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Symmetry Analysis and Conservation Laws to the (2+1)-Dimensional Coupled Nonlinear Extension of the Reaction-Diffusion Equation*

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Abstract *In this paper, a detailed Lie symmetry analysis of the (2+1)-dimensional coupled nonlinear extension of the reaction-diffusion equation is presented. The general finite transformation group is derived via a simple direct method, which is equivalent to Lie point symmetry group actually. Similarity reduction and some exact solutions of the original equation are obtained based on the optimal system of one-dimensional subalgebras. In addition, conservation laws are constructed by employing the new conservation theorem.*

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Key words: (2+1)-dimensional coupled nonlinear reaction-diffusion equation, Lie symmetry, invariant solutions, optimal system, conservation laws

1 Introduction

In nonlinear science, especially in integrable systems, to study the symmetry of partial differential equations (PDEs) is one of the most important and essential task because of the existence of infinitely many symmetries. For a given nonlinear system, Lie symmetry method, proposed by Sophus Lie[1−2] during the nineteenth century is a standard method to find the corresponding Lie point symmetry algebras and groups. Several universal applications of Lie symmetry groups in differential equations were discussed in the literatures.[1−12] such as reduction of order of ordinary differential equations and dimension of PDEs, construction of invariant solutions, mapping solutions to other solutions and the detection of linearizing transformations, derivation of conservation laws and so on.

There have been several generalizations of the classical Lie group method for symmetry reductions. Bluman and Cole[3] proposed the method of conditional symmetries, since all solutions of the classical determining equations necessarily satisfy the nonclassical determining equations, and the solution set may be larger in the nonclassical case. From the theory, these methods were further generalized by Olver and Rosenau[4] to include weak symmetries and, even more generally, side conditions or differential constraints. More recently, motivated by the fact that symmetry reductions of the Boussinesq equation are known that are not obtained using the classical Lie group method, Clarkson and Kruskal[5] (CK) introduced a simple direct method to find all the possible similarity reductions of a nonlinear system without using any group theory. Lou and Ma[6] modified CK's direct method to find the generalized Lie and non-Lie symmetry groups for some well-known nonlinear equations.

On the other hand, the concept of conservation laws plays an important role in the study of nonlinear science.[13−20] The mathematical idea of conservation laws comes from the formulation of familiar physical laws such as for mass, energy, and momentum. In the study of PDEs, construction of explicit forms of the conserved quantity is meaningful, since they are used for the development of appropriate numerical methods and for mathematical analysis, particularly, existence, uniqueness and stability analysis of solutions.[13−15] In addition, the existence of a large number of conservation laws of a partial differential equation (system) is a strong indication of its integrability. The famous Noether’s theorem[17] provides a systematic and effective way of determining conservation laws for Euler–Lagrange differential equations once their Noether symmetries are known, but the application of this theorem depends upon the knowledge of a suitable Lagrangian. To find conservation laws of differential equations without classical Lagrangians, researchers have made various generalizations of Noether's theorem from different aspects.[18] Among these extended methods, the new conservation theorem, also called nonlocal conservation theorem, recently introduced by Ibragimov[19] is one of the most frequently used approaches.

Very recently, while investigating an integrable (2+1)-dimensional (modified) Heisenberg ferromagnet (HF)

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model[21] by using the prolongation structure theory, Zhai *et al.*[22] have constructed its corresponding geometrical equivalent counterparts, such as the (2+1)-dimensional nonlinear Schrödinger equation and the coupled (2+1)-dimensional integrable equations, presented through the motion of Minkowski space curves endowed with an additional spatial variable. These last coupled (2+1)-dimensional integrable equations, namely, the (2+1)-dimensional coupled nonlinear extension of the reaction-diffusion (CNLERD) equation, may be given by[22–23]

$$\begin{aligned}&u_t + u_{xy} + \delta w u = 0, \quad v_t - v_{xy} - \delta w v = 0\,,\\&w_x + (uv)_y = 0\,,\end{aligned}\tag{1}$$

where $\delta^2 = 1$, u, v and w are physical observables and subscripts denote partial differentiation. By means of the technique of Painlevé analysis, the complete integrability of the CNLERD equation (1), has been investigated by Thomas and Victor *et al.*[24–25] At the same time, they have unearthed and discussed a great variety of localized coherent structures with the aid of the Hirota's formalism and the multilinear variable separation approach. When $\delta = -1$, Bekir[26] and Malik[27] obtained some exact travelling wave solutions of Eq. (1) by using the tanh-coth method and (G'/G)-expansion method.

The outline of this paper is as follows. In Sec. 2, by using the classical method, we perform the Lie point symmetry of the CNLERD equation (1) and present the commutator relations of the generators associated with the symmetry. In Sec. 3, with the help of a simple direct method, the finite transformation group of Eq. (1) is derived, which is equivalent to Lie point symmetry group generated via the standard approaches. In Sec. 4, after an optimal system of one-dimensional subalgebras is constructed, we give the corresponding similarity reduction and provide some exact solutions of the original equation (1). Section 5 is devoted to construct conservation laws for the CNLERD equation (1) using the new conservation theorem. Some conclusions and discussions are given in the last section.

2 Lie Point Symmetry by Classical Symmetry Method

In this section we perform Lie symmetry analysis for the (2+1)-dimensional CNLERD equation (1). Let us consider a one-parameter Lie group of infinitesimal transformation

$$\begin{aligned}&x \to x + \epsilon X(x,y,t,u,v,w)\,,\\&y \to y + \epsilon Y(x,y,t,u,v,w)\,,\\&t \to t + \epsilon T(x,y,t,u,v,w)\,,\\&u \to u + \epsilon U(x,y,t,u,v,w)\,,\\&v \to v + \epsilon V(x,y,t,u,v,w)\,,\\&w \to w + \epsilon W(x,y,t,u,v,w)\,,\end{aligned}\tag{2}$$

with a small parameter $\epsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$\boldsymbol{v} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + W\frac{\partial}{\partial w}\,.\tag{3}$$

Then the invariance of system (1) under transformation (2) leads to the expressions for the functions $\{X, Y, T, U, V, W\}$ of the form

$$\begin{aligned}&X = c_6 xt + c_4 x + f\,,\\&Y = c_6 yt + c_5 y - c_4 y + c_3 t + c_1\,,\\&T = c_6 t^2 + c_5 t + c_2\,,\\&U = (c_6 xy + c_3 x + \dot{f} y + g)u\,,\\&V = -(c_6 xy + c_3 x + \dot{f} y + g + 2c_6 t + 2c_4)v\,,\\&W = -(\ddot{f} y + \dot{g} + c_6)\delta - (2c_6 t + c_5)w\,,\end{aligned}\tag{4}$$

where f and g are arbitrary functions of t, c_i ($i = 1, 2, \ldots, 6$) are arbitrary constants, and the dot over the function means its derivative with respect to time t. The presence of the arbitrary function and constants leads to an infinite-dimensional Lie algebra of symmetries. A general element of this algebra is written as

$$\begin{aligned}\boldsymbol{v} = {}& c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + c_4\boldsymbol{v}_4 + c_5\boldsymbol{v}_5 + c_6\boldsymbol{v}_6\\&+ \boldsymbol{v}_7(f) + \boldsymbol{v}_8(g)\,,\end{aligned}\tag{5}$$

where

$$\begin{aligned}&\boldsymbol{v}_1 = \frac{\partial}{\partial y}, \quad \boldsymbol{v}_2 = \frac{\partial}{\partial t}\,,\\&\boldsymbol{v}_3 = t\frac{\partial}{\partial y} + xu\frac{\partial}{\partial u} - xv\frac{\partial}{\partial v}\,,\\&\boldsymbol{v}_4 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2v\frac{\partial}{\partial v}\,,\\&\boldsymbol{v}_5 = y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - w\frac{\partial}{\partial w}\,,\\&\boldsymbol{v}_6 = xt\frac{\partial}{\partial x} + yt\frac{\partial}{\partial y} + t^2\frac{\partial}{\partial t} + xyu\frac{\partial}{\partial u}\\&\qquad - (xy + 2t)v\frac{\partial}{\partial v} - (\delta + 2wt)\frac{\partial}{\partial w}\,,\\&\boldsymbol{v}_7(f) = f\frac{\partial}{\partial x} + \dot{f}yu\frac{\partial}{\partial u} - \dot{f}yv\frac{\partial}{\partial v} - \delta\ddot{f}y\frac{\partial}{\partial w}\,,\\&\boldsymbol{v}_8(g) = gu\frac{\partial}{\partial u} - gv\frac{\partial}{\partial v} - \delta\dot{g}\frac{\partial}{\partial w}\,,\end{aligned}\tag{6}$$

construct a basis for the vector space. The associated Lie algebra among these vector fields are given by Table 1, where the entry in the j-th row and the k-th column represents the commutator $[\boldsymbol{v}_j, \boldsymbol{v}_k]$.

