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Nonlocal symmetries and explicit solutions of the AKNS system

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ABSTRACT

In this letter, based on the Lax pair of the Ablowitz–Kaup–Newell–Segur (AKNS) system, the nonlocal symmetry is obtained and successfully localized to a Lie point symmetry by introducing an appropriate auxiliary dependent variable. For the closed prolongation, the construction for the one-dimensional optimal system is presented in detail. Furthermore, using the obtained optimal system, we give the reductions and the explicit analytic interaction solutions between cnoidal waves and solitary waves. For some interesting solutions, the figures are given to show their properties.

1. Introduction

The Lie group method [1–10] is one of the most important methods for constructing exact solutions of nonlinear partial differential equations (PDEs). In addition to the classical and non-classical Lie symmetries, there exist so-called nonlocal symmetries [11–18] which enlarge the class of symmetries and are connected with integrable models. However, it is difficult to find the nonlocal symmetries of nonlinear PDEs. Recently, Lou et al. [19,20] obtained nonlocal symmetries which are related to the Darboux transformation (DT) and gave explicit analytic solutions through the localization procedure. Differently from the above method, we can obtain the nonlocal symmetries through a direct assumption method which can give both local and nonlocal symmetries.

For the full symmetry group that leaves the PDE invariant, there is no need to list all possible group-invariant solutions. One can minimize these solutions to find nonequivalent branches, which leads to the concept of optimal systems [21–25]. In this letter, we obtain the nonlocal symmetry by using a direct method and localize it to a Lie point symmetry by introducing an appropriate auxiliary dependent variable. For the prolonged system, the one-dimensional optimal system is presented. Based on the optimal system, some reductions and explicit solutions of the Ablowitz–Kaup–Newell–Segur (AKNS) system are derived.

This letter is arranged as follows. In Section 2, the nonlocal symmetries of the AKNS system are obtained by using the Lax pair. In Section 3, we transform the nonlocal symmetries into Lie point symmetries. Then, the finite symmetry transformations are obtained by solving the initial value problem. In Section 4, an optimal system is constructed to classify the group-invariant solutions of the AKNS system. In Section 5, based on the optimal system, some symmetry reductions and explicit solutions of the AKNS system are given by using the Lie point symmetry of the prolonged system. Finally, some conclusions and discussions are given in Section 6.

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2. Nonlocal symmetries of the AKNS system

The well-known AKNS system [14] reads

$$\begin{aligned} u_t &= \frac{1}{2} i u_{xx} + i u^2 v, \\ v_t &= -\frac{1}{2} i v_{xx} - i v^2 u. \end{aligned} \tag{1}$$

The Lax pair of Eq. (1) has the form

$$\begin{aligned} \begin{pmatrix} p \\ q \end{pmatrix}_x &= \begin{pmatrix} 0 & v \\ -u & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \\ \begin{pmatrix} p \\ q \end{pmatrix}_t &= \begin{pmatrix} -\frac{1}{2} iuv & -\frac{1}{2} i v_x \\ -\frac{1}{2} i u_x & \frac{1}{2} iuv \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \end{aligned} \tag{2}$$

To seek for the nonlocal symmetries, we adopt a method which can obtain the nonlocal symmetries directly. Practice shows that this method can obtain not only the nonlocal symmetries but also the general Lie point symmetries of the given equations.

First of all, the symmetries σ_1, σ_2 of the AKNS system are defined as solutions of their linearized equations

$$\begin{aligned} &\sigma_{1t} - \frac{1}{2} i \sigma_{1xx} - i u^2 \sigma_2 - 2iuv\sigma_1 = 0, \\ &\sigma_{2t} + \frac{1}{2} i \sigma_{2xx} + 2iuv\sigma_2 + i v^2 \sigma_1 = 0, \end{aligned} \tag{3}$$

which means that Eq. (1) is form invariant under the infinitesimal transformations

$$u \to u + \epsilon \sigma_1, \qquad v \to v + \epsilon \sigma_2, \tag{4}$$

with the infinitesimal parameter ϵ.

