

# Nonlocal symmetry, optimal systems, and explicit solutions of the mKdV equation\*

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The nonlocal symmetry of the mKdV equation is obtained from the known Lax pair; it is successfully localized to Lie point symmetries in the enlarged space by introducing suitable auxiliary dependent variables. For the closed prolongation of the nonlocal symmetry, the details of the construction for a one-dimensional optimal system are presented. Furthermore, using the associated vector fields of the obtained symmetry, we give the reductions by the one-dimensional sub-algebras and the explicit analytic interaction solutions between cnoidal waves and kink solitary waves, which provide a way to study the interactions among these types of ocean waves. For some of the interesting solutions, the figures are given to show their properties.

**Keywords:** nonlocal symmetry, optimal system, prolonged system, explicit solutions

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## 1. Introduction

Nonlocal symmetries were considered in a number of publications.<sup>[1–8]</sup> To search for nonlocal symmetries of the nonlinear systems is an interesting undertaking, since the nonlocal symmetries can enlarge the class of symmetries and they are closely connected with integrable models. In a number of cases, the nonlocal symmetries arise naturally when acting by so-called recursion operators<sup>[9,10]</sup> on local symmetries. For example,<sup>[11]</sup> acting by Lenard's recursion operator  $D^2 + 2u/3 + u_x D^{-1}/3$  on KdV scale symmetry

$$tu_{xxx} + \left(tu + \frac{1}{3}x\right)u_x + \frac{2}{3}u,$$

the nonlocal symmetry, which depends on  $\int u dx$ , will be obtained. The concept of potential symmetry<sup>[12,13]</sup> was first explicitly formulated by Bluman *et al.* and was subsequently applied in investigations of important classes of PDEs. Galas<sup>[4]</sup> obtained the nonlocal Lie–Bäcklund symmetries by introducing the pseudo-potentials. More recently, Lou, Hu and Chen<sup>[14,15]</sup> obtained nonlocal symmetries that were related to the Darboux transformation and Bäcklund transformation; different classes of explicit solutions were given, such as the rational solution hierarchy, the Bessel function solution hierarchy, soliton-cnoidal wave solutions and so on.

Moreover, it appears that the nonlocal symmetries are difficult to construct explicit solutions of nonlinear systems, since the finite symmetry transformations and similarity reductions cannot be directly calculated. Hence, nonlocal symmetries need to be transformed into local symmetries<sup>[16–20]</sup>, which are

equivalent to Lie point symmetries of the prolonged systems of DEs. In this paper, we show that when the nonlocal symmetries are localized, it is quite easy to obtain new exact solutions from symmetry reduction approaches.

A basic problem concerning the group invariant solution is its classification. Since a Lie group usually contains infinitely many subgroups of the same dimension, a classification of them up to some equivalence relation is necessary. This article extends the technique and applies it to the prolonged system of nonlocal symmetries. Based on the optimal system, some reductions<sup>[21–25]</sup> and explicit solutions of mKdV equation are derived.

This paper is arranged as follows. In Section 2, the nonlocal symmetries of the mKdV equation are obtained using the Lax pair. In Section 3, we transform the nonlocal symmetries into Lie point symmetries by extending the original system. The finite symmetry transformation can be obtained by solving the initial value problem. In Section 4, an optimal system is constructed to classify the group-invariant solutions of mKdV equation. In Section 5, some symmetry reductions and explicit solutions of the mKdV are obtained by using the Lie point symmetry of an extending system. Finally, some conclusions and discussion are given in Section 6.

## 2. Nonlocal symmetries of mKdV equation

In this paper, we take the well known mKdV equation as an example. The mKdV equation is

$$u_t = u_{xxx} - 6u^2u_x. \quad (1)$$

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The corresponding Lax pair has been obtained in Ref. [26] as

$$\psi_{xx} = -2u\psi_x, \quad \psi_t = -2(u_x + u^2)\psi_x, \quad (2)$$

which means that the integrable conditions of Eq. (2),  $\psi_{xxt} = \psi_{txx}$  are just the mKdV Eq. (1).