We now consider a point transformation

$$G : (x,y,t,u,v,w) \mapsto (\chi, \zeta, \gamma, P, Q, R)\,.\tag{7}$$

From the transformation (2), we have the corresponding one-parameter group of symmetries of the (2+1)-dimensional CNLERD equation

$$\begin{aligned}
&G_1:(x,y,t,u,v,w)\mapsto(x,y+\epsilon,t,u,v,w),\quad G_2:(x,y,t,u,v,w)\mapsto(x,y,t+\epsilon,u,v,w)\,,\\
&G_3:(x,y,t,u,v,w)\mapsto(x,y+\epsilon t,t,u\,\mathrm{e}^{\epsilon x},v\,\mathrm{e}^{-\epsilon x},w)\,,\\
&G_4:(x,y,t,u,v,w)\mapsto(x\,\mathrm{e}^{\epsilon},y\,\mathrm{e}^{-\epsilon},t,u,v\,\mathrm{e}^{-2\epsilon},w),\quad G_5:(x,y,t,u,v,w)\mapsto(x,y\,\mathrm{e}^{\epsilon},t\,\mathrm{e}^{\epsilon},u,v,w\,\mathrm{e}^{-\epsilon})\,,\\
&G_6:(x,y,t,u,v,w)\mapsto\Big(\frac{x}{1-\epsilon t},\frac{y}{1-\epsilon t},\frac{t}{1-\epsilon t},u\,\mathrm{e}^{\epsilon xy/(1-\epsilon t)},(1-\epsilon t)^2v\,\mathrm{e}^{-\epsilon xy/(1-\epsilon t)}w(1-\epsilon t)^2-\epsilon\delta(1-\epsilon t)\Big)\,,\\
&G_7:(x,y,t,u,v,w)\mapsto(x+\epsilon f,y,t,u\,\mathrm{e}^{\epsilon\dot f y},v\,\mathrm{e}^{-\epsilon\dot f y},w-\epsilon\delta\ddot f y)\,,\\
&G_8:(x,y,t,u,v,w)\mapsto(x,y,t,u\,\mathrm{e}^{\epsilon g},v\,\mathrm{e}^{-\epsilon g},w-\epsilon\delta\dot g)\,.
\end{aligned}\tag{8}$$

So the entire symmetry group is obtained by composing one-dimensional subgroups G_i $(i=1,2,\ldots,8)$. When G is an element of this group, if $\{u(x,y,t),\ v(x,y,t),\ w(x,y,t)\}$ is a solution of CNLERD equation, then $\{P(\chi,\zeta,\gamma),\ Q(\chi,\zeta,\gamma),\ R(\chi,\zeta,\gamma)\}$ is also a solution of CNLERD equation.

Table 1 The commutation relation between infinitesimal generators of point symmetries.

$[\boldsymbol{v}_j,\boldsymbol{v}_k]$	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4$	$\boldsymbol{v}_5$	$\boldsymbol{v}_6$	$\boldsymbol{v}_7(f)$	$\boldsymbol{v}_8(g)$
$\boldsymbol{v}_1$	0	0	0	$-\boldsymbol{v}_1$	$\boldsymbol{v}_1$	$\boldsymbol{v}_3$	$\boldsymbol{v}_8(\dot f)$	0
$\boldsymbol{v}_2$		0	$\boldsymbol{v}_1$	0	$\boldsymbol{v}_2$	$\boldsymbol{v}_4+2\boldsymbol{v}_5$	$\boldsymbol{v}_7(\dot f)$	$\boldsymbol{v}_8(\dot g)$
$\boldsymbol{v}_3$			0	$-\boldsymbol{v}_3$	0	0	$\boldsymbol{v}_8(t\dot f-f)$	0
$\boldsymbol{v}_4$				0	0	0	$-\boldsymbol{v}_7(f)$	0
$\boldsymbol{v}_5$					0	$\boldsymbol{v}_6$	$\boldsymbol{v}_7(t\dot f)$	$\boldsymbol{v}_8(t\dot g)$
$\boldsymbol{v}_6$						0	$\boldsymbol{v}_7(t^2\dot f-tf)$	$\boldsymbol{v}_8(t^2\dot g)$
$\boldsymbol{v}_7(f)$							0	0
$\boldsymbol{v}_8(g)$								0

Since each group G_i is a symmetry group, it implies that if $\{u=p(x,y,t),v=q(x,y,t),w=r(x,y,t)\}$ is a solution of CNLERD equation, so are the functions:

$$\begin{aligned}
&(u^{(1)},v^{(1)},w^{(1)})=(p(x,y-\epsilon,t),q(x,y-\epsilon,t),r(x,y-\epsilon,t))\,,\\
&(u^{(2)},v^{(2)},w^{(2)})=(p(x,y,t-\epsilon),q(x,y,t-\epsilon),r(x,y,t-\epsilon))\,,\\
&(u^{(3)},v^{(3)},w^{(3)})=(\,\mathrm{e}^{\epsilon x}p(x,y-\epsilon t,t),\ \mathrm{e}^{-\epsilon x}q(x,y-\epsilon t,t),r(x,y-\epsilon t,t))\,,\\
&(u^{(4)},v^{(4)},w^{(4)})=(p(\,\mathrm{e}^{-\epsilon}x,\ \mathrm{e}^{\epsilon}y,t),\ \mathrm{e}^{-2\epsilon}q(\,\mathrm{e}^{-\epsilon}x,\ \mathrm{e}^{\epsilon}y,t),r(\,\mathrm{e}^{-\epsilon}x,\ \mathrm{e}^{\epsilon}y,t))\,,\\
&(u^{(5)},v^{(5)},w^{(5)})=(p(x,\ \mathrm{e}^{-\epsilon}y,\ \mathrm{e}^{-\epsilon}t),q(x,\ \mathrm{e}^{-\epsilon}y,\ \mathrm{e}^{-\epsilon}t),\ \mathrm{e}^{-\epsilon}r(x,\ \mathrm{e}^{-\epsilon}y,\ \mathrm{e}^{-\epsilon}t))\,,\\
&(u^{(6)},v^{(6)},w^{(6)})=\Big(\exp\Big\{\frac{\epsilon xy}{1+\epsilon t}\Big\}p\Big(\frac{x}{1+\epsilon t},\frac{y}{1+\epsilon t},\frac{t}{1+\epsilon t}\Big),\frac{1}{(1+\epsilon t)^2}\exp\Big\{-\frac{\epsilon xy}{1+\epsilon t}\Big\}q\Big(\frac{x}{1+\epsilon t},\frac{y}{1+\epsilon t},\frac{t}{1+\epsilon t}\Big),\\
&\qquad\frac{1}{(1+\epsilon t)^2}r\Big(\frac{x}{1+\epsilon t},\frac{y}{1+\epsilon t},\frac{t}{1+\epsilon t}\Big)-\frac{\epsilon\delta}{1+\epsilon t}\Big)\,,\\
&(u^{(7)},v^{(7)},w^{(7)})=(\,\mathrm{e}^{\epsilon\dot f y}p(x-\epsilon f,y,t),\ \mathrm{e}^{-\epsilon\dot f y}q(x-\epsilon f,y,t),r(x-\epsilon f,y,t)-\epsilon\delta\ddot f y)\,,\\
&(u^{(8)},v^{(8)},w^{(8)})=(\,\mathrm{e}^{\epsilon g}p(x,y,t),\ \mathrm{e}^{-\epsilon g}q(x,y,t),r(x,y,t)-\epsilon\delta\dot g)\,.
\end{aligned}$$

3 Symmetry Group by a Simple Direct Method

According to the symmetry group direct method,[6] we can take the simplified symmetry transformation ansatz as

$$u=\alpha_1+\beta_1U(\xi,\eta,\tau),\quad v=\alpha_2+\beta_2V(\xi,\eta,\tau),\quad w=\alpha_3+\beta_3W(\xi,\eta,\tau)\,,\tag{9}$$

where α_i,β_i $(i=1,2,3),\xi,\eta$ and τ are functions of $\{x,y,t\}$ to be determined. Requiring $U(\xi,\eta,\tau),V(\xi,\eta,\tau)$ and $W(\xi,\eta,\tau)$ also satisfy the CNLERD equation but with different independent variables $\{\xi,\eta,\tau\}$

$$U_\tau+U_{\xi\eta}+\delta WU=0,\quad V_\tau-V_{\xi\eta}-\delta WV=0,\quad W_\xi+(UV)_\eta=0\,.\tag{10}$$

