J. Math. Anal. Appl. 400 (2013) 624–634

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Binary Bell polynomial manipulations on the integrability of a generalized (2 + 1)-dimensional Korteweg–de Vries equation

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ARTICLE INFO

Article history:
Received 25 June 2012
Available online 21 November 2012
Submitted by G. Bluman

Keywords:
Binary Bell polynomial
Lax pair
Darboux covariant Lax pair
Bäcklund transformation
Infinite conservation law
N-soliton solution

ABSTRACT

This paper investigates the integrability of a generalized (2 + 1)-dimensional Korteweg–de Vries equation. With the aid of binary Bell polynomials, its bilinear formalism, bilinear Bäcklund transformation, Lax pair and Darboux covariant Lax pair are succinctly constructed, which can be reduced to the ones of several integrable equations such as the Korteweg–de Vries equation and the Calogero–Bogoyavlenskii–Schiff equation. Moreover, the infinite conservation laws of the generalized (2 + 1)-dimensional Korteweg–de Vries equation are found by virtue of binary Bell polynomials. All conserved densities and fluxes are given with explicit recursion formulas.

1. Introduction

The integrability of nonlinear evolution equations (NLEEs) has been investigated quite intensively in recent years, which can be regarded as a pretest and the first step of their exact solvability. There are many significant properties, such as bilinear formalism, bilinear Bäcklund transformation (BT), Lax pair, infinite conservation laws, infinite symmetries and Hamiltonian structure, that can characterize the integrability of NLEEs. There have been many methods proposed to deal with NLEEs, such as the inverse scattering transformation method [1], Darboux transformation method [2,3], Bäcklund transformation method [3–5], and Hirota's bilinear method [6,7]. Among these methods, Hirota's bilinear method is a direct method to derive the multisoliton solutions, bilinear BT and some other properties of a given NLEE [8–13]. The key of Hirota's bilinear method is the construction of bilinear formalism for a given NLEE; however, this process is not as one would wish. It relies on a particular skill by choosing suitable variable transformations, such as rational transformation and logarithmic transformation, but there is no general rule to find the transformations.

The advent of Lambert and co-workers' work [14–20] has had a significant impact on the integrability of NLEEs, which not only provides a procedure to construct a bilinear formalism but also the bilinear BT, Lax pair and Darboux covariant Lax pair for NLEEs in a lucid and systematic way by applying binary Bell polynomials. In terms of the binary Bell polynomial approach, one may obtain, on the one hand, such results as the Bell polynomial expressions (in the P or $\mathcal{Y}$ polynomial form), Bell polynomial-type BTs and Lax pairs, and, on the other hand, the connection between the Bell polynomial and Hirota bilinear method can be revealed, namely, the Bell polynomial expressions can be cast into a bilinear form, and the Bell polynomial-type BTs can be mapped into bilinear BTs. Then, both the Bell polynomial-type and bilinear BTs can lead to the corresponding Lax pairs. Moreover, a Darboux covariant Lax pair can also be obtained under a certain gauge

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doi:10.1016/j.jmaa.2012.11.028

transformation, whose form is invariant. For instance, Darboux covariant Lax pairs of the celebrated Korteweg–de Vries (KdV) equation and Kadomtsev–Petviashvili (KP) equation have been investigated by using binary Bell polynomials [18]. More recently, Fan extended this method to non-isospectral and variable-coefficient NLEEs, and proposed the generalized super Bell polynomials [21–24]. Tian extended this method to the generalized Lotka–Volterra equation, etc. [25,26].

The purpose of this paper is to extend the binary Bell polynomial approach to the following generalized (2 + 1)-dimensional KdV equation [27]:

$$4u_t - h_1\left(4uu_y + 2u_x\partial_x^{-1}u_y + u_{xxy}\right) - h_2\left(6uu_x + u_{xxx}\right) = 0, \tag{1.1}$$

which is derived from the following Lax pair:

$$L\varphi = \lambda\varphi, \qquad \varphi_t = T\varphi, \tag{1.2}$$

with

$$L = \partial_x^2 + u, \tag{1.3a}$$

$$T = h_1\partial_x^2\partial_y + h_2\partial_x^3 + h_3\partial_x^2 + \left[\frac{h_1}{2}\int u_y dx + \frac{3}{2}h_2 u\right]\partial_x + h_1 u\partial_y + \left[\frac{3}{4}h_1 u_y + \frac{3}{4}h_2 u_x + h_3 u\right], \tag{1.3b}$$

where λ is the spectrum parameter and $h_i = h_i(t), i = 1, 2, 3$. Eq. (1.1) includes some celebrated physical equations as special reductions, such as the KdV equation ($h_1 = 0, h_2 = -4$)

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.4}$$

and the Calogero–Bogoyavlenskii–Schiff (CBS) equation ($h_1 = -4, h_2 = 0$) [28]

$$u_t + 4uu_y + 2u_x\partial_x^{-1}u_y + u_{xxy} = 0. \tag{1.5}$$

For $h_1 = -4, h_2 = -4$, Eq. (1.1) can be reduced to the constant-coefficient (2 + 1)-dimensional KdV equation, whose exact solutions and BT are obtained by applying the singular manifold method [27]. To our knowledge, the bilinear formalism of Eq. (1.1) has not been obtained up to now. In the present paper, we will give the bilinear formalism of Eq. (1.1) and then investigate its integrability including N-soliton solutions, bilinear BT, Lax pair, Darboux covariant Lax pair and infinite conservation laws.

The structure of the paper is as follows. In Section 2, we give the Bell polynomials preliminary that will be used in this paper. These results will then be applied to construct the bilinear representations, N-soliton solutions, bilinear BT, Lax pair, Darboux covariant Lax pair and infinite conservation laws to Eq. (1.1) in Sections 3–6, respectively. Finally, Section 7 presents our conclusions.

