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General projective Riccati equation method and exact solutions for generalized KdV-type and KdV–Burgers-type equations with nonlinear terms of any order

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Abstract

Applying the improved generalized method, which is a direct and unified algebraic method for constructing multiple travelling wave solutions of nonlinear partial differential equations and implemented in a computer algebraic system, we consider the KdV-type equations and KdV–Burgers-type equations with nonlinear terms of any order. As a result, we can not only successfully recover the previously known travelling wave solutions found by existing various tanh methods and other sophisticated methods, but also obtain some new formal solutions. The solutions obtained include kink-shaped solitons, bell-shaped solitons, singular solitons and periodic solutions.

1. Introduction

It is known that to find exact solutions of differential equations is always one of the central theme of perpetual interest in mathematics and physics. With the development of soliton theory, many powerful methods have been presented, such as inverse scattering transform method [1], Bäcklund transformation [2,3], Darbour transformation [4], various tanh methods [5–8], homogeneous balance method [9] and similarity reductions method [10,11] etc.

Solitary wave solutions of a nonlinear partial differential equation (NLPDEs) $E(u) = 0$ in the unknown $u(x, y, z, \ldots, t)$ are solutions of the ordinary differential equation obtained by the travelling wave reduction $u(x, y, z, \ldots, t) \to u(\xi = x + \lambda_1 y + \lambda_2 z + \cdots + \lambda_n t)$. Particularly, some general ansatz have been proposed in order to obtain new formal solutions for given NLPDE.

In [12], Conte et al. presented a general ansatz to seek more new solitary wave solutions of some NLPDEs that can be expressed as a polynomial in two elementary functions which satisfy a project Riccati equations [13]. Recently, Yan developed Conte's method and presented the general projective Riccati equation method. The key idea of Yan's method is to extend the projective Riccati equation (see Ref. [14] for detail)

$$\sigma'(\xi) = -\sigma(\xi)\tau(\xi), \quad \tau'(\xi) = -\tau^2(\xi) - \frac{\mu}{K}\sigma(\xi) + 1, \ \ \mu, K = \text{constant}, \tag{1}$$

$$\left(\frac{1}{\tau(\xi)} - \frac{\mu}{K}\right)^2 - \frac{\tau^2(\xi)}{\sigma^2(\xi)} = K^{-2}. \tag{2}$$

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to be more general form

$$\sigma'(\xi) = \epsilon\sigma(\xi)\tau(\xi), \quad \tau'(\xi) = R + \epsilon\tau^2(\xi) - \mu\sigma(\xi), \;\; \epsilon = \pm 1, \;\; R, \mu = \text{constant}, \tag{3}$$

$$\tau^2(\xi) = -\epsilon\left[R - 2\mu\sigma(\xi) + \frac{\mu^2 - 1}{R}\sigma^2(\xi)\right], \quad (R \neq 0), \tag{4}$$

where $' = d/\xi$. When $\epsilon = -1$, $R = 1$, $\mu \to \mu/K$, (3) becomes (1).

The present work is motivated by the desire to improve the work made in [12,14] to study some systems of NLPDEs with nonlinear term of any order. Up to now, some systems of NLPDEs with nonlinear term of any order have been studied [15–17]. All of these models are interesting both from the mathematics as well as the physics points of view. Mathematically, these equations are generalized. From the physics point of view, these models include many famous known physical models and can describe more complex physical phenomena in such diverse areas as nonlinear optics, hydrodynamics, condensed matter, plasma physics, quantum field theory and so on. However, to seek the exact solutions of these equations, generally, we could not use above-mentioned method directly. In this paper, by making a proper transformation, the method in [14] are improved so that it is able to deal with arbitrary balance constant. To illustrate the improved method, we consider the two-dimensional KdV–Burgers-type equation with high-order nonlinear terms [15],

$$(u_t + au^p u_x + bu^{2p} u_x + \gamma u_{xx} + \delta u_{xxx})_x + su_{yy} = 0, \tag{5}$$

where $a, b \neq 0$, $\gamma, \delta \neq 0$, p are all constants. When setting the parameters to be equal to various values, Eq. (5) includes many known KdV-type equations and KdV–Burgers-type equations in $(1+1)$-dimensional and $(2+1)$-dimensional cases (see, e.g., Refs. [2,18–26] for detail). Making use of the improved method described in Section 2, some new exact travelling wave solutions are found.