Substituting (9) into Eqs. (1), eliminating U_τ,V_τ,W_ξ and their higher-order derivatives by Eqs. (10), then setting the coefficients of the polynomials of U,V,W and their derivatives to be zero, we obtain a huge numbers of nonlinear PDEs with respect to differentiable functions: $\{\alpha_i,\beta_i(i=1,2,3),\xi,\eta,\tau\}$. By solving these equations, the general results are as follows:

$$\begin{aligned}
&\alpha_1=\alpha_2=0,\quad \alpha_3=-\frac{\delta}{\Delta_1}[(s_1t+s_2)\ddot\xi_0y+2s_1\dot\xi_0y+\Delta_2\dot\xi_0]-\frac{\delta\dot\beta_0}{\beta_0}-\frac{\delta s_1}{s_1t+s_2}\,,\\
&\beta_1=\beta_0\exp\Big(\frac{(s_1t+s_2)\dot\xi_0y}{\Delta_1}+\frac{s_1xy+\Delta_2x}{s_1t+s_2}\Big),\quad \beta_2=\frac{\Delta_1^2}{(s_1t+s_2)^2\beta_0}\exp\Big(-\frac{(s_1t+s_2)\dot\xi_0y}{\Delta_1}-\frac{s_1xy+\Delta_2x}{s_1t+s_2}\Big)\,,
\end{aligned}$$

$$\beta_3 = -\frac{\Delta_1}{(s_1t+s_2)^2}, \quad \xi = -\frac{\Delta_1 x}{s_1t+s_2} + \xi_0, \quad \eta = \frac{y+s_5t+s_6}{s_1t+s_2},$$
$$\tau = \frac{s_3t+s_4}{s_1t+s_2} \quad (s_1s_4 \neq s_2s_3), \quad \Delta = s_1s_4 - s_2s_3, \quad \Delta_2 = s_1s_6 - s_2s_5, \tag{11}$$

where $\xi_0 \equiv \xi_0(t)$ and $\beta_0 \equiv \beta_0(t)$ are arbitrary functions of t and s_i $(i = 1,2,3,4,5,6)$ are arbitrary constants. It is noteworthy to mention here that the independent variable t possess invariant property under the Möbious (conformal) transformation.

In summary, we can arrive at the following final transformation group theorem of Eqs. (1).

Theorem 1 If $\{U = U(x,y,t), V = V(x,y,t), W = W(x,y,t)\}$ is a solution of the CNLERD equation then so are $\{u,v,w\}$

$$u = \beta_0 \exp\Big(\frac{(s_1t+s_2)\dot{\xi}_0 y}{\Delta_1} + \frac{s_1xy+\Delta_2 x}{s_1t+s_2}\Big)U(\xi,\eta,\tau),$$
$$v = \frac{\Delta_1^2}{(s_1t+s_2)^2\beta_0}\exp\Big(-\frac{(s_1t+s_2)\dot{\xi}_0 y}{\Delta_1} - \frac{s_1xy+\Delta_2 x}{s_1t+s_2}\Big)V(\xi,\eta,\tau),$$
$$w = -\frac{\delta}{\Delta_1}[(s_1t+s_2)\ddot{\xi}_0 y + 2s_1\dot{\xi}_0 y + \Delta_2\dot{\xi}_0] - \frac{\delta\dot{\beta}_0}{\beta_0} - \frac{\delta s_1}{s_1t+s_2} - \frac{\Delta_1}{(s_1t+s_2)^2}W(\xi,\eta,\tau), \tag{12}$$

with Eqs. (11).

In order to see the equivalence between the Lie point symmetry group obtained in Theorem 1 and the known one from classical Lie group method, we need take the arbitrary functions α_0, β_0 and arbitrary constants s_i $(i = 1,2,3,4,5,6)$ to be different forms with respect to an infinitesimal parameter ϵ

$$s_1 = -c_6\epsilon, \quad s_2 = 1-(c_5-c_4)\epsilon, \quad c_3 = 1+c_4\epsilon, \quad s_4 = c_2\epsilon, \quad s_5 = c_3\epsilon, \quad s_6 = c_1\epsilon, \quad \xi_0 = \epsilon g, \quad \beta_0 = 1-\epsilon f,$$

then (12) can be written as

$$(u,v,w)^{\mathrm{T}} = (u,v,w)^{\mathrm{T}} + \epsilon\sigma = (u,v,w)^{\mathrm{T}} + \epsilon(\sigma(u),\sigma(v),\sigma(w))^{\mathrm{T}},$$
$$\sigma = \begin{pmatrix} (c_6xt+c_4x)u_x + (c_6yt+c_5y-c_4y+c_3t+c_1)u_y + (c_6t^2+c_5t+c_2)u_t - (c_6xy+c_3x)u \\ (c_6xt+c_4x)v_x + (c_6yt+c_5y-c_4y+c_3t+c_1)v_y + (c_6t^2+c_5t+c_2)v_t + (c_6xy+c_3x+2c_6t+2c_4)v \\ (c_6xt+c_4x)w_x + (c_6yt+c_5y-c_4y+c_3t+c_1)w_y + (c_6t^2+c_5t+c_2)w_t + c_6\delta + (2c_6t+c_5)w \end{pmatrix}$$
$$+ \begin{pmatrix} fu_x - (\dot{f}y+g)u \\ fv_x + (\dot{f}y+g)v \\ fw_x + \delta(\ddot{f}y+\dot{g}) \end{pmatrix},$$

which is exactly the same as one obtained by the standard Lie approach.

4 Similar Reductions and Some Similarity Solutions of the CNLERD Equation

As is well known, to find exact solutions and perform symmetry reductions of differential equations is one of most important application of the Lie group method. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups. It is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, which has led to the concept of an optimal system.[1,7] The idea of using the adjoint representation to classify group-invariant solutions was due to Ovsiannikov.[7] Some works on such optimal systems have been reported.[1,7–8]

Ignoring the discussion of the infinite-dimensional subalgebra, and selecting the following finite-dimensional vector fields

$$\boldsymbol{v}_7 \equiv \boldsymbol{v}_7(f{=}1) = \frac{\partial}{\partial x}, \quad \boldsymbol{v}_8 \equiv \boldsymbol{v}_8(g{=}1) = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \quad \boldsymbol{v}_9 \equiv \boldsymbol{v}_7(f{=}t) = t\frac{\partial}{\partial x} + yu\frac{\partial}{\partial u} - yv\frac{\partial}{\partial v}, \tag{13}$$

instead of the infinite-dimensional vector fields $\boldsymbol{v}_7(f)$ and $\boldsymbol{v}_8(g)$. Then, the associated Lie algebra among these vector fields $\boldsymbol{v}_i$, $(i = 1,\ldots,9)$ are given by Table 2, where the entry in the j-th row and the k-th column represents the commutator $[\boldsymbol{v}_j, \boldsymbol{v}_k]$.

To compute the adjoint representation, we use the Lie series in conjunction with the above commutator table. Applying the formula

$$\mathrm{Ad}(\exp(\epsilon\boldsymbol{v}))\boldsymbol{v}_0 = \boldsymbol{v}_0 - \epsilon[\boldsymbol{v},\boldsymbol{v}_0] - \frac{1}{2}\epsilon^2[\boldsymbol{v},[\boldsymbol{v},\boldsymbol{v}_0]] - \cdots,$$

we can construct Table 3 with the (i,j)-th entry indicating $\mathrm{Ad}(\exp(\epsilon\boldsymbol{v}_i))\boldsymbol{v}_j$.

Table 2 The commutation relation between infinitesimal generators of point symmetries.

$[\boldsymbol{v}_j,\boldsymbol{v}_k]$	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4$	$\boldsymbol{v}_5$	$\boldsymbol{v}_6$	$\boldsymbol{v}_7$	$\boldsymbol{v}_8$	$\boldsymbol{v}_9$
$\boldsymbol{v}_1$	0	0	0	$-\boldsymbol{v}_1$	$\boldsymbol{v}_1$	$\boldsymbol{v}_3$	0	0	$\boldsymbol{v}_8$
$\boldsymbol{v}_2$		0	0	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_4+2\boldsymbol{v}_5$	0	0	$\boldsymbol{v}_7$
$\boldsymbol{v}_3$			0	$-\boldsymbol{v}_3$	0	0	$-\boldsymbol{v}_8$	0	0
$\boldsymbol{v}_4$				0	0	0	$-\boldsymbol{v}_7$	0	$-\boldsymbol{v}_9$
$\boldsymbol{v}_5$					0	$\boldsymbol{v}_6$	0	0	$\boldsymbol{v}_9$
$\boldsymbol{v}_6$						0	$-\boldsymbol{v}_9$	0	0
$\boldsymbol{v}_7$							0	0	0
$\boldsymbol{v}_8$								0	0
$\boldsymbol{v}_9$									0

Table 3 Adjoint representation.