2. Bell polynomials preliminary

To begin with, we will provide the basic terminology and notation that are necessary for understanding the subsequent results. For details, refer to [14–20].

Definition 1. With the assumption that f is a $\mathbb{C}^\infty$ function of x and $f_r = \partial_x^r$, $r = 1, 2, \ldots,$ then

$$Y_{nx}(f) \equiv Y_n(f_1, \ldots, f_n) = e^{-f}\partial_x^n e^f, \quad n = 1, 2, \ldots, \tag{2.1}$$

is a polynomial in the derivatives of f with respect to x, which is called the one-dimensional Bell polynomial or Y-polynomial.

The subscripts in the notation $Y_{nx}(f)$ denote the highest-order derivatives of f with respect to the variable x, e.g.,

$$Y_x(f) = f_x, \qquad Y_{2x} = f_{2x} + f_x^2, \qquad Y_{3x} = f_{3x} + 3f_xf_{2x} + f_x^3, \ldots. \tag{2.2}$$

Based on the one-dimensional Bell polynomials, we consider the following multi-dimensional Bell polynomials.

Definition 2. Let $f = f(x_1, x_2, \ldots, x_n)$ be a $\mathbb{C}^\infty$ function with multi-variables and

$$f_{r_1x_1,\ldots,r_lx_l} = \partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}f, \quad r_1 = 0, \ldots, n_1; \ldots, r_l = 0, \ldots, n_l, \tag{2.3}$$

where l denotes arbitrary integers; then

$$Y_{n_1x_1,\ldots,n_lx_l}(f) \equiv Y_{n_1,\ldots,n_l}(f_{r_1x_1,\ldots,r_lx_l}) = e^{-f}\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}e^f \tag{2.4}$$

is a polynomial in the partial derivatives of f with respect to $x_1, \ldots, x_l$, which called a multi-dimensional Bell polynomial (Y-polynomial).

For the special case $f = f(x,t)$, the associated two-dimensional Bell polynomials defined by (2.4) read

$$Y_{x,t}(f) = f_{x,t} + f_x f_t, \qquad Y_{2x,t}(f) = f_{2x,t} + f_{2x}f_t + 2f_{x,t}f_x + f_x^2 f_t, \ldots. \tag{2.5}$$

By virtue of the above multi-dimensional Bell polynomials, the multi-dimensional binary Bell polynomials can be defined as follows.

Definition 3.

$$\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,w) \equiv Y_{n_1,\ldots,n_l}(f)\Big|_{f_{r_1x_1,\ldots,r_lx_l}=\begin{cases} v_{r_1x_1,\ldots,r_lx_l}, & \sum_{i=1}^{l} r_i \text{ is odd},\\ w_{r_1x_1,\ldots,r_lx_l}, & \sum_{i=1}^{l} r_i \text{ is even},\end{cases}} \tag{2.6}$$

where the vertical line means that the elements on the left-hand side are chosen according to the rule on the right-hand side, and v and w are both $\mathbb{C}^\infty$ functions of $(x_1, x_2, \ldots, x_l)$.

For example, the first few lowest-order binary Bell Polynomials are

$$\begin{gathered}\mathcal{Y}_x(v) = v_x, \qquad \mathcal{Y}_{2x}(v,w) = w_{2x} + v_x^2, \qquad \mathcal{Y}_{x,y}(v,w) = w_{x,y} + v_xv_y, \qquad \mathcal{Y}_{3x}(v,w) = v_{3x} + 3v_xw_{2x} + v_x^3,\\ \mathcal{Y}_{2x,y}(v,w) = v_{2x,y} + 2v_xw_{x,y} + v_x^2v_y + w_{2x}v_y, \ldots.\end{gathered} \tag{2.7}$$

Proposition 1. *The relations between the binary Bell polynomials and the standard Hirota D-operators can be given by the identity*

$$\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}\left(v = \ln\frac{F}{G}, w = \ln FG\right) = (F\cdot G)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G, \tag{2.8}$$

where $\sum_{i=1}^{l} n_i \geq 1$, and the Hirota D-operators are defined by

$$D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G = (\partial_{x_1} - \partial_{x_1'})^{n_1}\cdots(\partial_{x_l} - \partial_{x_l'})^{n_l}F(x_1,\ldots,x_l)\times G(x_1',\ldots,x_l')|_{x_1'=x_1,\ldots,x_l'=x_l}. \tag{2.9}$$

In the particular case when $F = G$, formula (2.8) can be rewritten as

$$F^{-2}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F^2 = \mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(0, q = 2\ln F) = \begin{cases} 0, & \sum_{i=1}^{l} n_i \text{ is odd},\\ P_{n_1x_1,\ldots,n_lx_l}(q), & \sum_{i=1}^{l} n_i \text{ is even},\end{cases} \tag{2.10}$$

which is also called a P-polynomial:

$$P_{n_1x_1,\ldots,n_lx_l}(q) = \mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(0, q = 2\ln F), \tag{2.11}$$

which vanishes unless $\sum_{i=1}^{l} n_i$ is even.

The first few P-polynomials are

$$P_{2x}(q) = q_{2x}, \qquad P_{x,y}(q) = q_{x,y}, \qquad P_{4x}(q) = q_{4x} + 3q_{2x}^2, \qquad P_{3x,y}(q) = q_{3x,y} + 3q_{x,y}q_{2x}, \ldots. \tag{2.12}$$

Formulas (2.8) and (2.10) play an important role in linking NLEEs with their corresponding bilinear equations; that is, once an NLEE can be expressed as a linear combination of P-polynomials, then its bilinear equation can be established directly.