To seek the nonlocal symmetries, we use a method that can obtain the nonlocal symmetries directly. The fact shows that this method not only can obtain the nonlocal symmetries but also the general Lie point symmetries of the given equations.

Firstly, the symmetry  $\sigma$  of the mKdV equation is defined as a solution of its linearized equation

$$\sigma_t - \sigma_{xxx} + 6u^2\sigma_x + 12u\sigma u_x = 0, \quad (3)$$

which means that equation (1) is form invariant under the transformation

$$u \rightarrow u + \varepsilon\sigma, \quad (4)$$

with the infinitesimal parameter  $\varepsilon$ .

The symmetry can be written as

$$\sigma = X(x, t, u, \psi, \psi_x)u_x + T(x, t, u, \psi, \psi_x)u_t - U(x, t, u, \psi, \psi_x). \quad (5)$$

This assumption shows that this kind of symmetry is neither a classical Lie point symmetry nor a Lie-Bäcklund symmetry because it depends on the auxiliary variables and their high order partial derivatives.

Substituting Eq. (5) into Eq. (3) and eliminating  $u_t, \psi_{xx}, \psi_t$  in terms of the closed system, the determining equations for the functions  $X, T, U$  are obtained. Calculated by computer algebra, the general solutions of them take the form

$$\sigma = \left( -6c_5t + \frac{c_1x}{3} + c_6 \right) u_x + (c_1t + c_2)u_t - \frac{3c_3\psi_x^2 - c_1u\psi_x + 3c_5\psi + 3c_4}{3\psi_x}, \quad (6)$$

where  $c_i$  ( $i = 1, \dots, 6$ ) are six arbitrary constants.

**Remark 1** The vector symmetry Eq. (6) contains two parts

$$\sigma_1 = \left( \frac{c_1x}{3} + c_6 \right) u_x + (c_1t + c_2)u_t + \frac{c_1}{3}u, \\ \sigma_2 = -6c_5tu_x - \frac{3c_3\psi_x^2 + 3c_5\psi + 3c_4}{3\psi_x}.$$

One can see that the first part is the classical Lie point symmetry and the second part is nonlocal symmetry.

### 3. Localization of nonlocal symmetry

For simplicity, we let  $c_1 = c_2 = c_4 = c_5 = c_6 = 0, c_3 = -1$  in formula (6), i.e.,

$$\sigma = \psi_x. \quad (7)$$

We know that nonlocal symmetries cannot be directly employed to construct explicit solutions. Hence, nonlocal symmetries need to be transformed into local ones. One may extend the original system to a closed prolonged system by introducing some additional dependent variables.

Then, to localize the nonlocal symmetry (7), we introduce the following variable:

$$\psi_1 = \psi_x, \quad (8)$$

whence, the field  $u$  has the symmetry transformation  $u \rightarrow u + \varepsilon\sigma$ . In other words, we have to solve the following linearized equations:

$$\sigma_t - \sigma_{xxx} + 6u^2\sigma_x + 12u\sigma u_x = 0, \\ \sigma_{xx}^\psi + 2u\sigma_x^\psi + 2\sigma\psi_x = 0, \\ \sigma_t^\psi + 2u_x\sigma_x^\psi + 2\sigma_x\psi_x + 2u^2\sigma_x^\psi + 4u\sigma\psi_x = 0, \\ \sigma^{\psi_1} - \sigma_x^\psi = 0, \quad (9)$$

where  $\sigma$  is given by Eq. (8).