Ad	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4$	$\boldsymbol{v}_5$	$\boldsymbol{v}_6$	$\boldsymbol{v}_7$	$\boldsymbol{v}_8$	$\boldsymbol{v}_9$
$\boldsymbol{v}_1$	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4+\epsilon\boldsymbol{v}_1$	$\boldsymbol{v}_5-\epsilon\boldsymbol{v}_1$	$\boldsymbol{v}_6-\epsilon\boldsymbol{v}_3$	$\boldsymbol{v}_7$	$\boldsymbol{v}_8$	$\boldsymbol{v}_9-\epsilon\boldsymbol{v}_8$
$\boldsymbol{v}_2$	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3-\epsilon\boldsymbol{v}_1$	$\boldsymbol{v}_4$	$\boldsymbol{v}_5-\epsilon\boldsymbol{v}_2$	$\boldsymbol{v}_6-\epsilon\boldsymbol{v}_4-2\epsilon\boldsymbol{v}_5+\epsilon^2\boldsymbol{v}_2$	$\boldsymbol{v}_7$	$\boldsymbol{v}_8$	$\boldsymbol{v}_9-\epsilon\boldsymbol{v}_7$
$\boldsymbol{v}_3$	$\boldsymbol{v}_1$	$\boldsymbol{v}_2+\epsilon\boldsymbol{v}_1$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4+\epsilon\boldsymbol{v}_3$	$\boldsymbol{v}_5$	$\boldsymbol{v}_6$	$\boldsymbol{v}_7+\epsilon\boldsymbol{v}_8$	$\boldsymbol{v}_8$	$\boldsymbol{v}_9$
$\boldsymbol{v}_4$	$\mathrm{e}^{-\epsilon}\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\mathrm{e}^{-\epsilon}\boldsymbol{v}_3$	$\boldsymbol{v}_4$	$\boldsymbol{v}_5$	$\boldsymbol{v}_6$	$\mathrm{e}^{\epsilon}\boldsymbol{v}_7$	$\boldsymbol{v}_8$	$\mathrm{e}^{\epsilon}\boldsymbol{v}_9$
$\boldsymbol{v}_5$	$\mathrm{e}^{\epsilon}\boldsymbol{v}_1$	$\mathrm{e}^{\epsilon}\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4$	$\boldsymbol{v}_5$	$\mathrm{e}^{-\epsilon}\boldsymbol{v}_6$	$\boldsymbol{v}_7$	$\boldsymbol{v}_8$	$\mathrm{e}^{-\epsilon}\boldsymbol{v}_9$
$\boldsymbol{v}_6$	$\boldsymbol{v}_1+\epsilon\boldsymbol{v}_3$	$\boldsymbol{v}_2+\epsilon\boldsymbol{v}_4+2\epsilon\boldsymbol{v}_5+\epsilon^2\boldsymbol{v}_6$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4$	$\boldsymbol{v}_5+\epsilon\boldsymbol{v}_6$	$\boldsymbol{v}_6$	$\boldsymbol{v}_7+\mathrm{e}^{\epsilon}\boldsymbol{v}_9$	$\boldsymbol{v}_8$	$\boldsymbol{v}_9$
$\boldsymbol{v}_7$	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3-\epsilon\boldsymbol{v}_8$	$\boldsymbol{v}_4-\epsilon\boldsymbol{v}_7$	$\boldsymbol{v}_5$	$\boldsymbol{v}_6-\epsilon\boldsymbol{v}_9$	$\boldsymbol{v}_7$	$\boldsymbol{v}_8$	$\boldsymbol{v}_9$
$\boldsymbol{v}_8$	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4$	$\boldsymbol{v}_5$	$\boldsymbol{v}_6$	$\boldsymbol{v}_7$	$\boldsymbol{v}_8$	$\boldsymbol{v}_9$
$\boldsymbol{v}_9$	$\boldsymbol{v}_1+\epsilon\boldsymbol{v}_8$	$\boldsymbol{v}_2+\epsilon\boldsymbol{v}_7$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4-\epsilon\boldsymbol{v}_9$	$\boldsymbol{v}_5+\epsilon\boldsymbol{v}_9$	$\boldsymbol{v}_6$	$\boldsymbol{v}_7$	$\boldsymbol{v}_8$	$\boldsymbol{v}_9$

Following Ovsiannikov,[7] one calls two subalgebras $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ of a given Lie algebra equivalent if one can find an element g in the Lie group so that $\mathrm{Ad}g(\boldsymbol{v}_j)=\boldsymbol{v}_i$, where $\mathrm{Ad}g$ is the adjoint representation of g on $\boldsymbol{v}$. Given a nonzero vector

$$\boldsymbol{v}=a_1\boldsymbol{v}_1+a_2\boldsymbol{v}_2+a_3\boldsymbol{v}_3+a_4\boldsymbol{v}_4+a_5\boldsymbol{v}_5+a_6\boldsymbol{v}_6+a_7\boldsymbol{v}_7+a_8\boldsymbol{v}_8+a_9\boldsymbol{v}_9\,,$$

our task is to simplify as many of the coefficients a_i as possible by using adjoint maps to $\boldsymbol{v}$. In this way, omitting the detailed computation, we obtain an optical system $\mathcal{S}$ of the Lie algebra:

(1) $\boldsymbol{v}_1$, (2) $\boldsymbol{v}_2$, (3) $\boldsymbol{v}_7$, (4) $\boldsymbol{v}_8$, (5) $\boldsymbol{v}_9$, (6) $\boldsymbol{v}_1+a_7\boldsymbol{v}_7$ $(a_7\in R)$, (7) $\boldsymbol{v}_1+a_9\boldsymbol{v}_9$ $(a_9\neq 0)$,
(8) $\boldsymbol{v}_2+a_8\boldsymbol{v}_8$ $(a_8\in R)$, (9) $\boldsymbol{v}_2+a_9\boldsymbol{v}_9$ $(a_9\neq 0)$, (10) $\boldsymbol{v}_3+a_7\boldsymbol{v}_7$ $(a_7\neq 0)$, (11) $\boldsymbol{v}_3+a_9\boldsymbol{v}_9$ $(a_9\in R)$,
(12) $\boldsymbol{v}_4+a_8\boldsymbol{v}_8$ $(a_8\in R)$, (13) $\boldsymbol{v}_5+a_7\boldsymbol{v}_7$ $(a_7\in R)$, (14) $\boldsymbol{v}_5+a_8\boldsymbol{v}_8$ $(a_8\in R)$, (15) $\boldsymbol{v}_6+a_7\boldsymbol{v}_7$ $(a_7\neq 0)$,
(16) $\boldsymbol{v}_6+a_8\boldsymbol{v}_8$ $(a_8\in R)$, (17) $\boldsymbol{v}_1+\boldsymbol{v}_6+a_7\boldsymbol{v}_7$ $(a_7\neq 0)$, (18) $\boldsymbol{v}_2+\boldsymbol{v}_3+a_9\boldsymbol{v}_9$ $(a_7\in R)$,
(19) $\boldsymbol{v}_2+\boldsymbol{v}_6+a_8\boldsymbol{v}_8$ $(a_8\in R)$, (20) $\boldsymbol{v}_3+\boldsymbol{v}_5+a_7\boldsymbol{v}_7$ $(a_7\in R)$, (21) $\boldsymbol{v}_4+\boldsymbol{v}_5+a_8\boldsymbol{v}_8$ $(a_8\in R)$.

After determining the infinitesimal generators, the similarity variables can be found by solving the characteristic equations

$$\frac{\mathrm{d}x}{X}=\frac{\mathrm{d}y}{Y}=\frac{\mathrm{d}t}{T}=\frac{\mathrm{d}u}{U}=\frac{\mathrm{d}v}{V}=\frac{\mathrm{d}w}{W}\,.$$

Note that the symmetry $\boldsymbol{v}_8$ in the optical system $\mathcal{S}$ yields trivial group-invariant solutions, we do not list it here. All other symmetry reductions are presented in Table 4.

The symmetries $\boldsymbol{v}_2$ and $\boldsymbol{v}_2+a_8\boldsymbol{v}_8$ $(a_8\in R)$ lead the CNLERD equation to the steady cases of the original system, respectively.

From the reduced equations related to the generator $\boldsymbol{v}_1$, we obtain a simple solution of Eqs. (1)

$$u=F(x)\exp\Big[-\delta\int H(t)\mathrm{d}t\Big],\quad v=G(x)\exp\Big[\delta\int H(t)\mathrm{d}t\Big],\quad w=H(t)\,,\tag{14}$$

where $F(x)$ and $G(x)$ are arbitrary functions of y and $H(t)$ is an arbitrary function of t.

The reduction of the generator $\boldsymbol{v}_7$ gives a solution of Eq. (1)

$$u=F(y)\exp\Big[-\delta\int H(y,t)\mathrm{d}t],\quad v=F(y)^{-1}\exp[\delta\int H(y,t)\mathrm{d}t\Big],\quad w=H(y,t)\,,\tag{15}$$

where $H(y,t)$ is an arbitrary function of x and t and $H(y)$ is an arbitrary function of y.

Table 4 Reduction of the CNLERD equation.