The symmetry can be written as

$$\begin{aligned} \sigma_1 &= X(x,t,u,v,p,q) u_x + T(x,t,u,v,p,q) u_t - U(x,t,u,v,p,q), \\ \sigma_2 &= X(x,t,u,v,p,q) v_x + T(x,t,u,v,p,q) v_t - V(x,t,u,v,p,q). \end{aligned} \tag{5}$$

Substituting Eq. (5) into Eq. (3) and eliminating $u_t, v_t, p_x, p_t, q_x, q_t$ in terms of the closed system yields a system of determining equations for the functions X, T, U, V, which can be solved by virtue of Maple to give

$$\begin{aligned} &X(x,t,u,v,p,q) = c_3 x + c_1 t + c_2, \qquad T(x,t,u,v,p,q) = 2c_3 t + c_4, \\ &U(x,t,u,v,p,q) = (c_1 i x - 2c_3 - c_6) u - c_6 q^2, \qquad V(x,t,u,v,p,q) = (-c_1 i x + c_6) v + c_5 p^2, \end{aligned} \tag{6}$$

where c_i $(i = 1, \dots, 6)$ are six arbitrary constants and $i^2 = -1$. In Ref. [26], Kazuhiro gave the nonlocal symmetry of the AKNS hierarchy with implicit form and the symmetry algebra was isomorphic to a loop algebra. Here, we construct the nonlocal symmetry in explicit form using an algebra method. Compared with the results of Ref. [14], we not only give the nonlocal symmetries but also the Lie point symmetries.

3. Localization of the nonlocal symmetry

As we know, the nonlocal symmetries cannot be used to construct explicit solutions of differential equations (DEs) directly. Hence, we need to transform the nonlocal symmetries into local ones [19,20]. In this section, we will find a related system which possesses a Lie point symmetry that is equivalent to the nonlocal symmetry.

For simplicity, we let $c_1 = c_2 = c_3 = c_4 = 0$, $c_5 = c_6 = 1$ in formula (6), i.e.,

$$\sigma_1 = q^2, \qquad \sigma_2 = -p^2. \tag{7}$$

To localize the nonlocal symmetry (7), we have to solve the following linearized equations:

$$\sigma_{3x} - v\sigma_4 - \sigma_2 q = 0, \qquad \sigma_{4x} + u\sigma_3 + \sigma_1 p = 0, \tag{8}$$

with σ_1, σ_2 given by (7). It is not difficult to verify that the solutions of (8) have the following forms:

$$\sigma_3 = pf, \qquad \sigma_4 = qf, \tag{9}$$

where f is given by

$$f_x = -pq, \qquad f_t = \frac{1}{2}i(up^2 + vq^2). \tag{10}$$

It is easy to obtain the following result:

$$\sigma_5 = \sigma_f = f^2. \tag{11}$$

The results (9) shows us that the nonlocal symmetry (7) in the original space x, t, u, v has been successfully localized to a Lie point symmetry in the enlarged space x, t, u, v, p, q, f.

Another interesting point one can see is that the introduced auxiliary dependent variable f just satisfies the Schwartzian form of the AKNS system

$$\left(\frac{\phi_t}{\phi_x}\right)_t = \left(\frac{3\phi_t^2}{2\phi_x^2} - \frac{1}{4}\{\phi; x\}\right)_x, \tag{12}$$

where $\{\phi; x\} = (p_{xxx}/p_x) - 3/2(p_{xx}/p_x)^2$ is the Schwartzian derivative.

After succeeding in making the nonlocal symmetry (7) equivalent to the Lie point symmetry (10) of the related prolonged system, we can construct the explicit solutions naturally by Lie group theory. With the Lie point symmetry (10), by solving the following initial value problem:

$$\begin{aligned} &\frac{d\bar{u}}{d\varepsilon} = q^2, \quad \bar{u}|_{\varepsilon=0} = u; \quad \frac{d\bar{v}}{d\varepsilon} = -p^2, \quad \bar{v}|_{\varepsilon=0} = v; \quad \frac{d\bar{p}}{d\varepsilon} = pf, \quad \bar{p}|_{\varepsilon=0} = p; \\ &\frac{d\bar{q}}{d\varepsilon} = qf, \quad \bar{q}|_{\varepsilon=0} = q; \quad \frac{d\bar{f}}{d\varepsilon} = f^2, \quad \bar{f}|_{\varepsilon=0} = f, \end{aligned} \tag{13}$$