It is not difficult to verify that the solution of Eq. (9) with Eq. (8) has the form

$$\sigma^\psi = -\psi^2 + \psi, \quad \sigma^{\psi_1} = -2\psi\psi_1 + \psi_1. \quad (10)$$

Equation (10) shows that the nonlocal symmetry (8) in the original space  $x, t, u$  has been successfully localized to a Lie point symmetry in the enlarged space  $x, t, u, \psi, \psi_1$  with the vector form

$$V = \psi_x\partial u + (\psi - \psi^2)\partial\psi - (2\psi\psi_1 - \psi_1)\partial\psi_1. \quad (11)$$

After we succeed in making the nonlocal symmetry (8) equivalent to the Lie point symmetry (11) of the related prolonged system, the explicit solutions can be constructed naturally by Lie group theory. With the Lie point symmetry (11), by solving the following initial value problem

$$\frac{du'(\varepsilon)}{d\varepsilon} = \psi_1, \quad u'(0) = u, \\ \frac{d\psi'(\varepsilon)}{d\varepsilon} = -\psi^2 + \psi, \quad \psi'(0) = \psi, \\ \frac{d\psi_1'(\varepsilon)}{d\varepsilon} = -2\psi\psi_1 + \psi_1, \quad \psi_1'(0) = \psi_1, \quad (12)$$

the finite symmetry transformation can be calculated as

$$u'(\varepsilon) = \frac{u\psi e^{-\varepsilon} - u\psi + \psi_1 e^{-\varepsilon} - \psi_1 - u e^{-\varepsilon}}{\psi e^{-\varepsilon} - \psi - e^{-\varepsilon}}, \\ \psi'(\varepsilon) = -\frac{\psi}{\psi e^{-\varepsilon} - \psi - e^{-\varepsilon}}, \\ \psi_1'(\varepsilon) = \frac{\psi_1 e^{-\varepsilon}}{(\psi e^{-\varepsilon} - \psi - e^{-\varepsilon})^2}.$$

**Remark 2** For a given solution  $u$  of Eq. (1), the above finite symmetry transformation can give another solution  $u'$ .

To search for more similarity reductions of Eq. (1), we study Lie point symmetries of the whole prolonged equation system instead of the single Eq. (1). In order to find the Lie point symmetry, we assume that the symmetries have the vector form

$$V = X \frac{\partial}{\partial x} + T \frac{\partial}{\partial t} + U \frac{\partial}{\partial u} + \Psi \frac{\partial}{\partial \psi} + \Psi_1 \frac{\partial}{\partial \psi_1}, \quad (13)$$

where  $X, T, U, \Psi, \Psi_1$  are the functions with respect to  $x, t, u, \psi, \psi_1$ , which means that the closed system is invariant under the transformations

$$(x, t, u, \psi, \psi_1) \rightarrow (x + \varepsilon X, t + \varepsilon T, u + \varepsilon U, \psi + \varepsilon \Psi, \psi_1 + \varepsilon \Psi_1),$$

with a small parameter  $\varepsilon$ . Equivalently, the symmetries in the vector form (13) can be written as a function form

$$\begin{aligned} \sigma &= X(x, t, u, \psi, \psi_1)u_x + T(x, t, u, \psi, \psi_1)u_t \\ &\quad - U(x, t, u, \psi, \psi_1), \\ \sigma^\Psi &= X(x, t, u, \psi, \psi_1)\psi_x + T(x, t, u, \psi, \psi_1)\psi_t \\ &\quad - \Psi(x, t, u, \psi, \psi_1), \\ \sigma^{\Psi_1} &= X(x, t, u, \psi, \psi_1)\psi_{1x} + T(x, t, u, \psi, \psi_1)\psi_{1t} \\ &\quad - \Psi_1(x, t, u, \psi, \psi_1). \end{aligned} \quad (14)$$

Substituting Eq. (14) into Eq. (9) and eliminating  $u_t, \psi_{xx}, \psi_t, \psi_1$  in terms of the closed system, we get the determining equations for the functions  $X, T, U, \Psi, \Psi_1$ . Calculated by computer algebra, the general solutions of them take the form

$$\begin{aligned} X &= \frac{c_1 x}{3} + c_4, \quad T = c_1 t + c_2, \quad U = -\frac{c_1 u}{3} + c_3 \psi_1, \\ \Psi &= -c_3 \psi^2 + c_5 \psi + c_6, \quad \Psi_1 = -\frac{\psi_1(6c_3 \psi - 3c_5 + c_1)}{3}. \end{aligned} \quad (15)$$

In the next section, we will construct an optimal system to classify the group-invariant solutions of Eq. (1).