Case	Similarity variables	Reduced equations
(1) $\boldsymbol{v}_1$	$\chi = x, \quad \gamma = t, \quad u = P(\chi,\gamma),$ $v = Q(\chi,\gamma), \quad w = R(\chi,\gamma)$	$P_\gamma + \delta RP = 0, \quad Q_\gamma - \delta RQ = 0, \quad R_\chi = 0$
(2) $\boldsymbol{v}_2$	$\chi = x, \quad \gamma = y, \quad u = P(\chi,\gamma),$ $v = Q(\chi,\gamma), \quad w = R(\chi,\gamma)$	$P_{\chi\gamma} + \delta RP = 0, \quad Q_{\chi\gamma} + \delta RQ = 0, \quad R_\chi + (PQ)_\gamma = 0$
(3) $\boldsymbol{v}_7$	$\chi = y, \quad \gamma = t, \quad u = P(\chi,\gamma),$ $v = Q(\chi,\gamma), \quad w = R(\chi,\gamma)$	$P_\gamma + \delta RP = 0, \quad Q_\gamma - \delta RQ = 0, \quad (PQ)_\chi = 0$
(5) $\boldsymbol{v}_9$	$\chi = y, \quad \gamma = t, \quad u = P(\chi,\gamma)\exp(\frac{xy}{t}),$ $v = Q(\chi,\gamma)\exp(-\frac{xy}{t}), \quad w = R(\chi,\gamma)$	$\gamma P_\gamma + \chi P_\chi + P + \delta\gamma RP = 0,$ $\gamma Q_\gamma + \chi Q_\chi + Q - \delta\gamma RQ = 0, \quad (PQ)_\chi = 0$
(6) $\boldsymbol{v}_1 + a_7\boldsymbol{v}_7$ $(a_7 \in R)$	$\chi = \frac{-x+a_7 y}{a_7}, \quad \gamma = t, \quad u = P(\chi,\gamma),$ $v = Q(\chi,\gamma), \quad w = R(\chi,\gamma)$	$a_7 P_\gamma - P_{\chi\chi} + \delta a_7 RP = 0,$ $a_7 Q_\gamma + Q_{\chi\chi} - \delta a_7 RQ = 0, \quad R = a_7 PQ$
(7) $\boldsymbol{v}_1 + a_9\boldsymbol{v}_9$ $(a_9 \neq 0)$	$\chi = \frac{-x+a_9 yt}{a_9 t}, \quad \gamma = t, \quad w = R(\chi,\gamma)$ $u = P(\chi,\gamma)\exp(-\frac{x(x-2a_9 yt)}{2a_9 t^2}),$ $v = Q(\chi,\gamma)\exp(\frac{x(x-2a_9 yt)}{2a_9 t^2})$	$a_9\gamma P_\gamma - P_{\chi\chi} + a_9\chi P_\chi + a_9 P + \delta a_9\gamma RP = 0,$ $a_9\gamma Q_\gamma + Q_{\chi\chi} + a_9\chi Q_\chi + a_9 Q - \delta a_9\gamma RQ = 0,$ $R_\chi - a_9\gamma(PQ)_\chi = 0$
(8) $\boldsymbol{v}_2 + a_8\boldsymbol{v}_8$ $(a_8 \in R)$	$\chi = x, \quad \gamma = y, \quad u = P(\chi,\gamma)\exp(a_8 t),$ $v = Q(\chi,\gamma)\exp(-a_8 t), \quad w = R(\chi,\gamma)$	$a_8 P + P_{\chi\gamma} + \delta RP = 0,$ $a_8 Q + Q_{\chi\gamma} + \delta RQ = 0, \quad R_\chi + (PQ)_\gamma = 0$
(9) $\boldsymbol{v}_2 + a_9\boldsymbol{v}_9$ $(a_9 \neq 0)$	$\chi = -\frac{2x-a_9 t^2}{a_9}, \quad \gamma = y, \quad w = R(\chi,\gamma),$ $u = P(\chi,\gamma)\exp\left(\frac{a_9^2 t^2 y}{\sqrt{2a_9 x - a_9(2x-a_9 t^2)}}\right),$ $v = Q(\chi,\gamma)\exp\left(-\frac{a_9^2 t^2 y}{\sqrt{2a_9 x - a_9(2x-a_9 t^2)}}\right)$	$a_9^2\gamma P - 2P_{\chi\gamma} + \delta a_9 RP = 0,$ $a_9^2\gamma Q - 2Q_{\chi\gamma} + \delta a_9 RQ = 0,$ $2R_\chi - a_9(PQ)_\gamma = 0$
(10) $\boldsymbol{v}_3 + a_7\boldsymbol{v}_7$ $(a_7 \neq 0)$	$\chi = -\frac{xt-a_7 y}{a_7}, \ \gamma = t, \ u = P(\chi,\gamma)\exp(\frac{x^2}{2a_7}),$ $v = Q(\chi,\gamma)\exp(-\frac{x^2}{2a_7}), \quad w = R(\chi,\gamma)$	$a_7 P_\gamma - \gamma P_{\chi\chi} + \delta a_7 RP = 0,$ $a_7 Q_\gamma + \gamma Q_{\chi\chi} - \delta a_7 RQ = 0, \quad \gamma R_\chi - a_7(PQ)_\chi = 0$
(11) $\boldsymbol{v}_3 + a_9\boldsymbol{v}_9$ $(a_9 \in R)$	$\chi = -\frac{x-a_9 y}{a_9}, \quad \gamma = t, \quad w = R(\chi,\gamma),$ $u = P(\chi,\gamma)\exp(\frac{xy}{t}), \quad v = Q(\chi,\gamma)\exp(-\frac{xy}{t})$	$a_9\gamma P_\gamma - \gamma P_{\chi\chi} + a_9(\chi P_\chi + P) + \delta a_9\gamma RP = 0,$ $a_9\gamma Q_\gamma + \gamma Q_{\chi\chi} + a_9(\chi Q_\chi + Q) - \delta a_9\gamma RQ = 0,$ $R_\chi - a_9(PQ)_\chi = 0$
(12) $\boldsymbol{v}_4 + a_8\boldsymbol{v}_8$ $(a_8 \in R)$	$\chi = xy, \quad \gamma = t, \quad u = x^{a_8}P(\chi,\gamma),$ $v = x^{-a_8-2}Q(\chi,\gamma), \quad w = R(\chi,\gamma)$	$P_\gamma + \chi P_{\chi\chi} + (a_8+1)P_\chi + \delta RP = 0,$ $Q_\gamma - \chi Q_{\chi\chi} + (a_8+1)Q_\chi - \delta RQ = 0, \quad \chi R_\chi + (PQ)_\chi = 0$
(13) $\boldsymbol{v}_5 + a_7\boldsymbol{v}_7$ $(a_7 \in R)$	$\chi = y\exp(-\frac{x}{a_7}), \ \gamma = t\exp(-\frac{x}{a_7}), \ u = P(\chi,\gamma),$ $v = Q(\chi,\gamma), \quad w = R(\chi,\gamma)\exp(-\frac{x}{a_7})$	$a_7 P_\gamma - \chi P_{\chi\chi} - \gamma P_{\chi\gamma} - P_\chi + \delta a_7 RP = 0,$ $a_7 Q_\gamma + \chi Q_{\chi\chi} + \gamma Q_{\chi\gamma} + Q_\chi - \delta a_7 RQ = 0,$ $\chi R_\chi + \gamma R_\gamma + R - a_7(PQ)_\chi = 0$
(14) $\boldsymbol{v}_5 + a_8\boldsymbol{v}_8$ $(a_8 \in R)$	$\chi = x, \quad \gamma = \frac{t}{y}, \quad u = y^{a_8}P(\chi,\gamma),$ $v = y^{-a_8}Q(\chi,\gamma), \quad w = y^{-1}R(\chi,\gamma)$	$P_\gamma - \gamma P_{\chi\gamma} + a_8 P_\chi + \delta RP = 0,$ $Q_\gamma + \gamma Q_{\chi\gamma} + a_8 Q_\chi - \delta RQ = 0, \quad R_\chi - \gamma(PQ)_\gamma = 0$
(15) $\boldsymbol{v}_6 + a_7\boldsymbol{v}_7$ $(a_7 \neq 0)$,	$\chi = \frac{a_7+2xt}{2t^2}, \quad \gamma = -\frac{a_7 y}{t}, \quad w = \frac{a_7^2 R(\chi,\gamma)-\delta t}{t^2},$ $u = P(\chi,\gamma)\exp(\frac{(a_7+xt)y}{t^2})$ $v = Q(\chi,\gamma)\frac{a_7^2}{t^2}\exp(-\frac{(a_7+xt)y}{t^2})$	$a_7\gamma P_{\gamma\chi} - \gamma^2 P - \delta a_7^2\gamma RP = 0,$ $a_7\gamma Q_{\gamma\chi} - \gamma^2 Q - \delta a_7^2\gamma RQ = 0, \quad R_\chi - a_7(PQ)_\gamma = 0$
(16) $\boldsymbol{v}_6 + a_8\boldsymbol{v}_8$ $(a_8 \in R)$	$\chi = \frac{y}{x}, \quad \gamma = \frac{t}{x}, \quad w = \frac{tR(\chi,\gamma)-\delta x^2}{tx^2},$ $u = P(\chi,\gamma)\exp(\frac{xy-a_8}{t}),$ $v = x^{-2}Q(\chi,\gamma)\exp(-\frac{xy-a_8}{t}),$	$\gamma^2 P_\chi + \gamma^3 P_{\gamma\chi} + \gamma^2\chi P_{\chi\chi} - a_8 P - \delta\gamma^2 RP = 0,$ $3\gamma^2 Q_\chi + \gamma^3 Q_{\gamma\chi} + \gamma^2\chi Q_{\chi\chi} - a_8 Q - \delta\gamma^2 RQ = 0,$ $\gamma R_\gamma + \chi R_\chi + 2R - (PQ)_\chi = 0$
(17) $\boldsymbol{v}_1 + \boldsymbol{v}_6 + a_7\boldsymbol{v}_7$ $(a_7 \neq 0)$	$\chi = \frac{a_7+2xt}{2t^2}, \quad \gamma = \frac{x-a_7 y}{t}, \quad w = \frac{a_7^2 R(\chi,\gamma)-\delta t}{t^2}$ $u = P(\chi,\gamma)\exp(\frac{3(1+yt)xt+a_7(3yt+2)}{3t^3}),$ $v = \frac{a_7^2}{t^2}Q(\chi,\gamma)\exp(-\frac{3(1+yt)xt+a_7(3yt+2)}{3t^3})$	$a_7 P_{\gamma\gamma} + a_7 P_{\gamma\chi} + (2\chi-\gamma)P - \delta a_7^2 RP = 0,$ $a_7 Q_{\gamma\gamma} + a_7 Q_{\gamma\chi} + (2\chi-\gamma)Q - \delta a_7^2 RQ = 0,$ $R_\gamma + R_\chi - a_7(PQ)_\gamma = 0$
(18) $\boldsymbol{v}_2 + \boldsymbol{v}_3 + a_9\boldsymbol{v}_9$ $(a_9 \in R)$	$\chi = \frac{a_9 y-x}{a_9}, \quad \gamma = \frac{a_9 t^2-2x}{a_9}, \quad w = R(\chi,\gamma)$ $u = P(\chi,\gamma)\exp(\frac{(3x+3a_9 y-2a_9 t^2)t}{3}),$ $v = Q(\chi,\gamma)\exp(-\frac{(3x+3a_9 y-2a_9 t^2)t}{3})$	$P_{\chi\chi} + 2P_{\gamma\chi} + a_9^2(\gamma-\chi)P - \delta a_9 RP = 0,$ $Q_{\chi\chi} + 2Q_{\gamma\chi} + a_9^2(\gamma-\chi)Q - \delta a_9 RQ = 0$ $R_\chi + 2R_\gamma - a_9(PQ)_\chi = 0,$
(19) $\boldsymbol{v}_2 + \boldsymbol{v}_6 + a_8\boldsymbol{v}_8$ $(a_8 \in R)$	$\chi = \frac{y}{x}, \quad \gamma = \frac{t^2+1}{x^2}, \quad w = \frac{(t^2+1)R(\chi,\gamma)-\delta tx^2}{x^2(t^2+1)},$ $u = P(\chi,\gamma)\exp(\frac{xyt}{t^2+1} - a_8\tan^{-1}\frac{1}{t}),$ $v = Q(\chi,\gamma)\exp(-\frac{xyt}{t^2+1} + a_8\tan^{-1}\frac{1}{t})$	$\chi\gamma^2 P_{\chi\chi} + 2\gamma^3 P_{\gamma\chi} + \gamma^2 P_\chi - (\chi + a_8\gamma)P - \delta\gamma^2 RP = 0,$ $\chi\gamma^2 Q_{\chi\chi} + 2\gamma^3 Q_{\gamma\chi} + 3\gamma^2 Q_\chi - (\chi + a_8\gamma)Q - \delta\gamma^2 RQ = 0,$ $2\gamma R_\gamma + \chi R_\chi + 2R - (PQ)_\chi = 0$
(20) $\boldsymbol{v}_3 + \boldsymbol{v}_5 + a_7\boldsymbol{v}_7$ $(a_7 \in R)$	$\chi = t\exp(-\frac{x}{a_7}), \ \gamma = \frac{a_7 y-xt}{a_7}\exp(-\frac{x}{a_7}),$ $u = P(\chi,\gamma)\exp(\frac{x^2}{2a_7}), \ v = Q(\chi,\gamma)\exp(-\frac{x^2}{2a_7}),$ $w = R(\chi,\gamma)\exp(-\frac{x}{a_7})$	$\chi P_{\gamma\gamma} + \chi P_{\gamma\chi} + \gamma P_{\gamma\gamma} + P_\gamma - a_7 P_\chi - \delta a_7 RP = 0,$ $\chi Q_{\gamma\gamma} + \chi Q_{\gamma\chi} + \gamma Q_{\gamma\gamma} + Q_\gamma + a_7 Q_\chi - \delta a_7 RQ = 0,$ $\chi R_\chi + (\chi+\gamma)R_\gamma + R - a_7(PQ)_\gamma = 0$
(21) $\boldsymbol{v}_4 + \boldsymbol{v}_5 + a_8\boldsymbol{v}_8$ $(a_8 \in R)$	$\chi = y, \quad \gamma = \frac{t}{x}, \quad u = x^{a_8}P(\chi,\gamma),$ $v = x^{-a_8-2}Q(\chi,\gamma), \quad w = x^{-1}R(\chi,\gamma)$	$P_\gamma + a_8 R_\chi - \gamma P_{\gamma\chi} + \delta RP = 0,$ $Q_\gamma + (a_8+2)Q_\chi + \gamma Q_{\gamma\chi} - \delta RQ = 0, \quad \gamma R_\gamma + R - (PQ)_\chi = 0$

