0, A_1, and B_1 are constants to be determined later.

With the aid of *Maple*, substituting Eq. (18) along with Eqs. (1) and (2) into Eq. (16), yields a set of algebraic equations for $\sigma^j(\xi)\tau^i(\xi)$ $(j=0,1,\ldots;\,i=0,1)$. Setting the coefficients of these terms $\sigma^j(\xi)\tau^i(\xi)$ to zero yields a set of over-determined algebraic equations with respect to A_0, A_1, B_1, R, n, and λ (Note 1 that here we take $\epsilon=-1$).

$$2B_1A_1R(2p^2a_4RA_1^2+2B_1^2p^2a_4\mu^2-2B_1^2p^2a_4-a_0p-a_0+a_0\mu^2+a_0p\mu^2)=0\,, \tag{19}$$

$$R^2(3p^2a_3A_1B_1^2R-a_1B_1A_1R-a_0pB_1^2\mu R+12p^2a_4A_0A_1B_1^2R-4p^2a_4B_1^4\mu R+a_0pA_0A_1R-12p^2a_4A_0^2B_1^2\mu$$
$$-2p^2a_2B_1^2\mu+a_1A_0B_1\mu+3p^2a_3A_0^2A_1-6p^2a_3A_0B_1^2\mu+2p^2a_2A_0A_1+4p^2a_4A_0^3A_1)=0\,, \tag{20}$$

$$p^2B_1R^2(B_1^2Ra_3+4B_1^2Ra_4A_0+3a_3A_0^2+4a_4A_0^3+2a_2A_0)=0\,, \tag{21}$$

$$R(6R^2p^2a_4A_1^2B_1^2+R^2a_0A_1^2+3Rp^2a_3A_0A_1^2+Rp^2a_2A_1^2+6Rp^2a_4A_0^2A_1^2-2Ra_0pB_1^2$$
$$+3Ra_0pB_1^2\mu^2+3Ra_1A_1B_1\mu+6Rp^2a_4B_1^4\mu^2-6Rp^2a_3A_1B_1^2\mu-3Ra_0pA_0A_1\mu$$
$$+Ra_0B_1^2\mu^2-24Rp^2a_4A_0A_1B_1^2\mu-2Rp^2a_4B_1^4-6p^2a_4A_0^2B_1^2-p^2a_2B_1^2-3p^2a_3A_0B_1^2$$
$$-a_1A_0B_1\mu^2+a_1A_0B_1+3p^2a_3A_0B_1^2\mu^2+6p^2a_4A_0^2B_1^2\mu^2+p^2a_2B_1^2\mu^2)=0\,, \tag{22}$$

$$R(-12Rp^2a_4A1^2B_1^2\mu-2Ra_0A_1^2\mu-Ra_0pA_1^2\mu+Rp^2a_3A1^3+4Rp^2a_4A_0A_1^3$$
$$+4p^2a_4B_1^4\mu-2a_0pA_0A_1-12p^2a_4A_0A_1B_1^2+2a_0pA_0A_1\mu^2+2a_1A_1B_1+3a_0pB_1^2\mu$$
$$-2a_0B_1^2\mu^3-2a_1B_1A1\mu^2-4p^2a_4B_1^4\mu^3+12p^2a_4A_0A_1B_1^2\mu^2+3p^2a_3A_1B_1^2\mu^2$$
$$-3a_0pB_1^2\mu^3+2a_0B_1^2\mu-3p^2a_3A_1B_1^2)=0\,, \tag{23}$$

$$R^2(4p^2a_4A_1B_1^3R+a_0pB_1A_1R-a_0pA_0B_1\mu+12p^2a_4A_0^2A_1B_1+6p^2a_3A_0A_1B_1$$
$$+2p^2a_2A_1B_1-2p^2a_3B_1^3\mu-a_1A_0A_1-8p^2a_4A_0B_1^3\mu+a_1B_1^2\mu)=0\,, \tag{24}$$

$$R(3Rp^2a_3A_1^2B_1+12Rp^2a_4A_0A_1^2B_1-8Rp^2a_4A_1B_1^3\mu-2Ra_0B_1A_1\mu$$
$$-2Ra_0pA_1B_1\mu-Ra_1A_1^2+p^2a_3B_1^3\mu^2-2a_0pA_0B_1+2a_0pA_0B_1\mu^2+a_1B_1^2$$
$$-p^2a_3B_1^3-4p^2a_4A_0B_1^3+4p^2a_4A_0B_1^3\mu^2-a_1B_1^2\mu^2)=0\,, \tag{25}$$