This paper is organized as follows. In Section 2, we summarize the improved generalized method. In Section 3, we apply the improved method to G2DKdV–Burgers equation and bring out many solutions. Conclusions will be presented in finally.

2. Improved generalized method

In this section, we describe the improved generalized method as follows.

Consider a given NPLDE, say, two variables: x, t,

$$p(u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots) = 0, \tag{6}$$

under the transformation $u(x,t) = u(\xi)$, $\xi = x - \lambda t$, (3) reduces to be

$$G(u', u'', u''', \ldots) = 0. \tag{7}$$

Step 1. Balancing the highest order derivative term and the nonlinear terms in Eq. (6) or (7), we get a balance constant m (m is usually a positive integer). If m is a fraction or a negative integer, we make the following transformation

(1) when $m = p/q$ is a fraction, we let

$$u(\xi) = v^{p/q}(\xi), \tag{8}$$

then return to determine the balance constant m again;

(2) when m is a negative integer, we let

$$u(\xi) = v^{m}(\xi), \tag{9}$$

then return to determine balance constant m again.

Step 2. We express the solutions of Eq. (7) to be the following forms.

Type 1. When $R \neq 0$ in Eq. (3),

$$u(\xi) = A_0 + \sum_{i=1}^{m} \sigma^{i-1}[A_i\sigma(\xi) + B_i\tau(\xi)], \tag{10}$$

where $\sigma(\xi)$ and $\tau(\xi)$ satisfy Eqs. (3) and (4).

Type 2. When $R = \mu = 0$ in Eq. (3),

$$u(\xi) = \sum_{i=0}^{m} A_0 \tau^2(\xi), \tag{11}$$

where $\tau'(\xi) = \tau^2(\xi)$.

Step 3

(1) when $R \neq 0$, substituting (10) along with the conditions (3) and (4) into (7),

(2) when $R = \mu = 0$, substituting (11) along with $\tau'(\xi) = \tau^2(\xi)$ into (7), yields a set of algebraic equations for $\sigma^j(\xi)\tau^i(\xi)$, $j = 0, 1, \ldots; i = 0, 1)$ $(\tau^l\xi, l = 0, 1, \ldots)$. Setting the coefficients of these terms $\sigma^j\tau^i$ (or $\tau^l(\xi)$) to zero yields a set of over-determined algebraic equations in λ, A_i, B_i, R and μ.

Step 4. With the aid of Maple, solving the above set of equations, yields the values of A_i, B_i, R, λ.

Step 5. We know that Eq. (3) admits the following solutions:

Case 1. When $\epsilon = -1$, $R \neq 0$,

$$\begin{cases} \sigma_1(\xi) = \dfrac{R\,\mathrm{sech}(\sqrt{R}\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\xi) + 1}, & \tau_1(\xi) = \dfrac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\xi) + 1}, \\ \sigma_2(\xi) = \dfrac{R\,\mathrm{csch}(\sqrt{R}\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\xi) + 1}, & \tau_2(\xi) = \dfrac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\xi) + 1}. \end{cases} \tag{12}$$

Case 2. When $\epsilon = 1, R \neq 0$,

$$\begin{cases} \sigma_3(\xi) = \dfrac{R\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi) + 1}, & \tau_3(\xi) = \dfrac{\sqrt{R}\tan(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi) + 1}, \\ \sigma_4(\xi) = \dfrac{R\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi) + 1}, & \tau_4(\xi) = \dfrac{\sqrt{R}\cot(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi) + 1}. \end{cases} \tag{13}$$

Case 3. When $R = \mu = 0$,

$$\sigma_5(\xi) = \frac{C}{\xi} = C\epsilon\tau_5(\xi), \quad \tau_5(\xi) = \frac{1}{\epsilon\xi}, \tag{14}$$

where C is a constant.