For the corresponding reduction of the generator $\boldsymbol{v}_9$, one can derive directly a solution of Eq. (1)

$$u = F(y,t)\exp\Big(\frac{xy}{t}\Big), \quad v = \frac{1}{t^2F(y,t)}\exp\Big(-\frac{xy}{t}\Big), \quad w = -\frac{1}{\delta t} - \frac{tF_t(y,t)+yF_y(y,t)}{\delta tF(y,t)}\,, \tag{16}$$

where $F(y,t)$ is an arbitrary function of x and t.

To obtain the traveling wave solution of the CNLERD equation, we rewrite the reduction of the generator $\boldsymbol{v}_1+a_7\boldsymbol{v}_7$ ($a_7\in R$)

$$a_7P_\gamma - P_{\chi\chi} + \delta a_7P^2Q = 0, \quad a_7Q_\gamma + Q_{\chi\chi} - \delta a_7PQ^2 = 0, \quad R = a_7PQ\,, \tag{17}$$

which is the (1+1)-dimensional coupled integrable equation derived by Nakayama.[28] Considering the simple one-soliton solution of Eqs. (17)

$$P = -\lambda\theta + \lambda\tanh\Big(\chi - \frac{2\theta}{a_7}\gamma\Big), \quad Q = \frac{2\delta}{\lambda a_7^2}\Big[\theta + \tanh\Big(\chi - \frac{2\theta}{a_7}\gamma\Big)\Big], \quad R = -\frac{2\delta}{a_7}\mathrm{sech}^2\Big(\chi - \frac{2\theta}{a_7}\gamma\Big), \quad \theta = \pm 1\,, \tag{18}$$

then the traveling wave solution of the CNLERD equation can be given

$$u = -\lambda\theta + \lambda\tanh\Big(y - \frac{x+2\theta t}{a_7}\Big), \quad v = \frac{2\delta}{\lambda a_7^2}\Big[\theta + \tanh\Big(y - \frac{x+2\theta t}{a_7}\Big)\Big], \quad w = -\frac{2\delta}{a_7}\mathrm{sech}^2\Big(y - \frac{x+2\theta t}{a_7}\Big), \quad \theta = \pm 1\,, \tag{19}$$

Applying the Theorem 1, we can obtain the non-traveling wave solution of the CNLERD equation (1)

$$\begin{aligned}
u &= -\lambda\beta_0\exp\Big(\frac{(s_1t+s_2)\dot{\xi}_0y}{\Delta_1} + \frac{s_1xy+\Delta_2x}{s_1t+s_2}\Big)(\theta + \tanh\Lambda)\,,\\
v &= \frac{2\delta}{\lambda a_7^2}\frac{\Delta_1^2}{(s_1t+s_2)^2\beta_0}\exp\Big(-\frac{(s_1t+s_2)\dot{\xi}_0y}{\Delta_1} - \frac{s_1xy+\Delta_2x}{s_1t+s_2}\Big)(\theta + \tanh\Lambda)\,,\\
w &= -\frac{\delta}{\Delta_1}[(s_1t+s_2)\ddot{\xi}_0y + 2s_1\dot{\xi}_0y + \Delta_2\dot{\xi}_0] - \frac{\delta\dot{\beta}_0}{\beta_0} - \frac{\delta s_1}{s_1t+s_2} + \frac{2\delta\Delta_1}{a_7(s_1t+s_2)^2}\mathrm{sech}^2\Lambda\,,
\end{aligned} \tag{20}$$

with

$$\Lambda = \frac{\Delta_1x + a_7y - 2\theta(s_3t+s_4)}{a_7(s_1t+s_2)} + \frac{s_5t+s_6}{s_1t+s_2} - \frac{\xi_0}{a_7}\,,$$

where $\xi_0\equiv\xi_0(t)$ and $\beta_0\equiv\beta_0(t)$ are arbitrary functions of t and s_i ($i = 1,2,3,4,5,6$) are arbitrary constants.

5 Conservation Laws

In this section, we intend to construct conservation laws for the CNLERD equation (1). First, we recall the new conservation theorem given by Ibragimov in Ref. [19].