Proposition 2. *The binary Bell polynomials $\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,w)$ can be written as a combination of P-polynomials and Y-polynomials:*

$$\begin{aligned}(F\cdot G)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G &= \mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,w)|_{v=\ln F/G, w=\ln FG} = \mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v, v+q)|_{v=\ln F/G, q=2\ln G}\\ &= \sum_{n_1+\cdots+n_l=even}\sum_{r_1=0}^{n_1}\cdots\sum_{r_l=0}^{n_l}\prod_{i=1}^{l}\binom{n_i}{r_i}P_{n_1x_1,\ldots,n_lx_l}(q)\times Y_{(n_1-r_1)x_1,\ldots,(n_l-r_l)x_l}(v).\end{aligned} \tag{2.13}$$

Proposition 3. *In order to obtain the Lax pairs of corresponding NLEEs, we introduce the Hopf–Cole transformation $v = \ln\psi$, i.e., $\psi = \frac{F}{G}$; then the Y-polynomials can be written as*

$$Y_{n_1x_1,\ldots,n_lx_l}(v)\Big|_{v=\ln\psi}\frac{\psi_{n_1x_1,\ldots,n_lx_l}}{\psi}, \tag{2.14}$$

which provides the shortest way to the associated Lax systems of NLEEs.

3. Bilinear representation and N-soliton solutions

3.1. Bilinear representation

The invariance of Eq. (1.1) under the scale transformations

$$x \to \lambda x, \qquad y \to \lambda^{r_1} y, \qquad t \to \lambda^{r_2} t, u \to \lambda^{r_3} u, \tag{3.1}$$

with $r_3 = -2$, shows that a dimensionless field q can be related to the field u by setting

$$u = cq_{2x}, \tag{3.2}$$

with c being a free constant to be the appropriate choice such that Eq. (1.1) links with P-polynomials. Substituting (3.2) into (1.1), we can write the resulting equation in the form

$$4q_{2x,t} - h_1(4cq_{2x}q_{2x,y} + 2cq_{3x}q_{x,y} + q_{4x,y}) - h_2(6cq_{2x}q_{3x} + q_{5x}) = 0. \tag{3.3}$$

Further integrating Eq. (3.3) with respect to x yields

$$E(q) = 4q_{x,t} - h_2(q_{4x} + 3q_{2x}^2) - \frac{2}{3}h_1(q_{3x,y} + 3q_{2x}q_{x,y}) - \frac{1}{3}h_1\partial_x^{-1}\partial_y(q_{4x} + 3q_{2x}^2) = 0, \tag{3.4}$$

if we set $c = 1$ in terms of formula (2.12).

Then, we introduce an auxiliary variable z and impose a subsidiary constraint condition,

$$q_{4x} + 3q_{2x}^2 + q_{x,z} = 0, \tag{3.5}$$

which implies that Eq. (3.4) can be rewritten as the local form by eliminating the effect of the integration ∂_x^{-1}.

In terms of the subsidiary constraint condition (3.5), Eq. (3.4) reads

$$E(q) = 4q_{x,t} - h_2(q_{4x} + 3q_{2x}^2) - \frac{2}{3}h_1(q_{3x,y} + 3q_{2x}q_{x,y}) + \frac{1}{3}h_1 q_{y,z} = 0. \tag{3.6}$$

Now according to the P-polynomials (2.12), Eqs. (3.5) and (3.6) are then cast into a pair of equations in the form of P-polynomials:

$$P_{4x}(q) + P_{x,z}(q) = 0, \tag{3.7a}$$

$$4P_{x,t}(q) - h_2P_{4x}(q) - \frac{2}{3}h_1P_{3x,y}(q) + \frac{1}{3}h_1P_{y,z}(q) = 0. \tag{3.7b}$$

Finally, system (3.7) produces the bilinear representation of the generalized (2 + 1)-dimensional KdV equation (1.1):

$$(D_x^4 + D_xD_z)F \cdot F = 0, \tag{3.8a}$$

$$\left(4D_xD_t - h_2D_x^4 - \frac{2}{3}h_1D_yD_x^3 + \frac{1}{3}h_1D_yD_z\right)F \cdot F = 0, \tag{3.8b}$$

by using identity (2.8) and the following change of dependent variable:

$$q = 2\ln F \Leftrightarrow u = q_{2x} = 2(\ln F)_{2x}. \tag{3.9}$$

3.2. N-soliton solutions

Once the bilinear representation of Eq. (1.1) is given, associated soliton solutions are easily solved with the help of Hirota's bilinear method and symbolic computation. Here we leave out the computational process and give the N-soliton solutions directly.

The N-soliton solution of the generalized (2 + 1)-dimensional KdV equation (1.1) is

$$u = 2\left[\ln\left(\sum_{\mu=0,1} e^{\sum_{j=1}^{n}\mu_j\xi_j + \sum_{1\le j<l}^{n}\mu_j\mu_l A_{jl}}\right)\right]_{2x}, \qquad \xi_j = \kappa_j x + \iota_j y + \frac{\kappa_j^2(h_2\kappa_j + h_1\iota_j)t}{4} + \gamma_j, \qquad e^{A_{jl}} = \frac{(\kappa_j - \kappa_l)^2}{(\kappa_j + \kappa_l)^2},$$
$$j < l, j, l = 1, 2, 3, \ldots, \tag{3.10}$$

where $\sum_{\mu=0,1}$ indicates summation over all possible combinations of $\mu_j = 0, 1$ $(j = 1, 2, \ldots)$.