the finite symmetry transformation can be calculated as

$$\bar{u} = \frac{-\varepsilon q^2 + \varepsilon uf - u}{\varepsilon f - 1}, \quad \bar{v} = \frac{\varepsilon p^2 + \varepsilon vf - v}{\varepsilon f - 1}, \quad \bar{p} = -\frac{p}{\varepsilon f - 1}, \quad \bar{q} = -\frac{q}{\varepsilon f - 1}, \quad \bar{f} = -\frac{f}{\varepsilon f - 1}. \tag{14}$$

For a given solution u, v of Eq. (1), the above finite symmetry transformation will arrive at another solution $\bar{u}, \bar{v}$. To search for more similarity reductions of Eq. (1), we study the Lie point symmetries of the whole prolonged equation system instead of the single Eq. (1) and assume that the vector of the symmetries has the form

$$V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q} + F\frac{\partial}{\partial f}, \tag{15}$$

where X, T, U, P, Q, F are the functions with respect to x, t, u, p, q, f, which means that the closed system is invariant under the infinitesimal transformations

$$(x, t, u, p, q, f) \to (x + \epsilon X, t + \epsilon T, u + \epsilon U, p + \epsilon P, q + \epsilon Q, f + \epsilon F),$$

with

$$\begin{aligned} \sigma_1 &= X(x, t, u, v, p, q, f)u_x + T(x, t, u, v, p, q, f)u_t - U(x, t, u, v, p, q, f), \\ \sigma_2 &= X(x, t, u, v, p, q, f)v_x + T(x, t, u, v, p, q, f)v_t - V(x, t, u, v, p, q, f), \\ \sigma_3 &= X(x, t, u, v, p, q, f)p_x + T(x, t, u, v, p, q, f)p_t - P(x, t, u, v, p, q, f), \\ \sigma_4 &= X(x, t, u, v, p, q, f)q_x + T(x, t, u, v, p, q, f)q_t - Q(x, t, u, v, p, q, f), \\ \sigma_5 &= X(x, t, u, v, p, q, f)f_x + T(x, t, u, v, p, q, f)f_t - F(x, t, u, v, p, q, f). \end{aligned} \tag{16}$$

Moreover, $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ satisfy the following equations:

$$\begin{aligned} &\sigma_{1t} - \frac{1}{2}i\sigma_{1xx} - iu^2\sigma_2 - 2iuv\sigma_1 = 0, \quad \sigma_{2t} + \frac{1}{2}i\sigma_{2xx} + 2iuv\sigma_2 + iv^2\sigma_1 = 0, \\ &\sigma_{3x} - v\sigma_4 - q\sigma_2 = 0, \quad \sigma_{3t} + \frac{1}{2}iuv\sigma_3 + \frac{1}{2}iup\sigma_2 + \frac{1}{2}ivp\sigma_1 + \frac{1}{2}iv_x\sigma_4 + \frac{1}{2}iq\sigma_{2x} = 0, \\ &\sigma_{4x} + u\sigma_3 + p\sigma_1 = 0, \quad \sigma_{4t} + \frac{1}{2}iu_x\sigma_3 + \frac{1}{2}ip\sigma_{1x} - \frac{1}{2}iuv\sigma_4 - \frac{1}{2}iuq\sigma_2 - \frac{1}{2}iqv\sigma_1 = 0, \\ &\sigma_{5x} + p\sigma_4 + q\sigma_3 = 0, \quad \sigma_{5t} - \frac{1}{2}iq^2\sigma_2 - iqv\sigma_4 - iup\sigma_3 - \frac{1}{2}ip^2\sigma_1 = 0. \end{aligned} \tag{17}$$

Table 1
Optimal systems.