### 4. Optimal system of prolonged system

As it is mentioned in Ref. [17], the problem of classifying group-invariant solutions reduces to the problem of classifying subgroups of the full symmetry group under conjugation; the problem of finding an optimal of subgroups is equivalent to

that of finding an optimal system of subalgebras. In this section, using the method presented in Refs. [17] and [18], we will construct the optimal system of one-dimensional subalgebras of Eq. (1).

From Eq. (15), one can obtain the following six operators

$$\begin{aligned} v_1 &= \frac{1}{3}x\partial_x + t\partial_t - \frac{1}{3}u\partial_u - \frac{1}{3}\psi_1\partial\psi_1, \quad v_2 = \partial_t, \\ v_3 &= \psi_1\partial_u - \psi^2\partial\psi - 2\psi\psi_1\partial\psi_1, \quad v_4 = \partial_x, \\ v_5 &= \psi\partial\psi + \psi_1\partial\psi_1, \quad v_6 = \partial\psi. \end{aligned} \quad (16)$$

Applying the commutator operators  $[v_m, v_n] = v_m v_n - v_n v_m$ , we obtain the commutator table listed in Table 1 with the  $(i, j)$ -th entry indicating  $[v_i, v_j]$ .

Table 1. Lie bracket.

Lie	$v_1$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$
$v_1$	0	$-v_2$	0	$-\frac{1}{3}v_4$	0	0
$v_2$	$v_2$	0	0	0	0	0
$v_3$	0	0	0	0	$-v_3$	$2v_5$
$v_4$	$\frac{1}{3}v_4$	0	0	0	0	0
$v_5$	0	0	$v_3$	0	0	$-v_6$
$v_6$	0	0	$-2v_5$	0	$v_6$	0

To compute the adjoint representation, we use the Lie series in conjunction with the above commutator table. Applying the formula

$$\text{Ad}(\exp(\varepsilon v_i))v_j = v_j - \varepsilon[v_i, v_j] + \frac{1}{2}\varepsilon^2[v_i, [v_i, v_j]] - \dots$$

and Table 1, one can have the adjoint representation listed in Table 2 with the  $(i, j)$ -th entry indicating  $\text{Ad}(\exp(\varepsilon v_i))v_j$ .

Following Ref. [17], one calls two subalgebras  $v_2$  and  $v_1$  of a given Lie algebra equivalent if one can find an element  $g$  in the Lie group so that  $\text{Ad}g(v_1) = v_2$ ; where  $\text{Ad}g$  is the adjoint representation of  $g$  on  $v$ . Given a nonzero vector, for example

$$V = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 + a_6 v_6.$$

Our task is to simplify as many of the coefficients  $a_i$  as possible though judicious applications of adjoint maps to  $v$ . In this way, omitting the detailed computation, the results shown in Table 3 are obtained by the complicated computation.

In Table 3,  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  and  $\alpha_5$  are arbitrary constants. Next, we will discuss the reductions and solutions of Eq. (1) using the results in Table 3.

Table 2. Adjoint representation.

Lie	$v_1$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$
$v_1$	$v_1$	$e^\varepsilon v_2$	$v_3$	$e^{1/3\varepsilon} v_4$	$v_5$	$v_6$
$v_2$	$v_1 - \varepsilon v_2$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$
$v_3$	$v_1$	$v_2$	$v_3$	$v_4$	$v_5 + \varepsilon v_3$	$v_6 - 2\varepsilon v_5 - \varepsilon^2 v_3$
$v_4$	$v_1 - \frac{1}{3}\varepsilon v_4$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$
$v_5$	$v_1$	$v_2$	$e^{-\varepsilon} v_3$	$v_4$	$v_5$	$\varepsilon v_6$
$v_6$	$v_1$	$v_2$	$v_3 + 2\varepsilon v_5 - \varepsilon^2 v_6$	$v_4$	$v_5 - \varepsilon v_6$	$v_6$

Table 3. Optimal systems.