For $n = 1$, the single-soliton solution of the generalized (2 + 1)-dimensional KdV equation (1.1) can be written as

$$u = 2\left[\ln\left(1 + e^{\xi_1}\right)\right]_{2x}. \tag{3.11}$$

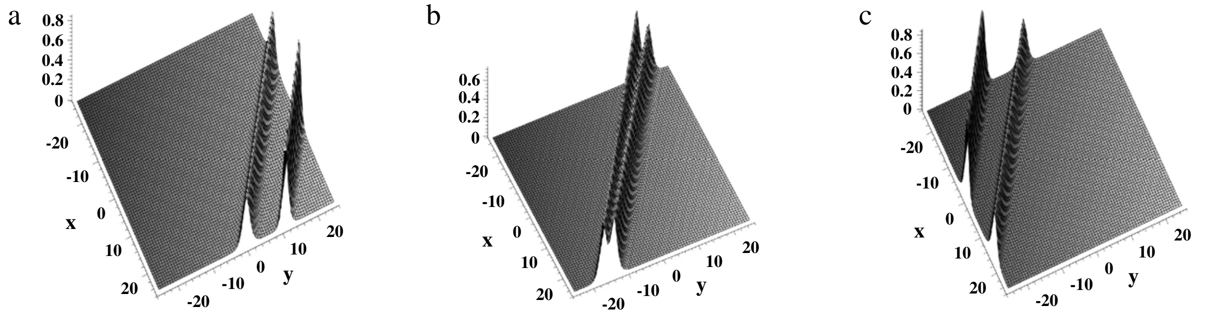

Fig. 1. The process of overtaking collision for two solitary waves (3.12) with $\kappa_1 = 1$, $\kappa_2 = 1.3$, $\iota_1 = 1.5$, $\iota_2 = 2$, $\gamma_1 = \gamma_2 = 0$ and (a): $t = -50$; (b): $t = 0$; (c): $t = 50$.

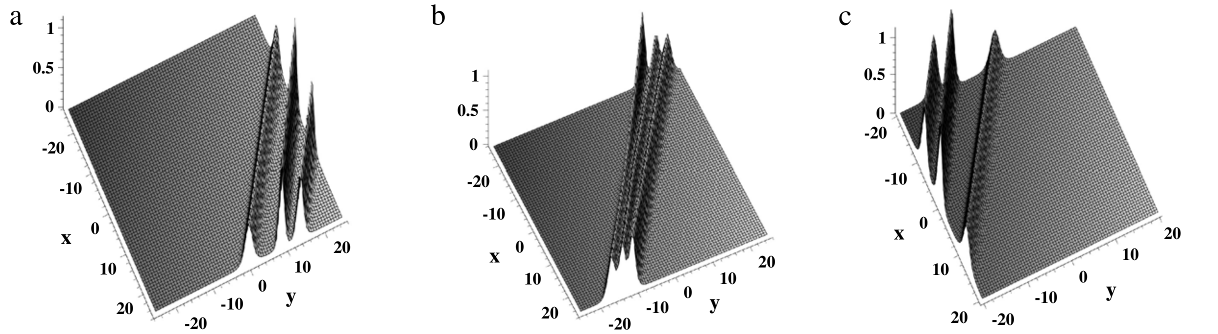

Fig. 2. The process of overtaking collision for three solitary waves (3.13) with $\kappa_1 = 1$, $\kappa_2 = 1.3$, $\kappa_3 = 1.5$, $\iota_1 = 1.5$, $\iota_2 = 2$, $\iota_3 = 2.5$, , $\gamma_1 = \gamma_2 = \gamma_3 = 0$ and (a): $t = -50$; (b): $t = 0$; (c): $t = 50$.

For $n = 2$, the two-soliton solution reads

$$u = 2\left[\ln\left(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1+\xi_2+A_{12}}\right)\right]_{2x}. \tag{3.12}$$

For $n = 3$, the three-soliton solution reads

$$u = 2\left[\ln\left(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_1+\xi_2+A_{12}} + e^{\xi_1+\xi_3+A_{13}} + e^{\xi_2+\xi_3+A_{23}} + e^{\xi_1+\xi_2+\xi_3+A_{12}+A_{13}+A_{23}}\right)\right]_{2x}. \tag{3.13}$$

When $n > 1$, soliton collisions such as overtaking will occur. Fig. 1 shows the two solitary collisions at $t = -50, 0, 50$, respectively. Fig. 2 shows the three solitary collisions at $t = -50, 0, 50$, respectively.

4. Bilinear Bäcklund transformation and associated Lax pair

4.1. Bilinear Bäcklund transformation

The bilinear BT is useful in constructing solutions and also serves as a characteristic of integrability for a given system. In this section, we shall consider the bilinear BT and Lax pair of generalized $(2 + 1)$-dimensional KdV equation (1.1).

Suppose that $q = 2\ln G$ and $\bar{q} = 2\ln F$ are two different solutions of Eq. (3.4). Then, we consider the two-field condition

$$\begin{aligned} E(\bar{q}) - E(q) = {} & 4(\bar{q} - q)_{x,t} - h_2\left[(\bar{q} - q)_{4x} + 3(\bar{q} - q)^2_{2x}\right] - \frac{2}{3}h_1\left[(\bar{q} - q)_{3x,y} + 3(\bar{q} - q)_{2x}(\bar{q} - q)_{x,y}\right] \\ & - \frac{1}{3}h_1\partial_x^{-1}\partial_y\left[(\bar{q} - q)_{4x} + 3(\bar{q} - q)^2_{2x}\right] = 0. \end{aligned} \tag{4.1}$$

