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A series of new double periodic solutions to a (2+1)-dimensional asymmetric Nizhnik–Novikov–Veselov equation*

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By means of a new general ansätz and with the aid of symbolic computation, a new algebraic method named Jacobi elliptic function rational expansion is devised to uniformly construct a series of new double periodic solutions to (2+1)-dimensional asymmetric Nizhnik–Novikov–Veselov (ANNV) equation in terms of rational Jacobi elliptic function.

Keywords: (2+1)-dimensional asymmetric Nizhnik–Novikov–Veselov equation, Jacobi elliptic functions, travelling wave solution, soliton solution, periodic solution

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1. Introduction

There have been a great number of activities aiming to find methods of exactly solving nonlinear evolution equations. The solutions to the equations may well describe various phenomena in physics and other fields and thus may give a good insight into the physical aspects of the problems. Many powerful methods have been presented, such as the inverse scattering transform method,[1] Darboux transformation, Cole–Hopf transformation, Hirota method,[2] Bäcklund transformation,[2,3] Painlevé method,[2,4,5] homogeneous balance method (HBM),[6−13] tanh method,[14,15] and generalized hyperbolic-function method.[16,17] Recently, Liu *et al*[18] presented the Jacobi elliptic function expansion method where three Jacobi elliptic functions were used to express exact solutions to some nonlinear evolution equations. Fan and Hon Benny[19] extended Jacobi elliptic function method to some nonlinear evolution equations and, in particular, special-type of nonlinear equations for constructing their doubly periodic wave solutions. Such equations cannot be directly dealt with by the method and require some kinds of prepossessing techniques. Yan[20] further developed an extended Jacobi elliptic function expansion method by using 12 Jacobi elliptic functions. Based on the above idea and by means of a new general ansätz, a new Jacobi elliptic function rational expansion method is presented and it is more powerful than the above existing Jacobi elliptic function method[18−20] to uniformly construct more new exact doubly-periodic solutions to nonlinear evolution equations in terms of rational Jacobi elliptic functions. As its illustration, we apply the proposed method to a (2+1)-dimensional ANNV equation,[21−26] which reads

$$u_t - u_{xxx} + \alpha(uv)_x = 0, \quad (1)$$

$$u_x + \beta v_y = 0. \quad (2)$$

where α, $\beta \neq 0$ are all constants. This equation system is also named the (2+1)-dimensional KdV equation or Boiti–Leon–Manna–Pempinelli (BLMP) equation by Boiti and Leon[22] where the idea of the weak Lax pair is used. The ANNV equation (1) can also be obtained from the Katomtsev–Petviashvili (KP) equation[23] with the inner parameter-dependent sym-

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metry constraint. For more details about the results of this system, the reader can find the remarkable achievements in Refs.[21–26].

This paper is organized as follows. In Section 2, we summarize the Jacobi elliptic function rational expansion method. In Section 3, we apply the generalized method to a (2+1)-dimensional ANNV equation and bring out many solutions. Conclusions are presented in Section 4.

2. Summary of the rational elliptic function expansion method.

In the following we outline the main steps of our general method.

Step 1 A given nonlinear partial differential equation system with some physical fields $u_i\,(x,y,t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0, \tag{3}$$

through the wave transformation

$$u_i\,(x,y,t) = u_i\,(\xi)\,, \qquad \xi = x + ly - \lambda\, t, \tag{4}$$

where k, l and λ are constants to be determined later, can be reduced to a nonlinear ordinary differential equation:

$$G_i(\ u_i,\ u_i',\ u_i'', \ldots) = 0. \tag{5}$$

Step 2 We introduce a new ansätz in terms of finite rational elliptic function expansion in the following forms:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\left(a_{ij}\frac{\mathrm{sn}^j(\xi)}{(\mu\mathrm{cn}(\xi)+1)^j} + b_{ij}\frac{\mathrm{sn}^{j-1}(\xi)\mathrm{cn}(\xi)}{(\mu\mathrm{cn}(\xi)+1)^j}\right). \tag{6}$$