Cases	Optimal systems
(a1) $a_3 = a_6 = 0, a_1 \neq 0, a_5 \neq 0$	$v_1 + a_5 v_5$
(a2) $a_3 = 0, a_1 \neq 0$	$v_2$
(a3) $a_3 = a_5 = 0, a_1 \neq 0$	$v_1 + \alpha_1 v_6$
(b1) $a_1 \neq 0, a_5 \neq 0$	$v_4$
(c1) $a_1 = a_5 = 0, a_3 \neq 0, a_6 \neq 0$	$\alpha_2 v_2 + v_3 + \alpha_3 v_4 + \alpha_4 v_6$
(d1) $a_1 = a_3 = a_6 = 0, a_5 \neq 0$	$\alpha_2 v_2 + \alpha_5 v_4 + v_5$
(d1) $a_2 = a_4 = a_6 = 0, a_1 \neq 0, a_5 \neq 0,$	$v_1 + a_1 v_5$
(e1) $a_1 = a_3 = a_5 = 0, a_6 \neq 0$	$\alpha_2 v_2 + \alpha_5 v_4 + v_6$
(f1) $a_3 = a_6 = 0$	$\alpha_2 v_2 + \alpha_5 v_4$

### 5. Symmetry reduction of mKdV equation

In this section, we take the case (c1) in Table 3 as the example, other cases can be solved using the same method. In this case, without loss of generality, we let  $\alpha_3 = k_1 \alpha_2, \alpha_4 = k_2 \alpha_2$

$$V = \alpha_2 \partial t + k_1 \alpha_2 \partial x + \psi_1 \partial u - (\psi^2 - k_2 \alpha_2) \partial \psi - 2\psi \psi_1 \partial \psi_1. \tag{17}$$

By solving the following characteristic equation:

$$\frac{dx}{k_1 \alpha_2} = \frac{dt}{\alpha_2} = \frac{du}{\psi_1} = \frac{d\psi}{-(\psi^2 - k_2 \alpha_2)} = \frac{d\psi_1}{-2\psi \psi_1}, \tag{18}$$

one can obtain

$$u = \frac{\sqrt{k_2 \alpha_2} F_3(\xi) - F_2(\xi) \tanh \left[ \frac{\sqrt{k_2 \alpha_2} (x + F_1(\xi))}{k_2 \alpha_2} \right]}{\sqrt{k_2 \alpha_2}},$$

$$\psi = \sqrt{k_2 \alpha_2} \tanh \left[ \frac{\sqrt{k_2 \alpha_2} (x + F_1(\xi))}{k_2 \alpha_2} \right],$$

$$\psi_1 = -\frac{2F_2(\xi)}{\cosh \left[ \frac{2\sqrt{k_2 \alpha_2} (x + F_1(\xi))}{k_2 \alpha_2} \right] + 1}, \tag{19}$$

where  $\xi = (tk_1 - x)/k_1$ . Substituting Eq. (19) into the prolonged system yields

$$F_2(\xi) = \frac{k_2(F_1'(\xi) - k_1)}{k_1^2}, \quad F_3(\xi) = \frac{1}{2} \frac{F_1''(\xi)}{k_1(F_1'(\xi) - k_1)}, \tag{20}$$

with  $F_1'(\xi) = \partial F_1(\xi)/\partial \xi$  and  $F_1(\xi)$  satisfying

$$2\alpha_2 k_1^3 F_1''' - 2\alpha_2 k_1^2 F_1' F_1''' + 3\alpha_2 k_1^2 F_1''^2 + 4k_2 F_1'^4 - 16k_1 k_2 F_1'^3 + (24k_2 k_1^2 - \alpha_2 k_1^5) F_1'^2 + (2\alpha_2 k_1^6 - 16k_2 k_1^3) F_1' = 0. \tag{21}$$