The two-field condition (4.1) can be regarded as an ansatz for a bilinear BT and may produce the required transformation under appropriate additional constraints. To this end, we introduce two new variables,

$$v = \frac{(\bar{q} - q)}{2} = \ln\left(\frac{F}{G}\right), \qquad w = \frac{(\bar{q} + q)}{2} = \ln(FG), \tag{4.2}$$

on account of which, condition (4.1) can be rewritten as

$$E(\bar{q}) - E(q) = 8v_{x,t} - 2h_2 v_{4x} - 12h_2 w_{2x} v_{2x} - 2h_1 v_{3x,y} - 4h_1 w_{2x} v_{x,y} - 4h_1 2w_{x,y} v_{2x} - 4h_1 \partial_x^{-1}\Big[w_{2x,y} v_{2x} + w_{2x} v_{2x,y}\Big]$$
$$= 2\partial_x\Big[4\mathcal{Y}_t(v) - h_2\mathcal{Y}_{3x}(v,w) - h_1\mathcal{Y}_{2x,y}(v,w)\Big] + R(x,w) = 0, \tag{4.3}$$

where

$$R(v,w) = 2h_1\partial_x[(w_{2x}+v_x^2)v_y] - 6h_2 w_{2x}v_{2x} + 6h_2 w_{3x}v_x + 6h_2 v_x^2 v_{2x} - 4h_1 w_{2x} v_{x,y} + 4h_1 w_{2x,y} v_x$$
$$- 4h_1\partial_x^{-1}(w_{2x}v_{2x,y} + w_{2x,y}v_{2x}). \tag{4.4}$$

In order to decouple the two-field condition (4.3) into a pair of constraints, we impose such a constraint which enables us to express $R(v,w)$ as the x-derivative of a combination of $\mathcal{Y}$-polynomials. The simplest possible choice of such a constraint may be

$$\mathcal{Y}_{2x}(v,w) = w_{2x} + v_x^2 = \lambda, \tag{4.5}$$

where λ is an arbitrary constant. In terms of the constraint (4.5), $R(v,w)$ can be expressed as

$$R(v,w) = 2h_1\lambda v_{x,y} - 6h_2 w_{2x}v_{2x} + 6h_2 w_{3x}v_x + 6h_2 v_x^2 v_{2x} - 4h_1 w_{2x}v_{x,y} + 4h_1 w_{2x,y}v_x - 4h_1\lambda v_{x,y} + 4h_1 v_x^2 v_{x,y}$$
$$= -6\lambda\partial_x\big(h_2\mathcal{Y}_x(v) + h_1\mathcal{Y}_y(v)\big), \tag{4.6}$$

under the following relations:

$$w_{2x} = \lambda - v_x^2, \qquad w_{3x} = -2v_x v_{2x}, \quad \text{and} \quad w_{2x,y} = -2v_x v_{x,y}.$$

Then, combining relations (4.3)–(4.6), we deduce a coupled system of $\mathcal{Y}$-polynomials,

$$\mathcal{Y}_{2x}(v,w) - \lambda = 0, \tag{4.7a}$$

$$\partial_x\Big[4\mathcal{Y}_t(v) - h_2\mathcal{Y}_{3x}(v,w) - h_1\mathcal{Y}_{2x,y}(v,w)\Big] - 3\lambda\partial_x\Big[h_2\mathcal{Y}_x(v) + h_1\mathcal{Y}_y(v)\Big] = 0, \tag{4.7b}$$

where the second equation is useful for constructing conservation laws later. Based on identity (2.13), system (4.7) immediately leads to the bilinear BT

$$\Big[D_x^2 - \lambda\Big]F\cdot G = 0, \tag{4.8a}$$

$$\Big[4D_t - h_2D_x^3 - h_1D_x^2D_y - 3\lambda(h_2D_x + h_1D_y) - \mu\Big]F\cdot G = 0, \tag{4.8b}$$

where we have integrated the second equation in system (4.7b) with respect to x, and μ is an arbitrary constant.

4.2. Lax pair

In this section, we shall derive the Lax pairs of Eq. (1.1) by applying system (4.7).

In order to linearize the Bell system (4.7) into a Lax pair, we make the Hopf–Cole transformation $v = \ln\psi$; then, on account of formula (2.14), we have

$$\mathcal{Y}_t(v) = \frac{\psi_t}{\psi}, \qquad \mathcal{Y}_y(v) = \frac{\psi_y}{\psi}, \qquad \mathcal{Y}_{2x}(v,w) = q_{2x} + \frac{\psi_{2x}}{\psi}, \qquad \mathcal{Y}_{3x}(v,w) = \frac{3q_{2x}\psi_x}{\psi} + \frac{\psi_{3x}}{\psi},$$
$$\mathcal{Y}_{2x,y}(v,w) = \frac{2q_{x,y}\psi_x}{\psi} + \frac{q_{2x}\psi_y}{\psi} + \frac{\psi_{2x,y}}{\psi}, \ldots. \tag{4.9}$$

Thus, system (4.7) is then linearized into a Lax pair with double parameters λ and μ:

$$L_1\psi = (\partial_x^2 + q_{2x})\psi = \lambda\psi, \tag{4.10a}$$

$$4\psi_t + L_2\psi = \Big[4\partial_t - 3h_2q_{2x}\partial_x - h_2\partial_x^3 - 2h_1q_{x,y}\partial_x - h_1q_{2x}\partial_y - h_1\partial_x^2\partial_y - 3\lambda(h_2\partial_x + h_1\partial_y)\Big]\psi = \mu\psi, \tag{4.10b}$$

or equivalently

$$\psi_{2x} + (u-\lambda)\psi = 0, \tag{4.11a}$$

$$4\psi_t - 3h_2u\psi_x - h_2\psi_{3x} - 2h_1\left(\int u_y dx\right)\psi_x - h_1u\psi_y - h_1\psi_{2x,y} - 3\lambda(h_2\psi_x + h_1\psi_y) - \mu\psi = 0. \tag{4.11b}$$

It is easy to check that, for the equations

$$L_1\psi = \lambda\psi, \qquad 4\psi_t + L_2\psi = \mu\psi, \tag{4.12}$$

their integrability condition

$$0 = L_{1,t} - [L_1, L_2] = -\frac{1}{4}\Big[4u_t - h_1(4uu_y + 2u_x\partial_x^{-1}u_y + u_{xxy}) - h_2(6uu_x + u_{xxx})\Big] \tag{4.13}$$

exactly gives the generalized (2 + 1)-dimensional KdV equation (1.1) by replacing q_{2x} with u.