Notice that

$$\begin{aligned}\frac{\mathrm{d}u_i}{\mathrm{d}\xi} =& \sum_{j=1}^{m_i}\frac{\mathrm{dn}\,(\xi)\left(\left(a_{ij}\mathrm{sn}^{j-1}\,(\xi)\,j - b_{ij}\mathrm{sn}^{j-2}\,(\xi)\,\mu + b_{ij}\mathrm{sn}^{j-2}\,(\xi)\,j\mu\right)\mathrm{cn}\,(\xi) + a_{ij}\mathrm{sn}^{j-1}\,(\xi)\,j\mu\right)}{(\mu\,\mathrm{cn}\,(\xi)+1)^{j+1}} \\ &+ \sum_{j=1}^{m_i}\frac{\mathrm{dn}\,(\xi)\left(-b_{ij}\mathrm{sn}^{j-2}\,(\xi) + b_{ij}\mathrm{sn}^{j-2}\,(\xi)\,j - b_{ij}\mathrm{sn}^{j}\,(\xi)\,j\right)}{(\mu\,\mathrm{cn}\,(\xi)+1)^{j+1}},\end{aligned} \tag{7}$$

where sn(ξ), cn(ξ), dn(ξ), ns(ξ), cs(ξ), ds(ξ), etc are Jacobi elliptic functions, which are double periodic and possess the following properties:

(1) properties of triangular function

$$\mathrm{cn}^2(\xi) + \mathrm{sn}^2(\xi) = \mathrm{dn}^2(\xi) + \mathrm{m}^2\mathrm{sn}^2(\xi) = 1, \tag{8}$$

$$\mathrm{ns}^2(\xi) = 1 + \mathrm{cs}^2(\xi), \quad \mathrm{ns}^2(\xi) = \mathrm{m}^2 + \mathrm{ds}^2(\xi), \tag{9}$$

(2) derivatives of the Jacobi elliptic functions

$$\begin{aligned}\mathrm{sn}'(\xi) &= \mathrm{cn}(\xi)\mathrm{dn}(\xi), \\ \mathrm{cn}'(\xi) &= -\mathrm{sn}(\xi)\mathrm{dn}(\xi), \\ \mathrm{dn}'(\xi) &= -\mathrm{m}^2\mathrm{sn}(\xi)\mathrm{cn}(\xi),\end{aligned} \tag{10}$$

$$\begin{aligned}\mathrm{ns}'(\xi) &= -\mathrm{ds}(\xi)\mathrm{cs}(\xi), \\ \mathrm{ds}'(\xi) &= -\mathrm{cs}(\xi)\mathrm{ns}(\xi), \\ \mathrm{cs}'(\xi) &= -\mathrm{ns}(\xi)\mathrm{ds}(\xi),\end{aligned} \tag{11}$$

where m is a modulus. The Jacobi–Glaisher functions for an elliptic function can be found in Refs.[27–29].

Step 3 The underlying mechanism for occurrence of a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions is that different effects, which act upon the change of wave forms in many nonlinear equations, i.e. dispersion, dissipation and nonlinearity, either separately or jointly in various combinations are able to balance out. We define the degree of $u_i\,(\xi)$ as $D[u_i\,(\xi)] = n_i$, which gives rise to the degrees of other expressions as

$$\begin{aligned}&D[u_i^{(\alpha)}] = n_i + \alpha, \\ &D[u_i^{\beta}(u_j^{(\alpha)})^s] = n_i\beta + (\alpha + n_j)s.\end{aligned} \tag{12}$$

Therefore we can obtain the value of m_i in Eq.(6). If n_i is a nonnegative integer, then we first perform the transformation $u_i = \omega^{n_i}$.

Step 4 Substitute Eq.(6) into Eqs.(5) (7), (10) and (11) and then set all coefficients of $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$, ($i = 1, 2, \ldots$; $j = 0, 1$) to be zero to obtain an over-determined system of nonlinear algebraic equations with respect to λ, l, μ, a_{i0}, a_{ij} and b_{ij} ($i = 1, 2, \ldots$; $j = 1, 2, \ldots, m_i$).

Step 5 Solving the over-determined system of nonlinear algebraic equations by use of *Maple*, we end up with the explicit expressions for λ, l, μ, a_{i0}, a_{ij} and b_{ij} $(i=1,2,\ldots;\ j=1,2,\ldots,m_i)$, from which λ, l, μ, a_{i0}, a_{ij} and b_{ij} $(i=1,2,\ldots;\ j=1,2,\ldots,m_i)$ can be obtained. In this way, we can acquire double periodic solutions with Jacobi elliptic function.