One can simplify Eq. (21) using  $N(\xi)$  to replace  $F'$ ; the reduction equation is

$$2\alpha_2 k_1^3 N'' - 2\alpha_2 k_1^2 N N'' + 3\alpha_2 k_1^2 N'^2 + 4k_2 N^4 - 16k_1 k_2 N^3 + (24k_2 k_1^2 - \alpha_2 k_1^5) N^2 + (2\alpha_2 k_1^6 - 16k_2 k_1^3) N = 0. \tag{22}$$

Equation (22) can be solved in terms of solutions of the equation

$$N_\xi^2 = \lambda_1 N^4 + \lambda_2 N^3 + \lambda_3 N^2 + \lambda_4 N + \lambda_5, \tag{23}$$

where

$$\lambda_1 = \frac{4k_2}{\alpha_2 k_1^2}, \quad \lambda_2 = \frac{\alpha_2 k_1^2 - 12k_1 k_2}{\alpha_2 k_1^2},$$

$$\lambda_3 = \frac{2\alpha_2 k_1^5 + 12k_1^2 k_2 - 3\alpha_2 k_1^3}{\alpha_2 k_1^2},$$

$$\lambda_4 = \frac{3\alpha_2 k_1^4 - 4k_1^3 k_2 - 3\alpha_2 k_1^6}{\alpha_2 k_1^2}, \quad \lambda_5 = \frac{\alpha_2 k_1^7 - \alpha_2 k_1^5}{\alpha_2 k_1^2}.$$

After summarizing the above formulas, we obtain the explicit solution of the mKdV equation

$$u = \frac{1}{2} \frac{N'}{k_1(N - k_1)} - \frac{k_2(N - k_1)}{\sqrt{k_2 \alpha_2} k_1^2} \times \tanh \left[ \frac{\sqrt{k_2 \alpha_2} (x + \int N d\xi)}{k_2 \alpha_2} \right], \tag{24}$$

where  $N$  satisfies Eq. (23).

**Remark 3** We know that the general solution of Eq. (23) can be written out in terms of Jacobi elliptic functions. Hence, the solution expressed by Eq. (24) is just the explicit exact interaction between the soliton and the cnoidal periodic wave.

To show more clearly this kind of solution, we offer two special cases of Eq. (24) by solving Eq. (23).

**Case 1** A simple solution of Eq. (23) is given as

$$N = b_0 + b_1 \text{sn}(l_1 \xi, n_1), \tag{25}$$

substituting Eq. (25) into Eq. (19) yields

$$u = \frac{1}{2} \frac{b_1 \text{cn}(l_1 \xi, n_1) \text{dn}(l_1 \xi, n_1)}{k_1 [b_0 + b_1 \text{sn}(l_1 \xi, n_1) - k_1]} - \frac{k_2 [b_0 + b_1 \text{sn}(l_1 \xi, n_1) - k_1]}{\sqrt{k_2 \alpha_2} k_1^2} \times \tanh \left[ \frac{\sqrt{k_2 \alpha_2} (x + \int_{\xi_0}^{\xi} b_0 + b_1 \text{sn}(l_1 \xi, n_1) d\xi)}{k_2 \alpha_2} \right], \tag{26}$$