Thus according to (8) (or (9)), (10)–(14) and the conclusions in Step 4, we can obtain many solutions for Eq. (6).

3. Applications

Let us consider the G2DKdV–Burgers equation, i.e., Eq. (5). According to the improved method, to seek travelling wave solutions of Eq. (5), we make the transformation

$$u(x, y, t) = v(\xi), \quad \xi = x + ny - \lambda t, \tag{15}$$

where n and λ are constants to be determined later, and thus Eq. (5) becomes

$$(-\lambda v_\xi + av^p v_\xi + bv^{2p} v_\xi + \gamma v_{\xi\xi} + \delta v_{\xi\xi\xi})_\xi + sn^2 v_{\xi\xi} = 0. \tag{16}$$

Integrating the above equation twice with regard to ξ, we obtain

$$\delta v''(\xi) + \gamma v'(\xi) + (sn^2 - \lambda)v(\xi) + \frac{a}{p+1} v^{p+1}(\xi) + \frac{b}{2p+1} v^{2p+1}(\xi) = 0, \tag{17}$$

with the integration constants taken to be zero. According to Step 1 in Section 2, if $\delta \neq 0$, $b \neq 0$ and $p \neq 0, \frac{1}{2}$, by balancing $v''(\xi)$ and $v^{2p+1)}(\xi)$ in Eq. (17), we get $m = 1/p$. Therefore we make the following transformation

$$v(\xi) = \varphi^{1/p}(\xi), \tag{18}$$

then substituting (18) into Eq. (17) yields

$$a_0[p\varphi(\xi)\varphi''(\xi) + (1-p)\varphi'^2(\xi)] + a_1\varphi(\xi)\varphi'(\xi) + p^2[a_2\varphi^2(\xi) + a_3\varphi^3(\xi) + a_4\varphi^4(\xi)] = 0, \tag{19}$$

where

$$a_0=(1+p)(1+2p)\delta,\quad a_1=p(1+p)(1+2p)\gamma,\\ a_2=(1+p)(1+2p)(sn^2-\lambda),\quad a_3=(1+2p)a,\quad a_4=(1+p)b. \tag{20}$$

According to Step 1 in Section 2, by balancing $\varphi(\xi)\varphi''(\xi)$ (or $\varphi'^2(\xi)$) and $\varphi^4(\xi)$ in Eq. (19), we get $m=1$. Therefore we suppose that Eq. (19) has the following formal solutions

$$\varphi(\xi)=A_0+A_1\sigma(\xi)+B_1\tau(\xi), \tag{21}$$

where $\sigma(\xi)$, $\tau(\xi)$ satisfies (3) and (4), where A_0, A_1, B_1 are constants to be determined later.

With the aid of Maple, substituting (21) along with (3) and (4) into (19), yields a set of algebraic equations for $\sigma^j(\xi)\tau^i(\xi)$ ($j=0,1,\ldots;\ i=0,1$). Setting the coefficients of these terms $\sigma^j\tau^i$ to zero yields a set of over-determined algebraic equations with respect to A_0, A_1, B_1, R, n and λ. (*Note 1*. Here we take $\epsilon=-1$)

$$2A_1B_1R(-a_0+2B_1^2p^2a_4\mu^2+a_0\mu^2+a_0p\mu^2-2B_1^2p^2a_4+2RA_1^2p^2a_4-a_0p)=0, \tag{22}$$

$$R(-Ra_0pA_1^2\mu+Rp^2a_3A_1^3-2Ra_0A_1^2\mu-12Rp^2a_4A_1^2B_1^2\mu+4Rp^2a_4A_0A_1^3-2a_1B_1A_1\mu^2-3\mu^3B_1^2pa_0-2a_0pA_0A_1\\ +2\mu B_1^2a_0+4\mu B_1^4a_4p^2+3\mu B_1^2pa_0-2\mu^3B_1^2a_0-4\mu^3B_1^4a_4p^2-12p^2a_4A_0A_1B_1^2-3p^2a_3A_1B_1^2+2a_1B_1A_1\\ +12p^2a_4A_0A_1B_1^2\mu^2+3p^2a3A_1B_1^2\mu^2+2a_0pA_0A_1\mu^2)=0, \tag{23}$$