Case		Optimal system
(a1)	$a_3 = a_4 = a_5 = a_6 = 0, a_1 \neq 0, a_2 \neq 0,$	$v_1 + a_2 v_2$
(a2)	$a_2 = a_3 = a_4 = a_5 = 0, a_1 \neq 0,$	$v_1 + \alpha_1 v_5 + \alpha_2 v_6$
(b1)	$a_1 = a_2 = 0, a_3 \neq 0, a_4 \neq 0,$	$v_3 + a_4 v_4 + \alpha_1 v_5 + \alpha_2 v_6$
(c1)	$a_1 = a_3 = a_4 = 0, a_2 \neq 0,$	v_5 or v_6
(c2)	$a_1 = a_3 = a_4 = a_5 = a_6 = 0, a_2 \neq 0,$	v_2
(d1)	$a_1 = a_2 = a_3 = 0, a_4 \neq 0,$	$v_4 + \alpha_1 v_5 + \alpha_2 v_6$
(e1)	$a_3 = a_4 = 0,$	$\alpha_1 v_5 + \alpha_2 v_6$

Substituting Eq. (16) into symmetry Eqs. (17) and eliminating $u_t, v_t, p_x, p_t, q_x, q_t, f_x, f_t$ in terms of the closed system, we arrive at a system of determining equations for the functions X, T, U, V, P, Q, and F, which can be solved by using Maple to give

$$\begin{aligned}
&X(x,t,u,v,p,q,f) = \frac{c_1 x}{2} + c_3, \qquad T(x,t,u,v,p,q,f) = c_1 t + c_2, \qquad U(x,t,u,v,p,q,f) = c_5 u + c_4 q^2, \\
&V(x,t,u,v,p,q,f) = (-c_1 - c_6)v - c_4 p^2, \qquad P(x,t,u,v,p,q,f) = -\frac{p}{2}(c_5 + c_1 - c_6 - 2c_4 f), \\
&Q(x,t,u,v,p,q,f) = \frac{q}{2}(c_5 + c_6 + 2c_4 f), \qquad F(x,t,u,v,p,q,f) = c_4 f^2 + c_6 f + c_7,
\end{aligned} \tag{18}$$

where $c_i,\ i = 1, 2, \ldots, 6$, are arbitrary constants.

In general, to each s-parameter subgroup of the full symmetry group, there will correspond a family of group-invariant solutions. Because there are always an infinite number of subgroups, there is no need to list possible group-invariant solutions to the system. Differently from Ref. [27], we will construct an optimal system to classify the group-invariant solutions of Eq. (1) in the next section.

4. Optimal system of the prolonged system

As stated in Ref. [2], the problem of finding an optimal system of subgroups is equivalent to finding an optimal system of subalgebras. In this section, we will construct the optimal system of one-dimensional subalgebras of Eq. (1) by using the method presented in Refs. [2,3]. From Eqs. (18), the associated vector fields for the one-parameter Lie group of infinitesimal transformations are seven generators given by

$$\begin{aligned}
&v_1 = \frac{1}{2}p\frac{\partial}{\partial p} + \frac{1}{2}q\frac{\partial}{\partial q} + f\frac{\partial}{\partial f}, \qquad v_2 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v} - \frac{1}{2}p\frac{\partial}{\partial p}, \\
&v_3 = q^2\frac{\partial}{\partial u} - p^2\frac{\partial}{\partial v} + pf\frac{\partial}{\partial p} + qf\frac{\partial}{\partial q} + f^2\frac{\partial}{\partial f}, \qquad v_4 = \frac{\partial}{\partial f}, \qquad v_5 = \frac{\partial}{\partial t}, \qquad v_6 = \frac{\partial}{\partial x}, \\
&v_7 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - \frac{1}{2}p\frac{\partial}{\partial p} + \frac{1}{2}q\frac{\partial}{\partial q}.
\end{aligned} \tag{19}$$

For simplicity, we omit the commutator table and the adjoint representation here. One can know that v_7 is the center of the group through calculation, so we do not have to consider it. Following Ref. [2], two subalgebras v_2 and v_1 of a given Lie algebra are equivalent if one can find an element g in the Lie group so that $Adg(v_1) = v_2$, where Adg is the adjoint representation of g on v. Given a nonzero vector, for example,

$$V = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 + a_6 v_6,$$

where $a_j, j = 1, 2, \ldots, 6$ are arbitrary constants. The key task is to simplify as many of the coefficients a_i as possible though judicious application of adjoint maps to v. In this way, one can get the following results with massive complex computations in Table 1, where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary constants.