5. Darboux covariant Lax pair

In this section, we obtain a kind of Darboux covariant Lax pair whose form is invariant under a certain gauge transformation. Now, we go back to the generalized (2+1)-dimensional KdV equation (1.1) and the associated linear system (4.10). Assume that ϕ is a solution for eigenvalue Eq. (4.10a). As is well known, the gauge transformation

$$T = \phi\partial_x\phi^{-1} = \partial_x - \sigma, \qquad \sigma = \partial_x \ln\phi, \tag{5.1}$$

maps the operator $L_1(q) - \lambda$ onto a similar operator

$$T\Big(L_1(q) - \mu\Big)T^{-1} = \bar{L}_1(\tilde{q}) - \lambda, \tag{5.2}$$

which satisfies the covariance condition

$$\tilde{L}_1(\tilde{q}) = L_1(\tilde{q} = q + \Delta q) \quad \text{with } \Delta q = 2\ln\phi. \tag{5.3}$$

But it can be verified that a similar property does not hold for the evolution Eq. (4.10b). However, one can find another third-order operator $L_{2,\mathrm{cov}}(q)$ with appropriate coefficients, such that $\partial_t + L_{2,\mathrm{cov}}(q)$ can be mapped, by gauge transformation (5.1), onto a similar operator $\tilde{L}_{2,\mathrm{cov}}(\tilde{q})$ which satisfies the covariance condition

$$\tilde{L}_{2,\mathrm{cov}}(\tilde{q}) = L_{2,\mathrm{cov}}(\tilde{q} = q + \Delta q). \tag{5.4}$$

Therefore, suppose that ϕ is a solution of the following Lax pair:

$$L_1\phi = \lambda\phi, \qquad 4\phi_t + L_{2,\mathrm{cov}}\phi = 0, \tag{5.5}$$

where

$$L_{2,\mathrm{cov}} = -4h_1\partial_y\partial_x^2 - 4h_2\partial_x^3 + b_1\partial_x + b_2\partial_y + b_3,$$

and b_1, b_2 and b_3 are functions to be determined. It suffices that we require the transformation T to map the operator $\partial_t + L_{2,\mathrm{cov}}$ onto the similar one

$$T(\partial_t + L_{2,\mathrm{cov}})T^{-1} = 4\partial_t + \tilde{L}_{2,\mathrm{cov}}, \tag{5.6}$$

where

$$\tilde{L}_{2,\mathrm{cov}} = -4h_1\partial_y\partial_x^2 - 4h_2\partial_x^3 + \tilde{b}_1\partial_x + \tilde{b}_2\partial_y + \tilde{b}_3, \tag{5.7}$$

and $\tilde{b}_1, \tilde{b}_2$ and $\tilde{b}_3$ satisfy the following covariant condition:

$$\tilde{b}_j = b_j(q) + \Delta b_j = b_j(q + \Delta q), \quad j = 1, 2, 3. \tag{5.8}$$

By virtue of (5.5) and (5.6), we can find that

$$\Delta b_1 = \tilde{b}_1 - b_1 = -12h_2\sigma_x - 4h_1\sigma_y, \tag{5.9a}$$

$$\Delta b_2 = \tilde{b}_2 - b_2 = -8h_1\sigma_x, \tag{5.9b}$$

$$\Delta b_3 = \tilde{b}_3 - b_3 = -8h_1\sigma_{x,y} + \sigma\Delta b_1 - 12h_2\sigma_{2x} + b_{1,x}, \tag{5.9c}$$

and

$$b_{2,x} - 4h_1\sigma_{2x} - 8h_1\sigma\sigma_x = 0, \tag{5.10a}$$

$$4\sigma_t - 4h_1\sigma_{2x,y} + b_1\sigma_x - 12h_1\sigma_x\sigma_y - 12h_2\sigma_x^2 - 4h_2\sigma_{3x} + b_2\sigma_y + \tilde{b}_3\sigma + b_{3,x} = 0. \tag{5.10b}$$

According to relation (5.8), it remains to determine b_1, b_2, and b_3 in the form of polynomial expressions in terms of derivatives of q:

$$b_j = F_j(q, q_x, q_y, q_{x,y}, q_{2x}, q_{2y}, q_{2x,y}, \ldots), \quad j = 1, 2, 3, \tag{5.11}$$

such that

$$\Delta F_j = F_j(q + \Delta q, q_x + \Delta q_x, \ldots) - F_j(q, q_x, q_y, \ldots) = \Delta b_j, \tag{5.12}$$

with $\Delta q_{kx,ly} = 2\Big(\ln q\Big)_{kx,ly}$, $k, l = 1, 2, \ldots$, and the Δb_j being determined by relations (5.9)–(5.10).

Thus, in order to satisfy the first condition

$$\Delta b_1 = \Delta F_1 = F_{1,q}\Delta q + F_{1,q_x}\Delta q_x + F_{1,q_y}\Delta q_y + \cdots = -12h_2\sigma_x - 4h_1\sigma_y = -6h_2\Delta q_{2x} - 2h_1\Delta q_{x,y}, \tag{5.13}$$

one chooses

$$b_1 = F_1(q_{2x}) + F_1(q_{x,y}) = -6h_2 q_{2x} - 2h_1 q_{x,y} + c_1, \tag{5.14}$$

with c_1 being an arbitrary constant.

Proceeding in the same way, the function b_2 can be determined as

$$b_2 = F_1(q_{2x}) = -4h_1 q_{2x} + c_2, \tag{5.15}$$

with c_2 being an arbitrary constant.