$$p^2R^2(a_4B_1^4R^2+3Ra_3A_0B_1^2+6Ra_4A_0^2B_1^2+Ra_2B_1^2+a_3A_0^3+a_2A_0^2+a_4A_0^4)=0\,, \tag{26}$$

$$p^2a_4B_1^4+p^2a_4B_1^4\mu^4+p^2a_4A_1^4R^2-2p^2a_4B_1^4\mu^2+a_0pB_1^2+a_0B_1^2-2a_0B_1^2\mu^2$$
$$-6p^2a_4A_1^2B_1^2R-a_0pA_1^2R+a_0pB_1^2\mu^4-a_0A_1^2R+a_0pA_1^2\mu^2R+a_0A_1^2\mu^2R$$
$$+a_0B_1^2\mu^4-2a_0pB_1^2\mu^2+6p^2a_4A1^2B_1^2\mu^2R=0\,. \tag{27}$$

By use of the *Maple* soft package "Charsets" by Wang Dong-Ming, which is based on the Wu-elimination method,[23] solving Eqs. (19) $\sim$ (27), we get the following results.

Case 1

$$A_0=B_1=a_1=a_3=\mu=0,\qquad a_0=\frac{RA_1^2p^2a_4}{1+p},\qquad a_2=-\frac{R^2A_1^2a_4}{1+p}\,. \tag{28}$$

Case 2

$$a_1=\frac{B_1p^2(\pm4a_4B_1R^{3/2}\pm2pa_4B_1R^{3/2}+a_3R+pa_3R)}{R(1+p)},\qquad a_0=-\frac{p^2a_4B_1^2}{1+p}\,,$$
$$A_0=\pm B_1\sqrt{R},\qquad A_1=\mu=0,\qquad a_2=\frac{2B_1(-2a_4B_1R^{3/2}\pm a_3R)}{\sqrt{R}}\,. \tag{29}$$

Case 3

$$a_1=\pm2\frac{(2B_1pa_4R^3+4a_4B_1R^3\pm a_3R^{5/2}\pm pa_3R^{5/2})B_1p^2}{(p+1)R^{5/2}},\qquad A_0=\pm B_1\sqrt{R}\,,$$

$$A_1=\pm\frac{\sqrt{R(\mu^2-1)}B_1}{R},\qquad a_0=-\frac{4p^2a_4B_1^2}{1+p},\qquad a_2=-\frac{2B_1(2a_4B_1R^{3/2}\pm a_3R)}{\sqrt{R}}\,. \tag{30}$$

Case 4

$$a_1=A_0=B_1=0,\quad a_3=-\frac{A_1Ra_4\mu(2+p)}{\mu^2-1+p\mu^2-p},\quad a_0=-\frac{A_1^2Rp^2a_4}{\mu^2-1+p\mu^2-p},\quad a_2=\frac{A_1^2R^2a_4}{\mu^2-1+p\mu^2-p}\,. \tag{31}$$

From Eqs. (9), (12), (15), and (18) and Cases 1 $\sim$ 4, we obtain the following solutions for Eq. (3).

Family 1 From Eq. (28), we obtain the following solutions for the ZK equation, $u_t+bu^{2P}u_x+\delta u_{xxx}+su_{xyy}=0$, as follows.

$$u_{11}=\left\{A_1R\,\mathrm{sech}\big[\sqrt{R}\,(x+ly+\lambda t)\big]\right\}^{1/p}, \tag{32}$$

$$u_{12}=\left\{A_1R\,\mathrm{csch}\big[\sqrt{R}\,(x+ly+\lambda t)\big]\right\}^{1/p}, \tag{33}$$

where A_1 is an arbitrary constant and $R=(1+p)(1+2p)(\delta+sl^2)/A_1^2p^2b$, $\lambda=-(1+p)(1+2p)(\delta+sl^2)^2/bA_1^2p^4$.

Family 2 From Eq. (29), the GZK equation (3) has the following solutions.

$$u_{21}=\big[\mp\sqrt{R}B_1\pm B_1\sqrt{R}\tanh(\sqrt{R}\,\xi)\big]^{1/p}, \tag{34}$$

$$u_{22}=\big[\mp\sqrt{R}B_1\pm B_1\sqrt{R}\coth(\sqrt{R}\,\xi)\big]^{1/p}, \tag{35}$$

where

$$\xi=x+ly+\lambda t,\qquad \lambda=-\frac{2B_1\sqrt{R}[2B_1\sqrt{R}b(1+p)\pm a(1+2p)]}{(1+p)(1+2p)}\,,$$

$$B_1^2=-\frac{(1+p)(1+2p)(\delta+sl^2)}{bp^2},\qquad R=\frac{1}{4}\frac{[rl(1+p)(1+2p)-B_1pa(1+2p)]^2}{B_1^4b^2p^2(p+2)^2}\,.$$