5. Symmetry reduction of the AKNS system

In this section, we take the case $(b1)$ as an example; other cases can be solved in the same way. In this case, without loss of generality, we let $\alpha_2 = \alpha_1 k$, $a_4 = -\alpha_3$,

$$V = \alpha_1\frac{\partial}{\partial t} + k\alpha_1\frac{\partial}{\partial x} + q^2\frac{\partial}{\partial u} - p^2\frac{\partial}{\partial v} + pf\frac{\partial}{\partial p} + qf\frac{\partial}{\partial q} + (f^2 - \alpha_3)\frac{\partial}{\partial f}. \tag{20}$$

By solving the following characteristic equation:

$$\frac{dt}{\alpha_1} = \frac{dt}{\alpha_1 k} = \frac{du}{q^2} = \frac{dv}{p^2} = \frac{dp}{pf} = \frac{dq}{qf} = \frac{df}{f^2 - \alpha_3}, \tag{21}$$

we have

$$\begin{aligned}
f &= -\sqrt{-\alpha_3}\tanh\left(\frac{\sqrt{-\alpha_3}(t+F(\xi))}{a_1}\right),\\
p &= P(\xi)\sqrt{\tanh\left(\frac{\sqrt{-\alpha_3}(t+F(\xi))}{a_1}\right)-1}\sqrt{\tanh\left(\frac{\sqrt{-\alpha_3}(t+F(\xi))}{a_1}\right)+1}\\
q &= Q(\xi)\sqrt{\tanh\left(\frac{\sqrt{-\alpha_3}(t+F(\xi))}{a_1}\right)-1}\sqrt{\tanh\left(\frac{\sqrt{-\alpha_3}(t+F(\xi))}{a_1}\right)+1}\\
u &= \frac{-Q^2(\xi)}{\sqrt{-\alpha_3}}\tanh\left(\frac{\sqrt{-\alpha_3}(t+F(\xi))}{a_1}\right)+U(\xi), \qquad v = \frac{P^2(\xi)}{\sqrt{-\alpha_3}}\tanh\left(\frac{\sqrt{-\alpha_3}(t+F(\xi))}{a_1}\right)+V(\xi),
\end{aligned} \tag{22}$$

where $\xi = x - kt$.

Substituting Eqs. (22) into the prolonged system yields

$$\begin{aligned}
&P(\xi) = e^{-\frac{1}{2}\int_{\xi_0}^{\xi}\frac{-2i+2ikF_\xi-F_{\xi\xi}}{F_\xi}d\xi}, \qquad Q(\xi) = \frac{-\alpha_3 F_\xi}{a_1 P(\xi)}, \qquad U(\xi) = \frac{-\alpha_3(-2i+2ikF_\xi+F_{\xi\xi})}{2a_1P^2(\xi)},\\
&V(\xi) = \frac{a_1P^2(\xi)(-2i+2ikF_\xi-F_{\xi\xi})}{-2\alpha_3F_\xi^2},
\end{aligned} \tag{23}$$

where $F(\xi)$ satisfies the following equation:

$$a_1^2F_\xi^2F_{\xi\xi\xi\xi}-4a_1^2F_{\xi\xi\xi}F_{\xi\xi}F_\xi-8ka_1^2F_{\xi\xi}F_\xi+12a_1^2F_{\xi\xi}+4\alpha_3F_{\xi\xi}F_\xi^4+3a_1^2F_{\xi\xi}^3=0. \tag{24}$$

One can simplify Eq. (24) by replacing F_ξ with $W(\xi)$, and the reduction equation is

$$a_1^2W^2W_{\xi\xi\xi}-4a_1^2W_{\xi\xi}W_{\xi\xi}W-8ka_1^2W_\xi W+12a_1^2W_\xi+4\alpha_3W_\xi W^4+3a_1^2W_\xi^3=0. \tag{25}$$