We find that relation (5.9c) contains the term

$$b_{1,x} = -6h_2 q_{3x} - 2h_1 q_{2x,y}, \tag{5.16}$$

which should be eliminated such that Δb_3 admits the form (5.12). It follows from the eigenvalue equation in (5.5) that we can find the following relation:

$$q_{3x} = -\sigma_{2x} - 2\sigma\sigma_x, \qquad q_{2x,y} = -\sigma_{x,y} - 2\sigma\sigma_y. \tag{5.17}$$

Substituting (5.14) and (5.17) into (5.9c) yields

$$\Delta b_3 = -6(h_1\sigma_{x,y} + h_2\sigma_{2x}) = -3(h_1\Delta q_{2x,y} + h_2\Delta q_{3x}). \tag{5.18}$$

It is can verified that the third condition

$$\Delta F_3 = F_{3,q}\Delta q + F_{3,q_x}\Delta q_x + F_{3,q_y}\Delta q_y + \cdots = \Delta b_3, \tag{5.19}$$

can be satisfied, if one chooses

$$b_3 = F_3(q_{2x,y}) + F_3(q_{3x}) = -3(h_2 q_{3x} + h_1 q_{2x,y}) + c_3, \tag{5.20}$$

in which c_3 is an arbitrary constant.

Setting $c_1 = c_2 = c_3 = 0$ in (5.14), (5.15) and (5.20), it follows from (5.5) that we find the following Darboux covariant evolution equation:

$$\begin{aligned} &4\phi_t + L_{2,\mathrm{cov}}\phi = 0, \\ &L_{2,\mathrm{cov}} = -4h_1\partial_x^2\partial_y - 4h_2\partial_x^3 - 2(3h_2q_{2x} + h_1q_{x,y})\partial_x - 4h_1q_{2x}\partial_y - 3(h_1q_{2x,y} + h_2q_{3x}), \end{aligned} \tag{5.21}$$

which is in agreement with Eq. (5.10). Moreover, the relation between the operator $L_{2,\mathrm{cov}}$ and the operator L_2 is given by

$$L_{2,\mathrm{cov}} = L_2 - 3(h_2\partial_x + h_1\partial_y)(L_1 - \lambda). \tag{5.22}$$

The integrability condition of the Darboux covariant Lax pair (5.5) precisely gives rise to Eq. (1.1) in Lax representation

$$[\partial_t + L_{2,\mathrm{cov}}, L_1] = -\frac{1}{4}\Big[4u_t - h_1(4uu_y + 2u_x\partial_x^{-1}u_y + u_{xxy}) - h_2(6uu_x + u_{xxx})\Big] = 0. \tag{5.23}$$

From the above, the higher operators can be obtained in a similar way step by step, which are Darboux covariant with respect to L_1, so as to obtain higher-order members of Eq. (1.1).

6. Infinite conservation laws

The conservation law plays an important role in nonlinear mathematical physics; for example, it describes the conservation of fundamental physical quantities, provides a method to study quantitative and qualitative properties of equations and their solutions, verifies complete integrability of nonlinear partial differential equations and is used to test numerical integrators, etc. Generally, the infinite conservation laws or conserved quantities for both continuous system and discrete system can be derived from Bäcklund transformations [29], from Lax pairs [29,30], from the formal solutions of eigenfunctions [31], from the scattering problem [29,32], from the trace identity [33], from the quasi-differential operator based on the Sato theory [34], etc. In this section, we derive the infinitely local conservation laws for generalized (2 + 1)-dimensional KdV equation (1.1) by using binary Bell polynomials.

In fact, the conservation laws actually have been hinted at in the two-field constraint system (4.7), which can be rewritten in the conserved form

$$\mathcal{Y}_{2x}(v,w)-\lambda=0, \tag{6.1a}$$

$$4\partial_t\mathcal{Y}_x(v)-\partial_x\Big[h_2\mathcal{Y}_{3x}(v,w)+h_1\mathcal{Y}_{2x,y}(v,w)+3\lambda h_2\mathcal{Y}_x(v)\Big]-\partial_y\Big[3\lambda h_1\mathcal{Y}_x(v)\Big]=0, \tag{6.1b}$$

by applying the relation

$$\partial_t\mathcal{Y}_x(v)=\partial_x\mathcal{Y}_t(v)=v_{x,t},\quad \text{and}\quad \partial_x\mathcal{Y}_y(v)=\partial_y\mathcal{Y}_x(v)=v_{x,y}.$$

By introducing a new potential function

$$\eta=\frac{\bar q_x-q_x}{2}, \tag{6.2}$$

it follows from the relation (4.2) that

$$v_x=\eta,\qquad w_x=q_x+\eta. \tag{6.3}$$

Substituting (6.3) into (6.1), we get a Riccati-type equation,

$$\eta_x+\eta^2+q_{2x}=\lambda, \tag{6.4}$$

and a divergence-type equation,

$$4\eta_t-\partial_x\Big[h_2\eta_{2x}+6h_2\lambda\eta-2h_2\eta^3+h_1\eta_{x,y}+2h_1\eta q_{x,y}+2h_1\eta\eta_y\Big]-4h_1\lambda\eta_y=0. \tag{6.5}$$

Suppose that $\lambda=\varepsilon^2$. Under the transformation $\eta=\bar\eta+\varepsilon$, we have

$$\bar\eta_x+\bar\eta^2+2\varepsilon\bar\eta+q_{2x}=0, \tag{6.6}$$

and

$$4\bar\eta_t-\partial_x\Big[h_2\bar\eta_{2x}-6\varepsilon h_2\bar\eta^2-2h_2\bar\eta^3+2h_1\bar\eta q_{x,y}+2\varepsilon h_1q_{x,y}+2h_1\varepsilon\bar\eta_y\Big]-\partial_y\Big[h_1\bar\eta_{2x}+2h_1\bar\eta\bar\eta_x+4\varepsilon^2h_1\bar\eta\Big]=0. \tag{6.7}$$