Since

$$\lim_{m\to 1}\mathrm{sn}(\xi)=\tanh(\xi),\quad \lim_{m\to 1}\mathrm{cn}(\xi)=\mathrm{sech}(\xi),\quad \lim_{m\to 1}\mathrm{dn}(\xi)=\mathrm{sech}(\xi), \tag{13}$$

$$\lim_{m\to 1}\mathrm{ns}(\xi)=\coth(\xi),\quad \lim_{m\to 1}\mathrm{cs}(\xi)=\mathrm{csch}(\xi),\quad \lim_{m\to 1}\mathrm{ds}(\xi)=\mathrm{csch}(\xi), \tag{14}$$

$$\lim_{m\to 0}\mathrm{sn}(\xi)=\sin(\xi),\quad \lim_{m\to 0}\mathrm{cn}(\xi)=\cos(\xi),\quad \lim_{m\to 0}\mathrm{dn}(\xi)=1, \tag{15}$$

$$\lim_{m\to 0}\mathrm{ns}(\xi)=\csc(\xi),\quad \lim_{m\to 0}\mathrm{cs}(\xi)=\cot(\xi),\quad \lim_{m\to 0}\mathrm{ds}(\xi)=\csc(\xi), \tag{16}$$

u_i degenerate respectively into the following forms:

(1) solitary wave solutions

$$u_i(\xi)=a_{i0}+\sum_{j=1}^{m_i}\Big(a_{ij}\frac{\tanh^j(\xi)}{(\mu\mathrm{sech}(\xi)+1)^j}+b_{ij}\frac{\tanh^{j-1}(\xi)\mathrm{sech}(\xi)}{(\mu\mathrm{sech}(\xi)+1)^j}\Big). \tag{17}$$

(2) trigonometric function formal solution

$$u_i(\xi)=a_{i0}+\sum_{j=1}^{m_i}\Big(a_{ij}\frac{\sin^j(\xi)}{(\mu\cos(\xi)+1)^j}+b_{ij}\frac{\sin^{j-1}(\xi)\cos(\xi)}{(\mu\cos(\xi)+1)^j}\Big). \tag{18}$$

So the Jacobi elliptic function rational expansion method is more powerful than the methods respectively by Liu *et al*,[18] and Fan and Hon Benny[19] and the method extended by Yan.[20] The solutions which contain solitary wave solutions, singular solitary solutions and triangular function formal solutions can be obtained by the extended method.

Remark1 If we replace the Jacobi elliptic functions sn(ξ), cn(ξ) in the ansätz (6) with other pairs of Jacobi elliptic functions such as sn(ξ) and dn(ξ); ns(ξ) and cs(ξ); ns(ξ) and ds(ξ); sc(ξ) and nc(ξ); dc(ξ) and nc(ξ); sd(ξ) and nd(ξ); cd(ξ) and nd(ξ),[27–29] other new double periodic wave solutions, solitary wave solutions, and triangular functional solutions can be obtained for some system, respectively. For simplicity, we omit them here. It is necessary to point out that the above combinations only require solving the recurrent coefficient relation or derivative relation for the terms of polynomial for computation closed.

3. Exact solutions of the (2+1)-dimensional asymmetric Nizhnik–Novikov–Veselov equation

Let us consider the (2+1)-dimensional asymmetric Nizhnik–Novikov–Veselov (ANNV) equation, i.e.

$$\begin{cases} u_t-u_{xxx}+\alpha(uv)_x=0,\\ u_x+\beta v_y=0,\end{cases} \tag{19}$$

According to the above method, to seek travelling wave solutions to Eqs.(19), we perform the transformation

$$u(x,y,t)=\phi(\xi),\quad v(x,y,t)=\sigma(\xi),\quad \xi=x+ly-\lambda t, \tag{20}$$

where l and λ are constants to be determined later, and thus Eqs.(19) become

$$\begin{cases} -\lambda\,\phi'-\phi'''+\alpha(\phi\sigma)'=0,\\ \phi'+\beta l\sigma'=0.\end{cases} \tag{21}$$