Family 3 From Eq. (30), the GZK equation (3) has the following solutions:

$$u_{31}=\Big[\mp\sqrt{R}B_1\pm\sqrt{R(\mu^2-1)}B_1\frac{\mathrm{sech}(\sqrt{R}\,\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\,\xi)+1}+B_1\frac{\sqrt{R}\tanh(\sqrt{R}\,\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\,\xi)+1}\Big]^{1/p}, \tag{36}$$

$$u_{32}=\Big[\mp\sqrt{R}B_1\pm\sqrt{R(\mu^2-1)}B_1\frac{\mathrm{csch}(\sqrt{R}\,\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\,\xi)+1}+B_1\frac{\sqrt{R}\coth(\sqrt{R}\,\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\,\xi)+1}\Big]^{1/p}, \tag{37}$$

where

$$\xi=x+ly+\lambda t,\qquad \lambda=-\frac{2B_1\sqrt{R}[2B_1\sqrt{R}b(1+p)\pm a(1+2p)]}{(1+p)(1+2p)}\,,$$

$$B_1^2=-\frac{(1+p)(1+2p)(\delta+sl^2)}{4bp^2},\qquad R=\frac{1}{16}\frac{[-rl(1+p)(1+2p)+2B_1pa(1+2p)]^2}{B_1^4b^2p^2(p+2)^2}\,.$$

Family 4 From Eq. (31), we obtain the following solution for the GZK equation, $u_t+au^pu_x+bu^{2p}u_x+\delta u_{xxx}+su_{xyy}=0$,

$$u_{41}=\Big[A_1\frac{R\,\mathrm{sech}(\sqrt{R}\,\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\,\xi)+1}\Big]^{1/p}, \tag{38}$$

$$u_{42}=\Big[A_1\frac{R\,\mathrm{csch}(\sqrt{R}\,\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\,\xi)+1}\Big]^{1/p}, \tag{39}$$

where

$$A_1=\frac{\mu(1+p)(2+p)(\delta+sl^2)}{ap^2},\qquad R=-\frac{(1+p)(1+2p)(\delta+sl^2)(\mu^2-1)}{bp^2A_1^2}\,,$$

$$\xi=x+ly+\lambda t,\qquad \lambda=\frac{bR^2A_1^2}{(\mu^2-1)(1+p)(1+2p)}\,.$$

The following periodic wave solutions are obtained under $\epsilon=1$.

Family 5 The ZK equation, $u_t+bu^{2P}u_x+\delta u_{xxx}+su_{xyy}=0$, has the following periodic solutions:

$$u_{51}=\left\{A_1R\sec\big[\sqrt{R}\,(x+ly+\lambda t)\big]\right\}^{1/p}, \tag{40}$$

$$u_{52}=\left\{A_1R\csc\big[\sqrt{R}\,(x+ly+\lambda t)\big]\right\}^{1/p}, \tag{41}$$

where A_1 is an arbitrary constant and $R=-(1+p)(1+2p)(\delta+sl^2)/A_1^2p^2b$, $\lambda=-(1+p)(1+2p)(\delta+sl^2)^2/bA_1^2p^4$.

Family 6 The GZK equation (3) has the following solutions:

$$u_{61}=\big[\mp\sqrt{-R}\,B_1\pm B_1\sqrt{-R}\tan(\sqrt{R}\,\xi)\big]^{1/p}\,, \tag{42}$$

$$u_{62}=\big[\mp\sqrt{-R}\,B_1\pm B_1\sqrt{-R}\cot(\sqrt{R}\,\xi)\big]^{1/p}\,, \tag{43}$$

where

$$\xi=x+ly+\lambda t,\qquad \lambda=\frac{2B_1R[2B_1\sqrt{-R}\,b(1+p)\pm a(1+2p)]}{\sqrt{-R}\,(1+p)(1+2p)}\,,$$

$$B_1^2=-\frac{(1+p)(1+2p)(\delta+sl^2)}{bp^2},\qquad R=-\frac{1}{4}\frac{[rl(1+p)(1+2p)+B_1pa(1+2p)]^2}{B_1^4b^2p^2(p+2)^2}\,.$$