Eq. (25) can be solved in terms of solutions of the equation

$$W_\xi^2 = \frac{1}{a_1^2}(-4\alpha_3W^4-2a_1^2W^3+2a_1^2W^2+8ka_1^2W+4a_1^2). \tag{26}$$

After summarizing the above formulas, we get the explicit solution of the AKNS system:

$$\begin{aligned}
u &= \frac{-\alpha_3^2F_\xi^2e^{\int_{\xi_0}^{\xi}\frac{-2i+2ikF_\xi-F_{\xi\xi}}{F_\xi}d\xi}}{a_1^2\sqrt{-\alpha_3}}\tanh\left(\frac{\sqrt{-\alpha_3}(t+F(\xi))}{a_1}\right)-\frac{\alpha_3(-2i+2ikF_\xi+F_{\xi\xi})e^{\int_{\xi_0}^{\xi}\frac{-2i+2ikF_\xi-F_{\xi\xi}}{F_\xi}d\xi}}{2a_1},\\
v &= \frac{e^{-\int_{\xi_0}^{\xi}\frac{-2i+2ikF_\xi-F_{\xi\xi}}{F_\xi}d\xi}}{\sqrt{-\alpha_3}}\tanh\left(\frac{\sqrt{-\alpha_3}(t+F(\xi))}{a_1}\right)+\frac{a_1e^{-\int_{\xi_0}^{\xi}\frac{-2i+2ikF_\xi-F_{\xi\xi}}{F_\xi}d\xi}(-2i+2ikF_\xi-F_{\xi\xi})}{-2\alpha_3F_\xi^2},
\end{aligned} \tag{27}$$

where $F_\xi = W$, and W satisfies Eq. (26).

Remark 1. We know that the general solution of Eq. (26) can be written in terms of Jacobi elliptic functions. Hence, the solution expressed by Eq. (27) is just the explicit exact interaction between the soliton and the cnoidal periodic wave.

To show more clearly this kind of solution, we offer a special case of Eq. (27) by solving Eq. (26).

A simple solution of Eq. (26) is given as

$$W = b_0 + b_1\mathrm{sn}(l_1\xi, m) + b_2\mathrm{sn}^2(l_1\xi, m). \tag{28}$$

Substituting Eq. (28) into Eq. (26) or Eq. (25) yields

$$\begin{aligned}
&a_1 = C_1, \qquad b_0 = C_2, \qquad b_1 = C_3, \qquad m = C_4, \qquad b_2 = 0,\\
&\alpha_3 = \frac{a_1^2m^2}{m^2b_0^4-b_0^2b_1^2-b_0^2b_1^2m^2+b_1^4}, \qquad k = \frac{b_0^2(2b_0^2m^2-b_1^2-b_1^2m^2)}{m^2b_0^4-b_0^2b_1^2-b_0^2b_1^2m^2+b_1^4},\\
&l = \frac{2b_1}{\sqrt{-m^2b_0^4+b_0^2b_1^2+b_0^2b_1^2m^2-b_1^4}}
\end{aligned} \tag{29}$$

with C_1, C_2 and C_3 being three arbitrary constants and $0 < C_4 < 1$. Here sn, cn, and dn are usual Jacobian elliptic functions with modulus m.