According to **Step 1** in Section 2, by balancing $\phi'''(\xi)$ and $(\sigma(\xi)\phi(\xi))'$, and $\sigma'(\xi)$ and $\phi'(\xi)$ in Eqs.(21), we assume that Eqs.(21) have the following formal solutions:

$$\begin{cases} \phi(\xi)=a_0+a_1\dfrac{\mathrm{sn}(\xi)}{\mu\mathrm{cn}(\xi)+1}+b_1\dfrac{\mathrm{cn}(\xi)}{\mu\mathrm{cn}(\xi)+1}+a_2\dfrac{\mathrm{sn}^2(\xi)}{(\mu\mathrm{cn}(\xi)+1)^2}+b_2\dfrac{\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{(\mu\mathrm{cn}(\xi)+1)^2},\\ \sigma(\xi)=A_0+A_1\dfrac{\mathrm{sn}(\xi)}{\mu\mathrm{cn}(\xi)+1}+B_1\dfrac{\mathrm{cn}(\xi)}{\mu\mathrm{cn}(\xi)+1}+A_2\dfrac{\mathrm{sn}^2(\xi)}{(\mu\mathrm{cn}(\xi)+1)^2}+B_2\dfrac{\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{(\mu\mathrm{cn}(\xi)+1)^2},\end{cases} \tag{22}$$

where a_0, a_1, b_1, a_2, b_2, A_0, A_1, A_2, B_1, B_2 are constants to be determined later.

With the aid of *Maple*, substituting Eqs.(22), (8), (9), (10) and (16) into (21), yields a set of algebraic equations for $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi), (i = 0,1,\ldots; \quad j = 0,1)$. Setting the coefficients of these terms $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$ to be zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, a_2, b_2, A_0, A_1, B_1, A_2, B_2, l and λ.

By use of the *Maple* soft package 'Charsets' by Wang Dongming, which is based on the Wu-elimination method,[30] to solve the over-determined algebraic equations, we obtain the following results.

Case 1

$$\begin{aligned}&\mu=0, \quad B_2=\pm 3\frac{\mathrm{i}m^2}{\alpha}, \quad l=-\frac{1}{6}\,\frac{\alpha\, a_0}{m^2\beta},\\ &b_2=\pm\,\frac{1}{2}\mathrm{i}a_0, \quad a_2=\frac{1}{2}\,a_0, \quad A_2=3\,\frac{m^2}{\alpha},\\ &\lambda=4+7\,m^2+\alpha\,A_0,\\ &a_1=b_1=A_1=B_1=0.\end{aligned} \tag{23}$$

Case 2

$$\begin{aligned}\mu=&0,\\ \lambda=&\frac{3\,a_0m^2+\alpha\,a_2A_0+4\,a_2+a_2m^2}{a_2},\\ l=&-\,\frac{1}{3}\,\frac{\alpha\,a_2}{m^2\beta},\\ B_2=&\pm\,\frac{3\mathrm{i}m^2}{\alpha},\\ A_2=&3\,\frac{m^2}{\alpha}, \quad b_2=\pm\mathrm{i}a_2,\\ a_1=&b_1=A_1=B_1=0.\end{aligned} \tag{24}$$

Case 3

$$\begin{aligned}&\mu=\pm\,1, \quad B_1=\pm 2\,A_2,\\ &a_2=\pm\,\frac{1}{2}\,b_1, \quad \lambda=\lambda, \quad l=l,\\ &a_1=b_2=A_1=B_2=0.\end{aligned} \tag{25}$$

Case 4

$$\begin{aligned}&\mu=\pm\,1, \quad B_1=\pm\frac{2\,\alpha\,A_2+3}{\alpha}, \quad l=-\,\frac{\alpha\,(-2\,a_2+b_1)}{3\beta}, \quad a_1=b_2=A_1=B_2=0,\\ &\lambda=-\,\frac{-8\,a_2m^2-5\,b_1+2\,\alpha\,a_2A_0-3\,a_0+2\,\alpha\,a_2A_2+7\,a_2-\alpha\,b_1A_0+4\,b_1m^2-\alpha\,b_1A_2}{-2\,a_2+b_1}.\end{aligned} \tag{26}$$

From Eqs.(20), (22) and Cases 1–4, we obtain the following solutions to Eqs.(19).

Family 1 From Eqs.(23), we obtain the following rational formal doubly periodic solutions to the ANNV:

$$u_1\,(x,y,t)=a_0+\frac{1}{2}\,a_0\mathrm{sn}^2\,(\xi)\pm\frac{1}{2}\mathrm{i}a_0\mathrm{sn}\,(\xi)\,\mathrm{cn}\,(\xi)\,, \tag{27}$$

$$v_1\,(x,y,t)=A_0+3\,\frac{m^2\mathrm{sn}^2\,(\xi)}{\alpha}\pm 3\frac{\mathrm{i}m^2\mathrm{sn}\,(\xi)\,\mathrm{cn}\,(\xi)}{\alpha}, \tag{28}$$

where $\xi = x + ly - \lambda t$, l and λ are determined by Eqs.(23), a_0 and A_0 are arbitrary constants.