Consider a k-th order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$,

$$F_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m\,, \tag{21}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second, ..., k-th order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_jD_i(u^\alpha), \ldots$, respectively, with the total derivative operator with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha\frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \ldots, n\,.$$

Theorem 2 Every Lie point, Lie–Bäcklund or non-local symmetry

$$X = \xi^i(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial u^\alpha}$$

admitted by the system (21) provides a conservation law $D_i(C^i) = 0$ for the system (21) and its adjoint system. The conserved vector is given by

$$\begin{aligned}
C^i &= \xi^iL + W^\alpha\Big[\frac{\partial L}{\partial u_i^\alpha} - D_j\Big(\frac{\partial L}{\partial u_{ij}^\alpha}\Big) + D_jD_k\Big(\frac{\partial L}{\partial u_{ijk}^\alpha}\Big) - \cdots\Big] + D_j(W^\alpha)\Big[\frac{\partial L}{\partial u_{ij}^\alpha} - D_k\Big(\frac{\partial L}{\partial u_{ijk}^\alpha}\Big) + \cdots\Big]\\
&\quad + D_jD_k(W^\alpha)\Big[\frac{\partial L}{\partial u_{ijk}^\alpha} - \cdots\Big] + \cdots\,,
\end{aligned} \tag{22}$$

with the adjoint system

$$F_\alpha^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(k)}, v_{(k)}) \equiv \frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m\,,$$

where $L = v^\alpha F_\alpha(x, u, u_{(1)}, \ldots, u_{(k)})$ is the formal Lagrangian, $W\alpha = \eta^\alpha - \xi^j u_j^\alpha$ is the Lie characteristic function, $v = (v^1, v^2, \ldots, v^m)$ are new dependent variables and the Euler–Lagrange operator, for each α, read

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2\cdots i_s}}, \quad \alpha = 1, \ldots, m\,.$$

To search for conservation laws of the system (1) by Theorem 2, we first construct its adjoint system

$$u_{1t} - u_{1xy} + v w_{1y} - \delta u_1 w = 0, \quad v_{1t} + v_{1xy} + u w_{1y} + \delta v_1 w = 0, \quad w_{1x} - \delta(u u_1 - v v_1) = 0\,, \tag{23}$$

and then construct the formal Lagrangian for the system (1)

$$L = u_1(u_t + u_{xy} + \delta w u) + v_1(v_t - v_{xy} - \delta w v) + w_1(w_x + (uv)_y)\,. \tag{24}$$

Supposing the Lie symmetry of the system (1)

$$\boldsymbol{v} = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \xi^3\frac{\partial}{\partial y} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v} + \eta^3\frac{\partial}{\partial w}\,,$$

we know the general formula of conservation laws

$$\begin{aligned}
C^1 &= \xi^1 L + W^1\frac{\partial L}{\partial u_t} + W^2\frac{\partial L}{\partial v_t}\,,\\
C^2 &= \xi^2 L - W^1 D_y\Big(\frac{\partial L}{\partial u_{xy}}\Big) - W^2 D_y\Big(\frac{\partial L}{\partial v_{xy}}\Big) + W^3\frac{\partial L}{\partial w_x} + D_y(W^1)\frac{\partial L}{\partial u_{xy}} + D_y(W^2)\frac{\partial L}{\partial v_{xy}}\,,\\
C^3 &= \xi^3 L + W^1\Big(\frac{\partial L}{\partial u_y} - D_x\Big(\frac{\partial L}{\partial u_{xy}}\Big)\Big) + W^2\Big(\frac{\partial L}{\partial v_y} - D_x\Big(\frac{\partial L}{\partial v_{xy}}\Big)\Big) + W^3\frac{\partial L}{\partial w_x} + D_x(W^1)\frac{\partial L}{\partial u_{xy}} + D_x(W^2)\frac{\partial L}{\partial v_{xy}}\,,
\end{aligned}\tag{25}$$

where the Lie characteristic functions are $W^1 = \eta^1 - \xi^j u_j$, $W^2 = \eta^2 - \xi^j v_j$ and $W^3 = \eta^3 - \xi^j w_j$.

In fact, due to the existence of the cross terms u_{xy} and v_{xy}, the general formula of conservation laws must be modified by two rules:[20] (i) There is only one derivative with respect to a cross term in one conservation quantity (for example, the terms $-W^1 D_y(\partial L/\partial u_{xy})$ and $D_y(W^1)(\partial L/\partial u_{xy})$ cannot appear at the same time in C^1). (ii) The location that one derivative with respect to a cross term appears at can not be the same in different conservation quantities (for example, if the term $-W^1 D_y(\partial L/\partial u_{xy})$ exist in C^2, the term $-W^1 D_x(\partial L/\partial u_{xy})$ cannot appear in C^3 and only $D_x(W^1)(\partial L/\partial u_{xy})$ can appear in C^3). Thus, there are only four formulas of conservation laws for the original system (1) and its adjoint system (23) as follows:

Formula 1

$$\begin{aligned}
&C^1 = \xi^1 L + W^1\frac{\partial L}{\partial u_t} + W^2\frac{\partial L}{\partial v_t}, \quad C^2 = \xi^2 L - W^1 D_y\Big(\frac{\partial L}{\partial u_{xy}}\Big) - W^2 D_y\Big(\frac{\partial L}{\partial v_{xy}}\Big) + W^3\frac{\partial L}{\partial w_x}\,,\\
&C^3 = \xi^3 L + W^1\frac{\partial L}{\partial u_y} + W^2\frac{\partial L}{\partial v_y} + W^3\frac{\partial L}{\partial w_x} + D_x(W^1)\frac{\partial L}{\partial u_{xy}} + D_x(W^2)\frac{\partial L}{\partial v_{xy}}\,.
\end{aligned}\tag{26}$$

Formula 2

$$\begin{aligned}
&C^1 = \xi^1 L + W^1\frac{\partial L}{\partial u_t} + W^2\frac{\partial L}{\partial v_t}, \quad C^2 = \xi^2 L + W^3\frac{\partial L}{\partial w_x} + D_y(W^1)\frac{\partial L}{\partial u_{xy}} + D_y(W^2)\frac{\partial L}{\partial v_{xy}}\,,\\
&C^3 = \xi^3 L + W^1\Big(\frac{\partial L}{\partial u_y} - D_x\Big(\frac{\partial L}{\partial u_{xy}}\Big)\Big) + W^2\Big(\frac{\partial L}{\partial v_y} - D_x\Big(\frac{\partial L}{\partial v_{xy}}\Big)\Big) + W^3\frac{\partial L}{\partial w_x}\,.
\end{aligned}\tag{27}$$

Formula 3

$$\begin{aligned}
&C^1 = \xi^1 L + W^1\frac{\partial L}{\partial u_t} + W^2\frac{\partial L}{\partial v_t}, \quad C^2 = \xi^2 L - W^1 D_y\Big(\frac{\partial L}{\partial u_{xy}}\Big) + W^3\frac{\partial L}{\partial w_x} + D_y(W^2)\frac{\partial L}{\partial v_{xy}}\,,\\
&C^3 = \xi^3 L + W^1\frac{\partial L}{\partial u_y} + W^2\Big(\frac{\partial L}{\partial v_y} - D_x\Big(\frac{\partial L}{\partial v_{xy}}\Big)\Big) + W^3\frac{\partial L}{\partial w_x} + D_x(W^1)\frac{\partial L}{\partial u_{xy}}\,.
\end{aligned}\tag{28}$$

Formula 4

$$\begin{aligned}
&C^1 = \xi^1 L + W^1\frac{\partial L}{\partial u_t} + W^2\frac{\partial L}{\partial v_t}, \quad C^2 = \xi^2 L - W^2 D_y\Big(\frac{\partial L}{\partial v_{xy}}\Big) + W^3\frac{\partial L}{\partial w_x} + D_y(W^1)\frac{\partial L}{\partial u_{xy}}\,,\\
&C^3 = \xi^3 L + W^1\Big(\frac{\partial L}{\partial u_y} - D_x\Big(\frac{\partial L}{\partial u_{xy}}\Big)\Big) + W^2\frac{\partial L}{\partial v_y} + W^3\frac{\partial L}{\partial w_x} + D_x(W^2)\frac{\partial L}{\partial v_{xy}}\,.
\end{aligned}\tag{29}$$

In the following paper, we only list the conservation laws for the system (1) according to Formula 1.