$$p^2B_1R^2(RB_1^2a_3+3a_3A_0^2+4RB_1^2a_4A_0+2a_2A_0+4a_4A_0^3)=0, \tag{24}$$

$$R(-2Ra_0B_1A_1\mu-8Rp^2a_4A_1B_1^3\mu+12Rp^2a_4A_0A_1^2B_1+3Rp^2a_3A_1^2B_1-2Ra_0pB_1A_1\mu-Ra_1A_1^2+2a_0pA_0B_1\mu^2\\ +4p^2a_4A_0B_1^3\mu^2-p^2a_3B_1^3+a_1B_1^2-2a_0pA_0B_1-4p^2a_4A_0B_1^3-a_1B_1^2\mu^2+p^2a_3B_1^3\mu^2)=0, \tag{25}$$

$$-6p^2a_4A_1^2B_1^2R+a_0A_1^2\mu^2R+6p^2a_4A_1^2B_1^2\mu^2R-2a_0B_1^2\mu^2+a_0B_1^2+p^2a_4B_1^4-a_0pA_1^2R+p^2a_4B_1^4\mu^4+a_0pA_1^2\mu^2R\\ -2a_0pB_1^2\mu^2+a_0pB_1^2-2p^2a_4B_1^4\mu^2+p^2a_4A_1^4R^2+a_0pB_1^2\mu^4-a_0A_1^2R+a_0B_1^2\mu^4=0, \tag{26}$$

$$R^2(-a_1B_1A_1R+12p^2a_4A_0A_1B_1^2R-a_0pB_1^2\mu R+3p^2a_3A_1B_1^2R+a_0pA_0A_1R-4p^2a_4B_1^4\mu R+a_1A_0B_1\mu\\ -12p^2a_4A_0^2B_1^2\mu-6p^2a_3A_0B_1^2\mu+2p^2a_2A_0A_1+4p^2a_4A_0^3A_1+3p^2a_3A_0^2A_1-2p^2a2B_1^2\mu)=0, \tag{27}$$

$$R^2(4p^2a_4A_1B_1^3R+a_0pB_1A_1R-2p^2a_3B_1^3\mu+a_1B_1^2\mu-a_1A_0A_1+2p^2a_2A_1B_1+6p^2a_3A_0A_1B_1+12p^2a_4A_0^2A_1B_1\\ -8p^2a_4A_0B_1^3\mu-a_0pA_0B_1\mu)=0, \tag{28}$$

$$R(-3p^2a_3A_0B_1^2-6p^2a_4A_0^2B_1^2+Rp^2a_2A_1^2-6Rp^2a_3A_1B_1^2\mu+3Ra_0pB_1^2\mu^2-a_1A_0B_1\mu^2+6p^2a_4A_0^2B_1^2\mu^2+a_1A_0B_1\\ -2Rp^2a_4B_1^4+3p^2a_3A_0B_1^2\mu^2+6Rp^2a_4B_1^4\mu^2+6R^2p^2a_4A_1^2B_1^2-3Ra_0pA_0A_1\mu+p^2a_2B_1^2\mu^2-p^2a_2B_1^2\\ +3Ra_1A_1B_1\mu-24Rp^2a_4A_0A_1B_1^2\mu+3Rp^2a_3A_0A_1^2+6Rp^2a_4A_0^2A_1^2+R^2a_0A_1^2+Ra_0B_1^2\mu^2-2Ra_0pB_1^2)=0, \tag{29}$$

$$p^2R^2(6Ra_4A_0^2B_1^2+3Ra_3A_0B_1^2+R^2a_4B_1^4+a2A_0^2+a_4A_0^4+a_3A_0^3+Ra_2B_1^2)=0. \tag{30}$$

By use of the Maple soft package "Charsets" by Dongming Wang, which based on the Wu-elimination method [27], solving Eqs. (22)–(30), we get the following results.