Family 2 From Eqs.(24), we obtain the following rational formal doubly periodic solutions to the ANNV:

$$u_2\,(x,y,t)=a_0+a_2\mathrm{sn}^2\,(\xi)\pm\mathrm{i}a_2\mathrm{sn}\,(\xi)\,\mathrm{cn}\,(\xi)\,, \tag{29}$$

$$v_2\,(x,y,t)=A_0+3\,\frac{m^2\mathrm{sn}^2\,(\xi)}{\alpha}\pm 3\frac{\mathrm{i}m^2\mathrm{sn}\,(\xi)\,\mathrm{cn}\,(\xi)}{\alpha}, \tag{30}$$

where $\xi = x + ly - \lambda t$, l and λ are determined by Eqs.(24), a_0, a_2 and A_0 are arbitrary constants.

Family 3 From Eqs.(25), we obtain the following rational formal doubly periodic solutions to the ANNV:

$$u_3\,(x,y,t)=a_0+\frac{b_1\mathrm{cn}\,(\xi)}{\pm\mathrm{cn}\,(\xi)+1}\pm\frac{1}{2}\,\frac{b_1\mathrm{sn}^2\,(\xi)}{(\pm\mathrm{cn}\,(\xi)+1)^2}, \tag{31}$$

$$v_3\,(x,y,t)=A_0\pm 2\,\frac{A_2\mathrm{cn}\,(\xi)}{\pm\mathrm{cn}\,(\xi)+1}+\frac{A_2\mathrm{sn}^2\,(\xi)}{(\pm\mathrm{cn}\,(\xi)+1)^2}, \tag{32}$$

where $\xi = x + ly - \lambda t$, l and λ are determined by Eqs.(25), a_0, a_2, b_1, A_0 and A_2 are arbitrary constants.

Family 4 From Eqs.(26), we obtain the following rational formal doubly periodic solutions to the ANNV:

$$u_4\,(x,y,t)=a_0+\frac{b_1\mathrm{cn}\,(\xi)}{\pm\mathrm{cn}\,(\xi)+1}+\frac{a_2\mathrm{sn}^2\,(\xi)}{(\pm\mathrm{cn}\,(\xi)+1)^2}, \tag{33}$$

$$v_4\,(x,y,t)=A_0\pm\frac{(2\,\alpha\,A_2+3)\,\mathrm{cn}\,(\xi)}{\alpha\,(\pm\mathrm{cn}\,(\xi)+1)}+\frac{A_2\mathrm{sn}^2\,(\xi)}{(\pm\mathrm{cn}\,(\xi)+1)^2}, \tag{34}$$

where $\xi = x + ly - \lambda t$, l and λ are determined by Eqs.(26), a_0, a_2, b_1, A_0 and A_2 are arbitrary constants.

4. Summary and conclusions

In this paper, we have presented a new Jacobi elliptic function rational expansion method. The method is more powerful than the methods proposed recently by Liu,[18] Fan and Hon Benny[19] and Yan.[20] The (2+1)-asymmetric Nizhnik–Novikov–Veselov (ANNV) equation is chosen to illustrate the method, so that two families of new Jacobi elliptic function solutions are obtained. When the modulus $m \to 1$, some of these obtained solutions degenerate into solitary wave solutions. Notice that if we replace the Jacobi elliptic functions $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ in the ansätz (6) with other pairs of Jacobi elliptic functions such as $\mathrm{sn}(\xi)$ and $\mathrm{dn}(\xi)$; $\mathrm{ns}(\xi)$ and $\mathrm{cs}(\xi)$; $\mathrm{ns}(\xi)$ and $\mathrm{ds}(\xi)$; $\mathrm{sc}(\xi)$ and $\mathrm{nc}(\xi)$; $\mathrm{dc}(\xi)$ and $\mathrm{nc}(\xi)$; $\mathrm{sd}(\xi)$ and $\mathrm{nd}(\xi)$; $\mathrm{cd}(\xi)$ and $\mathrm{nd}(\xi)$,[27−29] other new double periodic wave solutions, solitary wave solutions, and triangular functional solutions can be obtained for some system, respectively. It is necessary to point out that the above combinations only require solving the recurrent coefficient relation or derivative relation for the terms of polynomial for computation closed. The algorithm can be also applied to many nonlinear evolution equations in mathematical physics. Further work about various extensions and improvement of Jacobi function method needs us to find the more general ansätzes or the more general subequation.

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