Case 1 For the Lie point symmetry generator $\boldsymbol{v}_1=\partial/\partial y$, the corresponding Lie characteristic functions are $W^1=-u_y$, $W^2=-v_y$, and $W^3=-w_y$. The components of the conserved vector are expressed by

$$C^1=-u_1u_y-v_1v_y,\quad C^2=u_yu_{1y}-v_yv_{1y}-w_1w_y,\quad C^3=u_1u_t+v_1v_t+w_1w_x+\delta w(uu_1-vv_1)\,. \tag{30}$$

Case 2 For the Lie point symmetry generator $\boldsymbol{v}_2=\partial/\partial t$, the corresponding Lie characteristic functions are $W^1=-u_t$, $W^2=-v_t$, and $W^3=-w_t$. The components of the conserved vector are expressed by

$$\begin{aligned}&C^1=u_1u_{xy}-v_1v_{xy}+w_1(uv_y+vu_y+w_x)+\delta w(uu_1-vv_1),\quad C^2=u_tu_{1y}-v_tv_{1y}-w_tw_1\,,\\&C^3=-w_1(uv_t+vu_t)-u_1u_{xt}+v_1v_{xt}\,.\end{aligned} \tag{31}$$

Case 3 For the Lie point symmetry generator $\boldsymbol{v}_3=t(\partial/\partial y)+xu(\partial/\partial u)-xv(\partial/\partial v)$, the corresponding Lie characteristic functions are $W^1=xu-tu_y$, $W^2=-xv-tv_y$, and $W^3=-tw_y$. The components of the conserved vector are expressed by

$$\begin{aligned}&C^1=-t(u_1u_y+v_1v_y)+x(uu_1-vv_1),\quad C^2=t(u_yu_{1y}-v_yv_{1y}-w_1w_y)-x(uu_{1y}+vv_{1y})\,,\\&C^3=t(u_1u_t+v_1v_t+w_1w_x)+x(u_1u_x+v_1v_x)+\delta tw(uu_1-vv_1)+uu_1+vv_1\,.\end{aligned} \tag{32}$$

Case 4 For the Lie point symmetry generator $\boldsymbol{v}_4=x(\partial/\partial x)-y(\partial/\partial y)-2v(\partial/\partial v)$, the corresponding Lie characteristic functions are $W^1=-xu_x+yu_y$, $W^2=-2v-xv_x+yv_y$, and $W^3=-xw_x+yw_y$. The components of the conserved vector are expressed by

$$\begin{aligned}C^1&=-x(u_1u_x+v_1v_x)+y(u_1u_y+v_1v_y)-2vv_1\,,\\C^2&=x(u_1u_t+v_1v_t+u_1u_{xy}-v_1v_{xy}+u_xu_{1y}-v_xv_{1y}+w_1uv_y+w_1vu_y)+\delta xw(uu_1-vv_1)\\&\quad+y(v_yv_{1y}-u_yu_{1y}+w_1w_y)-2vv_{1y}\,,\\C^3&=x(v_1v_{xx}-u_1u_{xx}-w_1uv_x-w_1vu_x)-y(u_1u_t+v_1v_t+w_1w_x)-\delta yw(uu_1-vv_1)-u_1u_x-2w_1uv+3v_1v_x\,.\end{aligned} \tag{33}$$

Case 5 For the Lie point symmetry generator $\boldsymbol{v}_5=y(\partial/\partial y)+t(\partial/\partial t)-w(\partial/\partial w)$, the corresponding Lie characteristic functions are $W^1=-tu_t-yu_y$, $W^2=-tv_t-yv_y$, and $W^3=-w-tw_t-yw_y$. The components of the conserved vector are expressed by

$$\begin{aligned}&C^1=t(u_1u_{xy}-v_1v_{xy}+w_1vu_y+w_1uv_y+w_1w_x)+\delta tw(uu_1-vv_1)-y(u_1u_y+v_1v_y),\\&C^2=t(u_tu_{1y}-v_tv_{1y}-w_1w_t)+y(u_yu_{1y}-v_yv_{1y}-w_1w_y)-ww_1\,,\\&C^3=t(v_1v_{xt}-u_1u_{xt}-w_1vu_t-w_1uv_t)+y(u_1u_t+v_1v_t+w_1w_x)+\delta yw(uu_1-vv_1)\,.\end{aligned} \tag{34}$$

Case 6 For the Lie point symmetry generator $\boldsymbol{v}_6=xt(\partial/\partial x)+yt(\partial/\partial y)+t^2(\partial/\partial t)+xyu(\partial/\partial u)-(xy+2t)v(\partial/\partial v)-(\delta+2wt)(\partial/\partial w)$, the corresponding Lie characteristic functions are $W^1=xyu-t^2u_t-xtu_x-ytu_y$, $W^2=-(2t+xy)v-t^2v_t-xtv_x-ytv_y$, and $W^3=-\delta-2tw-t^2w_t-xtw_x-ytw_y$. The components of the conserved vector are expressed by

$$\begin{aligned}C^1&=t^2(u_1u_{xy}-v_1v_{xy})+t^2w_1(uv_y+vu_y+w_x)-tx(u_1u_x+v_1v_x)\\&\quad-ty(u_1u_y-v_1v_y)-2tvv_1+(xy+\delta t^2w)(uu_1-vv_1)\,,\\C^2&=t^2(u_tu_{1y}-v_tv_{1y}-w_1w_t)+tx(u_1u_t+v_1v_t+u_1u_{xy}-v_1v_{xy}+u_xu_{1y}-v_xv_{1y}+w_1uv_y+wvu_y\\&\quad+\delta w(uu_1-vv_1))+ty(u_yu_{1y}-v_yv_{1y}-w_1w_y)-xy(uu_{1y}+vv_{1y})-2t(vv_{1y}+ww_1)-\delta w_1\,,\\C^3&=-t^2(w_1uv_t+w_1vu_t+u_1u_{xt}-v_1v_{xt})-tx(w_1uv_x+w_1vu_x+u_1u_{xx}-v_1v_{xx})+ty(u_1u_t+v_1v_t\\&\quad+w_1w_x)+\delta tyw(uu_1-vv_1)+xy(u_1u_x+v_1v_x)+t(3v_1v_x-u_1u_x-2w_1uv)+y(uu_1+vv_1)\,.\end{aligned} \tag{35}$$

Case 7 For the Lie point symmetry generator

$$\boldsymbol{v}_7(f)=f\frac{\partial}{\partial x}+\dot{f}yu\frac{\partial}{\partial u}-\dot{f}yv\frac{\partial}{\partial v}-\delta\ddot{f}y\frac{\partial}{\partial w}\,,$$

the corresponding Lie characteristic functions are $W^1=\dot{f}yu-fu_x$, $W^2=-\dot{f}yv-fv_x$, and $W^3=-\delta\ddot{f}y-fw_x$. The components of the conserved vector are expressed by

$$\begin{aligned}&C^1=\dot{f}y(uu_1-vv_1)-f(u_1u_x+v_1v_x)\,,\\&C^2=-\dot{f}y(uu_{1y}+vv_{1y})-f(v_1v_{xy}-u_1u_{xy}-u_1u_t-v_1v_t-u_xu_{1y}+v_xv_{1y}-w_1vu_y-w_1uv_y)-\delta\ddot{f}yw_1\,,\\&C^3=\dot{f}y(u_1u_x+v_1v_x)-f(u_1u_{xx}-v_1v_{xx}+w_1uv_x+w_1vu_x)\,.\end{aligned} \tag{36}$$

Case 8 For the Lie point symmetry generator $\boldsymbol{v}_8(g) = gu(\partial/\partial u) - gv(\partial/\partial v) - \delta\dot{g}(\partial/\partial w)$, the corresponding Lie characteristic functions are $W^1 = gu$, $W^2 = -gv$, and $W^3 = -\delta\dot{g}$. The components of the conserved vector are expressed by

$$\begin{aligned} C^1 &= g(uu_1 - vv_1)\,, \\ C^2 &= -g(uu_{1y} + vv_{1y}) - \delta\dot{g}w_1\,, \\ C^3 &= g(u_1u_x + v_1v_x)\,. \end{aligned} \tag{37}$$

Remark 1 The components of the conserved vectors contain the arbitrary solutions $\{u, v, w\}$ and $\{u_1, v_1, w_1\}$ of the original system (1) and its adjoint system (23) and hence one can obtain an infinite number of conservation laws.

6 Summary and Discussions

In summary, we carry out a detailed Lie symmetry analysis of the (2+1)-dimensional CNLERD equation and investigate the algebraic structure of the symmetry groups for this equation. Also, we apply a simple direct method to derive the general finite transformation groups of the CNLERD equation, which is equivalent to Lie point symmetry groups generated via the standard approaches practically. Moreover, considering the close finite-dimensional vector fields associated the obtained Lie symmetry, the optimal system of one-dimensional subalgebras is presented. Based on this optimal system, we derive the corresponding similarity reduction by solving the characteristic equations and obtain some new exact solutions of the original equation by the general symmetry transformation.

In addition, as the other application of the obtained Lie symmetry, conservation laws for the CNLERD equation are constructed by employing the new conservation theorem. Due to the existence of cross terms, the new conservation theorem given by Ibragimov cannot be applied to the CNLERD equation directly. With the aid of two modification rules proposed by Zhang,[20] by which the new conservation theorem can be used to derive conservation laws for equations with cross terms, four formulas of conservation laws for the CNLERD equation are presented. By means of these formulas, infinitely many nontrivial conservation laws, including some time-dependent ones, for the original equation are derived. The investigation of other integrability properties such as Hamiltonian structure and generalized (nonlocal) symmetry of the (2+1)-dimensional CNLERD equation deserves further study.

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