Family 7 The GZK equation (3) possesses the following solutions:

$$u_{71}=\Big[\mp\sqrt{-R}\,B_1\pm\sqrt{-R(\mu^2-1)}\,B_1\frac{\sec(\sqrt{R}\,\xi)}{\mu\sec(\sqrt{R}\,\xi)+1}+B_1\frac{\sqrt{R}\tan(\sqrt{R}\,\xi)}{\mu\sec(\sqrt{R}\,\xi)+1}\Big]^{1/p}\,, \tag{44}$$

$$u_{72}=\Big[\mp\sqrt{R}\,B_1\pm\sqrt{-R(\mu^2-1)}\,B_1\frac{\csc(\sqrt{R}\,\xi)}{\mu\csc(\sqrt{R}\,\xi)+1}+B_1\frac{\sqrt{R}\cot(\sqrt{R}\,\xi)}{\mu\csc(\sqrt{R}\,\xi)+1}\Big]^{1/p}\,, \tag{45}$$

where

$$\xi=x+ly+\lambda t,\qquad \lambda=\frac{2B_1R[2B_1\sqrt{R}\,b(1+p)\pm a(1+2p)]}{\sqrt{R}\,(1+p)(1+2p)}\,,$$

$$B_1^2=-\frac{(1+p)(1+2p)(\delta+sl^2)}{4bp^2},\qquad R=-\frac{1}{16}\frac{[rl(1+p)(1+2p)+2B_1pa(1+2p)]^2}{B_1^4b^2p^2(p+2)^2}\,.$$

Family 8 The ZK equation $u_t+au^pu_x+bu^{2p}u_x+\delta u_{xxx}+su_{xyy}=0$, has the following solutions:

$$u_{81}=\Big[A_1\frac{R\sec(\sqrt{R}\,\xi)}{\mu\sec(\sqrt{R}\,\xi)+1}\Big]^{1/p}\,, \tag{46}$$

$$u_{82}=\Big[A_1\frac{R\csc(\sqrt{R}\,\xi)}{\mu\csc(\sqrt{R}\,\xi)+1}\Big]^{1/p}\,, \tag{47}$$

where

$$A_1=-\frac{\mu(1+p)(2+p)(\delta+sl^2)}{ap^2},\qquad R=\frac{(1+p)(1+2p)(\delta+sl^2)(\mu^2-1)}{bp^2A_1^2}\,,$$

$$\xi=x+ly+\lambda t,\qquad \lambda=\frac{bR^2A_1^2}{(\mu^2-1)(1+p)(1+2p)}\,.$$

Family 9 Rational solutions

When setting the solutions of Eq. (3) in the form (11), we obtain the following rational solutions for Eq. (3),

$$u_9=\Big[\pm\sqrt{-\frac{(1+P)(1+2p)(\delta+\rho l^2)}{bp^2}}\,\frac{1}{x+ly+\xi_0}\Big]^{1/p}\,, \tag{48}$$

where $l^2=-(1+2p)(\delta+sl^2)a^2/r^2(p+1)p^2b$, and ξ_0 is an arbitrary constant.

Remarks

(i) The solutions (34) reproduce the solution (28) in Ref. [15] and the solutions (3.24) of Ref. [16]; the solutions (32), (33) and (40), (41) are equal to the solutions (3.37), (3.38) and (3.39), (3.40) in Ref. [16], respectively.

(ii) When setting $\mu=0$ in Eqs. (36), (37), (44), and (45), the solutions (3.29) $\sim$ (3.32), (3.37), and (3.40) obtained in Ref. [16] can be recovered.

(iii) The other solutions obtained here, to our knowledge, are all new families of exact solutions of the GZK equation.

(iv) From our results, when setting $\epsilon=rl$ in the solutions of (34) $\sim$ (37) and (42) $\sim$ (45), we can obtain the exact travelling wave solutions for another generalized Zakharov–Kuzentsov equation,[15,16]

$$u_t+au^pu_x+bu^{2p}u_x+\epsilon u_{xx}+\delta u_{xxx}+su_{xyy}=0\,,$$
$$a,b,\epsilon\neq 0,\delta,\ \text{and } s \text{ are constants}\,, \tag{49}$$

which includes compound KdV type and compound KdV–Burgers type equations.[17,18]

4 Summary and Conclusion

In summary, by use of the general projective Riccati method, more general forms of solutions for GZK equations are obtained. Of course, this method can be extended to other nonlinear PDEs with nonlinear terms of any order. On the other hand, we will extend this method to seek soliton-like solutions for some PDEs in the forthcoming works.

Acknowledgments

We would like to express our sincere thanks to the referees for their valuable suggestions and help.

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