Case 1

$$a_1=\pm\frac{B_1p^2(4a_4B_1\sqrt{R}+2pa_4B_1\sqrt{R}-a_3-pa_3)}{1+p},\quad a_0=-\frac{p^2a_4B_1^2}{1+p},\quad A_0=\pm B_1\sqrt{R},\quad A_1=\mu=0,\\ a_2=-2B_1\sqrt{R}(2a_4B_1\sqrt{R}-a_3). \tag{31}$$

Case 2

$$a_1=\mp\left\{2p^2B_1\left[2B_1A_1R^{3/2}(2a_4+p)+A_1Ra_3(1+p)\pm\sqrt{B_1^2+RA_1^2}[2a_4B_1R(2+p)+a_3\sqrt{R}(1+p)]\right]\right\}\\ \Big/\left\{\mp RA_1(1+p)+\sqrt{B_1^2+RA_1^2}\sqrt{R}(1+p)\right\},$$

$$A_0=\pm B_1\sqrt{R},\quad \mu=\pm\frac{\sqrt{B_1^2+RA_1^2}}{B_1},\quad a_0=-\frac{4p^2a_4B_1^2}{1+p},\quad a_2=-2B_1\sqrt{R}(2a_4B_1\sqrt{R}-a_3). \tag{32}$$

Case 3

$$a_1 = A_0 = B_1 = 0, \quad a_3 = -\frac{A_1 R a_4 \mu(2+p)}{\mu^2 - 1 + p\mu^2 - p}, \quad a_0 = -\frac{A_1^2 R p^2 a_4}{\mu^2 - 1 + p\mu^2 - p}, \quad a_2 = \frac{A_1^2 R^2 a_4}{\mu^2 - 1 + p\mu^2 - p}. \tag{33}$$

Case 4

$$A_0 = B_1 = a_1 = a_3 = \mu = 0, \quad a_0 = \frac{R A_1^2 p^2 a_4}{1+p}, \quad a_2 = -\frac{R^2 A_1^2 a_4}{1+p}. \tag{34}$$

From (12), (15), (18), (21) and Cases 1–4, we obtain the following solutions for Eq. (5)
Family 1

$$u_{11} = \left[\mp \sqrt{R} B_1 \pm B_1 \sqrt{R} \tanh(\sqrt{R}\xi)\right]^{1/p}, \tag{35}$$

$$u_{12} = \left[\mp \sqrt{R} B_1 \pm B_1 \sqrt{R} \coth(\sqrt{R}\xi)\right]^{1/p}, \tag{36}$$

where

$$\xi = x + ny - \lambda t, \quad \lambda = sn^2 \mp \frac{2B_1\sqrt{R}[\mp 2B_1\sqrt{R}b(1+p) + a(1+2p)]}{(1+p)(1+2p)},$$

$$B_1^2 = -\frac{(1+p)(1+2p)\delta}{bp^2}, \quad R = \frac{1}{4}\frac{[-\gamma(1+p)(1+2p) + B_1 pa(1+2p)]^2}{B_1^4 b^2 p^2 (p+2)^2}.$$

Family 2

$$u_{21} = \left[\mp \sqrt{R} B_1 \pm \sqrt{R(\mu^2-1)} B_1 \frac{\operatorname{sech}(\sqrt{R}\xi)}{\mu \operatorname{sech}(\sqrt{R}\xi) + 1} + B_1 \frac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu \operatorname{sech}(\sqrt{R}\xi) + 1}\right]^{1/p}, \tag{37}$$

$$u_{22} = \left[\mp \sqrt{R} B_1 \pm \sqrt{R(\mu^2-1)} B_1 \frac{\operatorname{csch}(\sqrt{R}\xi)}{\mu \operatorname{csch}(\sqrt{R}\xi) + 1} + B_1 \frac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu \operatorname{csch}(\sqrt{R}\xi) + 1}\right]^{1/p}, \tag{38}$$

where

$$\xi = x + ny - \lambda t, \quad \lambda = sn^2 \mp \frac{2B_1\sqrt{R}[\mp 2B_1\sqrt{R}b(1+p) + a(1+2p)]}{(1+p)(1+2p)},$$

$$B_1^2 = -\frac{(1+p)(1+2p)\delta}{4bp^2}, \quad R = \frac{1}{16}\frac{[-\gamma(1+p)(1+2p) + 2B_1 pa(1+2p)]^2}{B_1^4 b^2 p^2 (p+2)^2}.$$