with

$$b_0 = \frac{(3n_1^2 + 1)(20l_1^2 - 4l_1^2 n_1^2)^{1/3}}{8(n_1^2 - 1)},$$

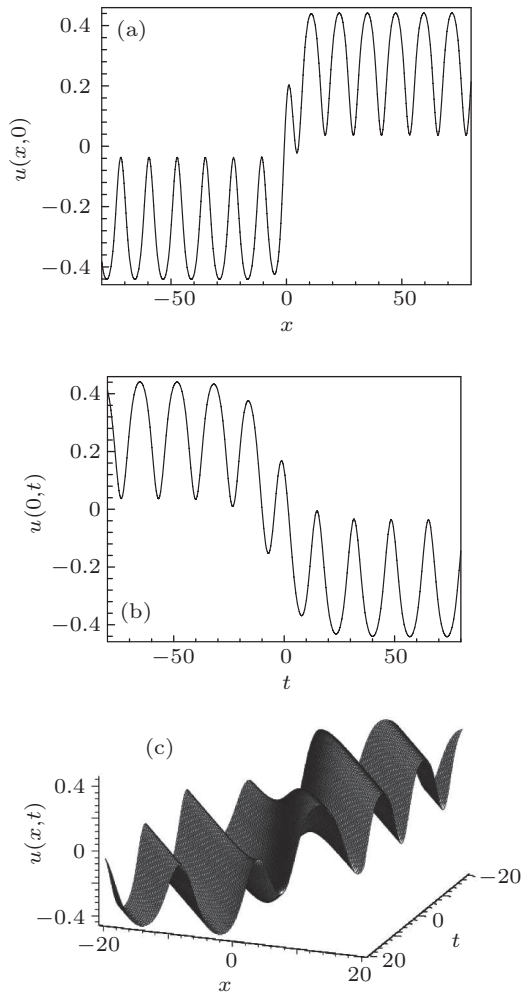
$$b_1 = -\frac{(n_1^2 - 5)n_1(20l_1^2 - 4l_1^2 n_1^2)^{1/3}}{8(n_1^2 - 1)},$$

$$k_1 = \frac{(20l_1^2 - 4l_1^2 n_1^2)^{1/3}}{2}, \quad k_2 = \frac{4\alpha_2 l_1^2 (n_1^4 - 2n_1^2 + 1)}{n_1^4 - 10n_1^2 + 25},$$

$$\xi = \frac{tk_1 - x}{k_1},$$

with  $\alpha_2$  and  $l_1$  being two arbitrary constants. Here, sn, cn and dn are usual Jacobian elliptic functions with modulus  $n_1$ . In order to study the structure of this solution, we provide some pictures, as follows.

In Fig. 1, we plot the interaction solutions between kink-shaped solitary waves and cnoidal waves expressed by Eq. (26) with parameters  $l_1 = 0.4, n_1 = 0.5, \alpha_2 = 0.2$ . We can see that the component  $u$  exhibits a kink-shaped soliton propagated on a cnoidal wave background.



**Fig. 1.** Interaction solutions to the mKdV equation. (a) Time-sliced view at  $t = 0$ ; (b) space-sliced view at  $x = 0$ ; (c) the corresponding two-dimensional image.

**Case 2** Another special solution of Eq. (23) reads

$$N = \frac{1}{b_2 + b_3 \text{sn}^2(l_2 \xi, n_2)}. \quad (27)$$

Substituting Eq. (27) into the Eq. (19) yields

$$u = \frac{b_3 l_2 \text{sn}(l_2 \xi, n_2) \text{cn}(l_2 \xi, n_2) \text{dn}(l_2 \xi, n_2)}{k_1 [b_2 + b_3 \text{sn}^2(l_2 \xi, n_2)] - k_1^2 [b_2 + b_3 \text{sn}^2(l_2 \xi, n_2)]^2} - \frac{k_2 \{ (b_2 + b_3 \text{sn}^2(l_2 \xi, n_2))^{-1} - k_1 \}}{\sqrt{k_2 \alpha_2 k_1^2}} \times \tanh \left[ \frac{\sqrt{k_2 \alpha_2} (x + \int_{\xi_0}^{\xi} (b_2 + b_3 \text{sn}^2(l_2 \xi, n_2))^{-1} d\xi)}{k_2 \alpha_2} \right], \quad (28)$$

where

$$b_2 = -\frac{n_2^2}{(n_2^2 - 1)(4l_2^2 - 8l_2^2 n_2^2)^{1/3}}, \quad b_3 = -\frac{(4l_2^2 - 8n_2^2 l_2^2)^{2/3}}{4l_2^2 (n_2^2 - 1)},$$