To proceed, inserting the expansion

$$\bar\eta=\sum_{n=1}^{\infty}\mathcal{I}_n(q,q_x,\ldots)\varepsilon^{-n}, \tag{6.8}$$

into (6.6) and equating the coefficients for powers of ε, we then obtain the recursion relations for the conserved densities:

$$\mathcal{I}_1=-\frac12q_{2x}=-\frac12u, \tag{6.9a}$$

$$\mathcal{I}_2=-\frac12\mathcal{I}_{1,x}=\frac14q_{3x}=\frac14u_x, \tag{6.9b}$$

$$\mathcal{I}_3=-\frac12(\mathcal{I}_{2,x}+\mathcal{I}_1^2)=-\frac18(u_{2x}+u^2), \tag{6.9c}$$

$$\mathcal{I}_4=-\frac12(I_{3,x}+2I_1I_2)=\frac1{16}(u_{3x}+4uu_x), \tag{6.9d}$$

$$\cdots, \tag{6.9e}$$

$$\mathcal{I}_{n+1}=-\frac12(\mathcal{I}_{n,x}+\sum_{k=1}^{n}\mathcal{I}_k\mathcal{I}_{n-k}),\quad n=2,3,\ldots. \tag{6.9f}$$

Substituting (6.8) into (6.7) yields

$$\begin{aligned}4\sum_{n=1}^{\infty}I_{n,t}\varepsilon^{-n}&-\partial_x\Bigg[h_2\sum_{n=1}^{\infty}I_{n,2x}\varepsilon^{-n}-6h_2\varepsilon\Big(\sum_{n=1}^{\infty}I_n\varepsilon^{-n}\Big)^2-2h_2\Big(\sum_{n=1}^{\infty}I_n\varepsilon^{-n}\Big)^3+2h_1q_{x,y}\sum_{n=1}^{\infty}I_n\varepsilon^{-n}+2h_1\varepsilon q_{x,y}\\&+2h_1\varepsilon\sum_{n=1}^{\infty}I_{n,y}\varepsilon^{-n}\Bigg]-\partial_y\Bigg[h_1\sum_{n=1}^{\infty}I_{n,2x}\varepsilon^{-n}+2h_1\sum_{n=1}^{\infty}I_n\varepsilon^{-n}\sum_{n=1}^{\infty}I_{n,x}\varepsilon^{-n}+4h_1\varepsilon^2\sum_{n=1}^{\infty}I_n\varepsilon^{-n}\Bigg]=0,\end{aligned} \tag{6.10}$$

which provides us with the infinite consequence of conservation laws:

$$4\mathcal{I}_{n,t}+\mathcal{F}_{n,x}+\mathcal{G}_{n,y}=0,\quad n=1,2,\ldots. \tag{6.11}$$

In Eq. (6.11), the conserved densities $\mathcal{I}_n's$ are given by formula (6.9), while the first fluxes $\mathcal{F}_n's$ are given explicitly by the recursion formulas

$$\mathcal{F}_1 = \frac{1}{2}\Big[h_2u_{2x} + 3h_2u^2 + 2h_1u\partial_x^{-1}u_y - h_1u_{x,y}\Big], \tag{6.12a}$$

$$\mathcal{F}_2 = -\frac{1}{4}\Big[h_2u_{3x} + 6h_2uu_x + 2h_1u_x\partial_x^{-1}u_y - h_1u_{2x,y} - 2h_1uu_y\Big], \tag{6.12b}$$

$$\cdots, \tag{6.12c}$$

$$\mathcal{F}_n = -h_2I_{n,2x} + 6h_2\sum_{k=1}^{n} I_kI_{n-k+1} + 2h_2\sum_{i+j+k=n} I_iI_jI_k - 2h_1\partial_x^{-1}u_yI_n - 2h_1I_{n,y}, \quad n = 3, 4, \ldots, \tag{6.12d}$$

and the second fluxes $\mathcal{G}_n's$ are

$$\mathcal{G}_1 = \frac{1}{2}\Big[2h_1u_{2x} + h_1u^2\Big], \tag{6.13a}$$

$$\mathcal{G}_2 = -\frac{1}{2}\Big[h_1u_{3x} + 3h_1uu_x\Big], \tag{6.13b}$$

$$\cdots, \tag{6.13c}$$

$$\mathcal{G}_n = -h_1I_{n,2x} - 2h_1\sum_{k=1}^{n} I_kI_{n-k} - 4h_1I_{n+2}, \quad n = 3, 4, \ldots. \tag{6.13d}$$

We present recursion formulas (6.12) and (6.13) for generating an infinite sequence of conservation laws; the first few conserved densities and associated fluxes are explicitly given. The first equation of conservation law Eq. (6.11) is exactly the generalized $(2+1)$-dimensional KdV equation (1.1).

7. Conclusions

The present investigation has been carried out on the generalized $(2+1)$-dimensional KdV equation (1.1), which is derived from a Lax pair. With binary Bell polynomials, expression (3.8) has been obtained, based on which the analytic N-soliton solution has been derived as expression (3.8) via symbolic computation. Then, its bilinear Bäcklund transformation, Lax pair, Darboux covariant Lax pair and infinite conservation laws are obtained. Compared with the Hirota bilinear method to derive the bilinear equations, bilinear Bäcklund transformations, Lax pairs and infinite conservation laws, the Bell polynomial manipulations employed here are shown to be more direct and systematic.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant No. 11075055), Innovative Research Team Program of the National Natural Science Foundation of China (61021004) and Shanghai Leading Academic Discipline Project (No. B412).

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