Family 3. From Eq. (33), we obtain the following solutions for the KdV-type equations: $(u_t + au^p u_x + bu^{2p} u_x + \delta u_{xxx})_x + su_{yy} = 0$,

$$u_{31} = \left[A_1 \frac{R \operatorname{sech}(\sqrt{R}\xi)}{\mu \operatorname{sech}(\sqrt{R}\xi) + 1}\right]^{1/p}, \tag{39}$$

$$u_{32} = \left[A_1 \frac{R \operatorname{csch}(\sqrt{R}\xi)}{\mu \operatorname{csch}(\sqrt{R}\xi) + 1}\right]^{1/p}, \tag{40}$$

where

$$A_1 = \frac{\mu\delta(1+p)(2+p)}{ap^2}, \quad R = -\frac{(1+p)(1+2p)\delta(\mu^2-1)}{bp^2 A_1^2}, \quad \lambda = sn^2 - \frac{bR^2 A_1^2}{(\mu^2-1)(1+p)(1+2p)}.$$

Family 4. From Eq. (34), we obtain the following solutions for the KdV-type equations, $(u_t + bu^{2p}u_x + \delta u_{xxx})_x + su_{yy} = 0$, as follows:

$$u_{41} = \left[\frac{(1+p)(1+2p)\delta}{A_1p^2b}\operatorname{sech}\sqrt{\frac{(1+p)(1+2p)\delta}{A_1^2p^2b}}\left[x+ny-\left(sn^2+\frac{(1+p)(1+2p)\delta^2b}{A_1^2p^4}\right)t\right]\right]^{1/p}, \tag{41}$$

$$u_{42} = \left[\frac{(1+p)(1+2p)\delta}{A_1p^2b}\operatorname{csch}\sqrt{\frac{(1+p)(1+2p)\delta}{A_1^2p^2b}}\left[x+ny-\left(sn^2+\frac{(1+p)(1+2p)\delta^2b}{A_1^2p^4}\right)t\right]\right]^{1/p}, \tag{42}$$

where A_1 is an arbitrary constant.

The following periodic wave solutions obtained is under $\epsilon = 1$.

Family 5

$$u_{51} = \left[\mp\sqrt{-R}B_1 \pm B_1\sqrt{R}\tan(\sqrt{R}\xi)\right]^{1/p}, \tag{43}$$

$$u_{52} = \left[\mp\sqrt{-R}B_1 \pm B_1\sqrt{R}\cot(\sqrt{R}\xi)\right]^{1/p}, \tag{44}$$

where

$$\xi = x+ny-\lambda t, \quad \lambda = sn^2 \mp \frac{2B_1R[\mp 2B_1\sqrt{-R}b(1+p)+a(1+2p)]}{\sqrt{-R}(1+p)(1+2p)},$$

$$B_1^2 = -\frac{(1+p)(1+2p)\delta}{bp^2}, \quad R = -\frac{1}{4}\frac{[\gamma(1+p)(1+2p)+B_1pa(1+2p)]^2}{B_1^4b^2p^2(p+2)^2}.$$