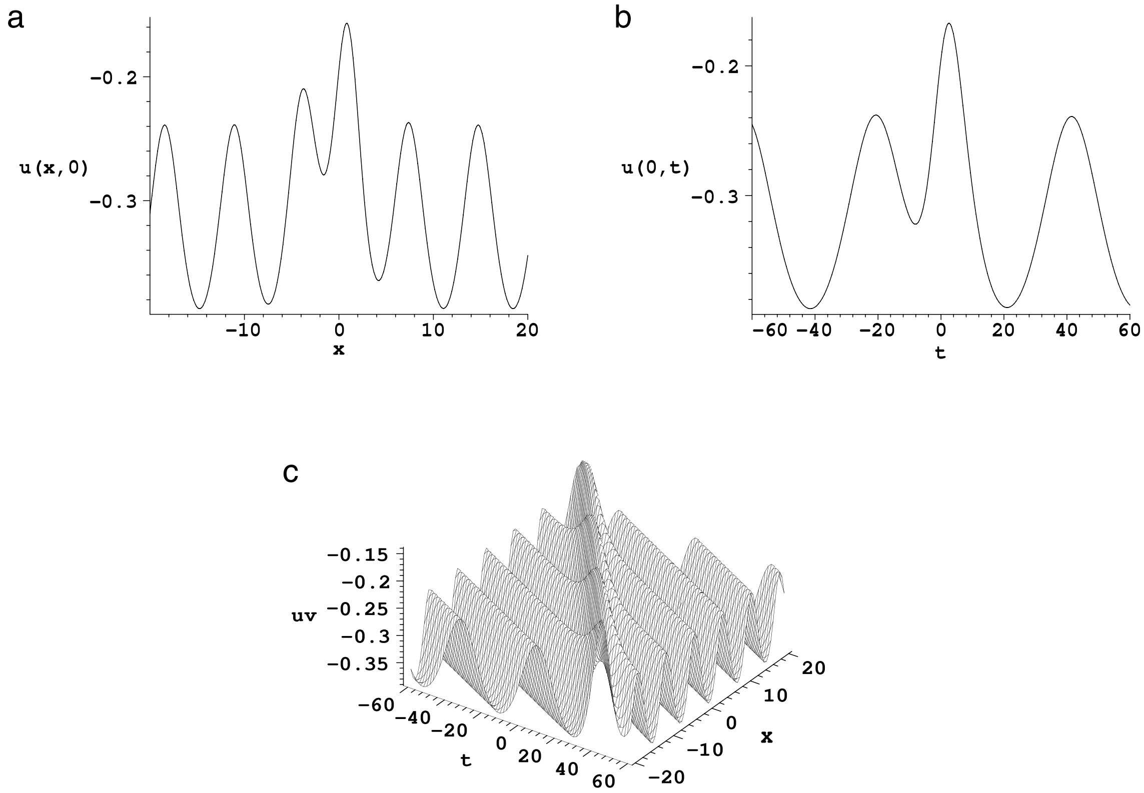

Fig. 1. Interaction solutions to the AKNS system. Part (a) shows the time-sliced view at $t = 0$; (b) shows the space-sliced view at $x = 0$ and (c) shows the corresponding two-dimensional image.

Substituting Eqs. (28), (29) and $F_\xi = W$ into Eq. (27), one can obtain the solutions of u, v. In order to study the properties of these solutions of the AKNS system, we give some pictures of u, v as shown in Fig. 1.

In Fig. 1, we plot the interaction solutions between solitary waves and cnoidal waves expressed by (27) with parameters $C_1 = 2, C_2 = 0.2, C_3 = 1, C_4 = 0.2$. We can see that the component uv exhibits a soliton propagating on a cnoidal wave background. In fact, it is of interest to study these types of solutions, for example, in describing localized states in optically refractive index gratings. In the ocean, there are some typical nonlinear waves such as solitary waves and cnoidal periodic waves.

Remark 2. If setting the module m degenerate to 1, the soliton + cnoidal wave solution degenerates to the soliton solution in which amplification of the amplitude has been experimentally observed and has practical applications in maritime security and coastal engineering. Some other types of solutions can be given by using the Jacobian elliptic equation; in order to save space, we do not give a detailed discussion.

6. Summary and discussion

In summary, the nonlocal symmetry of the AKNS system is obtained by using the Lax pair and successfully localized. On the basis of the prolonged system, one-dimensional subalgebras of a Lie algebra have been classified and the reductions of the AKNS system are given by using the associated vector fields. Using the reduction of the AKNS system, we successfully obtain the moving direction of a soliton on a cnoidal background wave which can be applicable to explain some physical processes. This method can be applied to other interesting integrable models.

Moreover, infinitely many nonlocal symmetries and conservation laws can be constructed using the seed symmetry. From local and nonlocal conservation laws, one can seek new integrable systems. The above topics will be discussed in a future series of research works.

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