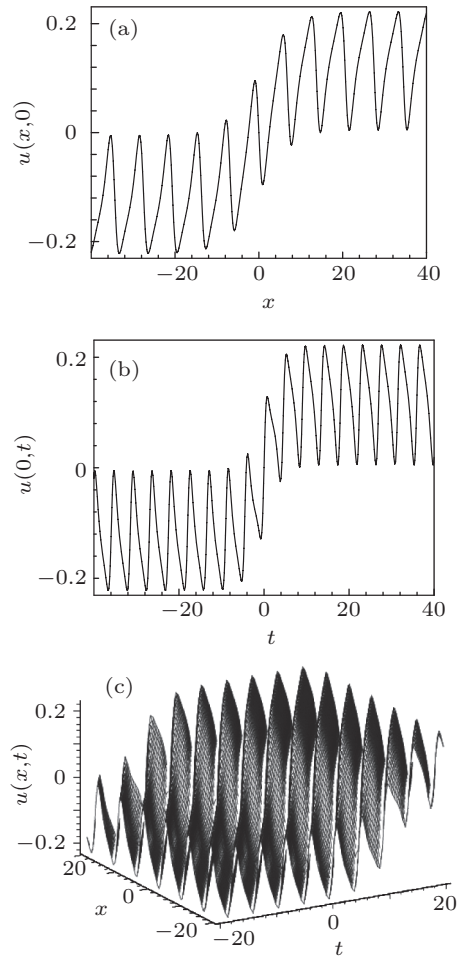
$$k_1 = (4l_2^2 - 8n_2^2 l_2^2)^{1/3},$$

$$k_2 = \frac{\alpha_2 l_2^2 n_2^2 (n_2^2 - 1)}{1 - 2n_2^2}, \quad \xi = \frac{tk_1 - x}{k_1},$$

with  $\alpha_2$  and  $l$  being two arbitrary constants. Here, sn, cn and dn are usual Jacobian elliptic functions with modulus  $n_2$ . In

order to study the structure of this solution, we give some pictures in Fig. 2.

From Fig. 2, we can see that this is the same kind solution which describes a kink-shaped soliton propagated on a cnoidal wave background for the mKdV equation. However, comparing with the first solution, it has different movement along the  $x$  axis. This kind of solution describing solitons moving on a cnoidal wave background instead of on the plane continuous wave background can be easily applicable to the analysis of physically interesting processes.



**Fig. 2.** An interaction wave to the mKdV equation with parameters:  $l_1 = 0.8$ ,  $n_1 = 0.65$ ,  $\alpha_2 = 0.6$ . (a) Time-sliced view at  $t = 0$ ; (b) space-sliced view at  $x = 0$ ; (c) the corresponding two-dimensional image.

## 6. Discussion and summary

In order to extend the applicability of nonlocal symmetry to obtain explicit solutions of the mKdV equation, the nonlocal symmetry is localized by introducing the  $x$  derivative of the spectral functions  $\psi_1 \equiv \psi_x$ . For the enlarged system  $\{u, \psi, \psi_1\}$ , the primary nonlocal symmetry is equivalent to a Lie point symmetry of a prolonged system. The one-dimensional subalgebras of a Lie algebra of a prolonged system have been classified. Then, the reductions and some solutions of the mKdV equation are given out by using the associated vector fields.



On the one hand, new explicit solutions are provided in this paper, for example, kink-sharp soliton + cnoidal wave solutions. In the real physics world, solitary waves must interact with other waves, such as the cnoidal waves, which are periodic and may be described by Jacobi elliptic functions. This would give more freedom in controlling solitons in a given environment. This kind of solution can be used to analyze the behavior of the soliton in optically induced lattices and to describe localized states in optically induced refractive index gratings.

Searching for nonlocal symmetries of integrable DEs and applying the nonlocal symmetries to construct explicit solutions are both of considerable interest and value. However, in general, the prolongation does not close, neither for the local nor for the nonlocal variables. There is not a universal way to estimate what kind of nonlocal symmetries can be spread to the Lie point symmetries of a related prolonged system. Moreover, one can construct infinitely many nonlocal symmetries by introducing some internal parameters from the seed symmetry and one may consider algebraic geometry solutions of the completely integrable finite-dimensional systems to achieve related solutions of the mKdV equation. Above topics will be discussed in the future series of research works.

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