Family 6

$$u_{61} = \left[\mp\sqrt{-R}B_1 \pm \sqrt{-R(\mu^2-1)}B_1\frac{\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1} + B_1\frac{\sqrt{R}\tan(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1}\right]^{1/p}, \tag{45}$$

$$u_{62} = \left[-\sqrt{R}B_1 \pm \sqrt{-R(\mu^2-1)}B_1\frac{\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1} - B_1\frac{\sqrt{R}\cot(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1}\right]^{1/p}, \tag{46}$$

where

$$\xi = x+ny-\lambda t, \quad \lambda = sn^2 \pm \frac{2B_1[\mp 2B_1\sqrt{-R}b(1+p)+a(1+2p)]}{\sqrt{-R}(1+p)(1+2p)},$$

$$B_1^2 = -\frac{(1+p)(1+2p)\delta}{4bp^2}, \quad R = -\frac{1}{16}\frac{[\gamma(1+p)(1+2p)+2B_1pa(1+2p)]^2}{B_1^4b^2p^2(p+2)^2}.$$

Family 7. The KdV-type equations, $(u_t + au^pu_x + bu^{2P}u_x + \delta u_{xxx})_x + su_{yy} = 0$, have the following solutions:

$$u_{71} = \left[A_1\frac{R\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1}\right]^{1/p}, \tag{47}$$

$$u_{72} = \left[A_1\frac{R\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1}\right]^{1/p}, \tag{48}$$

where

$$A_1 = -\frac{a(1+2p)(-1+\mu^2)}{(p+2)\mu bR}, \quad R = -\frac{(1+p)(1+2p)\delta(\mu^2-1)}{bp^2A_1^2}, \quad \lambda = sn^2 - \frac{bR^2A_1^2}{(\mu^2-1)(1+p)(1+2p)}.$$

Family 8. We obtain the solutions of the equation, $(u_t + bu^{2P}u_x + \delta u_{xxx})_x + su_{yy} = 0$, as follows:

$$u_{81} = \left[-\frac{(1+p)(1+2p)\delta}{A_1 p^2 b} \sec \sqrt{-\frac{(1+p)(1+2p)\delta}{A_1^2 p^2 b}} \left[x + ny + \left(sn^2 + \frac{(1+p)(1+2p)\delta^2 b}{A_1^2 p^4} \right) t \right] \right]^{1/p}, \tag{49}$$

$$u_{82} = \left[-\frac{(1+p)(1+2p)\delta}{A_1 p^2 b} \csc \sqrt{-\frac{(1+p)(1+2p)\delta}{A_1^2 p^2 b}} \left[x + ny + \left(sn^2 + \frac{(1+p)(1+2p)\delta^2 b}{A_1^2 p^4} \right) t \right] \right]^{1/p}, \tag{50}$$

where A_1 is an arbitrary constant.

Family 9. Rational solutions: When setting the solutions of Eq. (20) in the form (11), we obtain the following rational solutions for Eq. (5).

$$u_9 = \left[\pm \frac{\sqrt{-b\delta(3p + 2p^2 + 1)}}{bp} \cdot \frac{1}{x + ny - sn^2 t} \right]^{\frac{1}{p}}, \tag{51}$$

where $\gamma = \pm \frac{\sqrt{-(p+1)b\delta(1+2p)a}}{bp + b}$.

Remark

(1) The solutions obtained in this paper recover all of the solutions obtained by improved tanh method in [15]. For example, the solutions (35), (36), (43) and (44) are just the solutions (3.24)–(3.27) in [15] (see Ref. [15] for detail).
(2) The solutions obtained (37), (38), (45) and (46) are new and more general than the solutions (3.29)–(3.32) in [15]. It is easy to see that the solutions (3.29)–(3.32) in [15] can be recovered by setting $\mu = 0$ in (37), (38), (45) and (46).
(3) The solutions (39), (40), (47) and (48) are new families of exact solutions, which can not be obtained by the known various tanh methods.

4. Summary and conclusions

In summary, we have obtained many families of exact travelling wave solutions of general KdV-type and KdV–Burgers-type equations with nonlinear terms of any order, based upon the improved generally projective Riccati equations. Because Eq. (5) is more general than the equations studied in [12–26] and the method is more powerful, we find new and more general solutions, which include the previously known travelling wave solutions found by extended tanh method and other more sophisticated methods. We will extend the generally projective Riccati equations method to seek soliton-like solutions in the forthcoming works.

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