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Error Estimates for a Discontinuous Galerkin Method with Interior Penalties Applied to Nonlinear Sobolev Equations

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A Discontinuous Galerkin method with interior penalties is presented for nonlinear Sobolev equations. A semi-discrete and a family of fully-discrete time approximate schemes are formulated. These schemes are symmetric. Hp-version error estimates are analyzed for these schemes. For the semi-discrete time scheme a priori $L^\infty(H^1)$ error estimate is derived and similarly, $l^\infty(H^1)$ and $l^2(H^1)$ for the fully-discrete time schemes. These results indicate that spatial rates in H^1 and time truncation errors in L^2 are optimal.
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I. INTRODUCTION

It is well-known that discontinuous Galerkin methods have been very widely used for solving a large range of computational fluid problems [1–3]. They are preferred over standard continuous Galerkin methods because of their flexibility in approximating globally rough solutions, their local mass conservation, their possible definition on unstructured meshes, their potential for error control and mesh adaptation.

The idea of using penalty terms in a finite element method is not new. Baker [4] was the first one who used interior penalty with nonconforming elements for elliptic equations. Douglas and Dupont [5] analyzed a method which used interior penalties on the derivatives with conforming elements for linear elliptic and parabolic problems. Inspired by [4], Wheeler [6] presented an interior penalty method for second order linear elliptic equations. Closest to [6], Arnold [7] formulated a semi-discrete discontinuous Galerkin method with interior penalty for second order

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nonlinear parabolic equations. These methods [5–7] generalized a method by Nitsche [8] for treating Dirichlet boundary condition by the introduction of penalty terms on the boundary of the domain. Applications of these methods to flow in porous media were presented by Douglas, Wheeler, Darlow and Kendall in [9]. These methods frequently referred to as interior penalty Galerkin schemes.

In general, penalty terms are weighted L^2 inner products of the jumps in the function values across element edges. The primary motivation of including interior penalties is to impose approximate continuity. These terms enable closer approximation of solutions which varies in character from one element to another and allow the incorporation of partial knowledge of the solution into the scheme. Numerical experiments have clearly demonstrated the value of penalties for solving certain problems (see, e.g., [5]).

New applications of discontinuous Galerkin method with interior penalties to nonlinear parabolic equations were introduced and analyzed by Rivière and Wheeler ([2, 10, 11]). It was shown that the method in ([2, 10, 11]) was elementwise conservative, and a priori and a posteriori error estimate in higher dimensions were derived.

Sobolev equations arise in the flow of fluids through fissured rock [12], thermodynamics [13] and other applications. They are one of important partial differential equations in practical use. Numerical treatments for Sobolev equations can be found: [14] for semi-linear problems, [15–17] for nonlinear problems.

The purpose of this paper is to extend the discontinuous Galerkin method with interior penalties in [2, 7] to nonlinear Sobolev equations. We consider semi-discrete and a family of fully-discrete time schemes. These schemes are symmetric. Hp-version error estimates are analyzed for these schemes. More attentions are paid for treating a damping term $\nabla \cdot (b(u)\nabla u_t)$, which is a distinct character of Sobolev equations different from parabolic equation. To our knowledge, this paper appears to be the first trial to approximate nonlinear Sobolev equations by using the Discontinuous Galerkin method with interior penalties.

Because we focus mainly on using the interior penalties in this article, we derive at most 2-order time truncation errors for fully-discrete time DG schemes. As for higher order time discretization, the readers can refer to the TVD Runge-Kutta DG methods, which apply the explicit time discretizations introduced by Shu [18], Cockburn and Shu [19] to a space discretization that uses discontinuous basis functions.

The outline of the paper is as follows. In section II, we first briefly introduce nonlinear Sobolev equations and some definitions, then use discontinuous Galerkin method with interior penalties to present a semi-discrete and a family of fully-discrete time approximate schemes. We call these two schemes as **semi-discrete time DG scheme** and **fully-discrete time DG scheme**, respectively. In section III for Semi-discrete time DG scheme, the existence and uniqueness of the approximate solution are proved and a priori error estimate in $L^\infty(H^1)$ is derived. Then error estimates in $l^\infty(H^1)$ and $l^2(H^1)$ are obtained similarly in section IV for fully-discrete time DG scheme. Finally, conclusions and perspectives are described in the last section.

II. SOBOLEV EQUATIONS, PRELIMINARIES, THE DG SCHEMES

A. Sobolev Equations

We shall consider the following nonlinear Sobolev equations

$$\begin{cases} u_t - \nabla \cdot \{a(x,u)\nabla u + b(x,u)\nabla u_t\} = f(x,u), & \forall\, (x,t) \in \Omega \times (0,T], \\ \{a(x,u)\nabla u + b(x,u)\nabla u_t\} \cdot n = 0, & \forall\, (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & \forall\, x \in \Omega, \end{cases} \tag{2.1}$$

where Ω is a convex polygonal domain in IR^d, $d = 2, 3$, with boundary $\partial\Omega$ and n is the unit outward vector normal to $\partial\Omega$.

For $t \in [0, T]$ and $(x, p) \in \Omega \times IR$, the regularity assumptions on a, b, f and u are catalogued as **(A)** [15]:

- There exist constants a_0 and a^*, s.t. $0 < a_0 \le a(x, p), b(x, p) \le a^*$.
- Let $a(x, p), b(x, p)$, and $f(x, p)$ be continuously differentiable with respect to each variable and assume there exist uniform bounds for $|\frac{\partial a}{\partial p}|, |\frac{\partial b}{\partial p}|, |\frac{\partial f}{\partial p}| \le K$, K is a positive constant.
- $u \in C^2(\Omega \times [0, T])$ is a unique solution to Eq. (2.1), and $u \in L^2((0, T); H^s(\Omega))$, $u_t \in L^2((0, T); H^s(\Omega))$, for $s \ge 3$.
- ∇u and ∇u_t are bounded in $L^\infty(\Omega \times (0, T])$.

B. Preliminaries

Let $\mathcal{E}_h = \{E_1, E_2, \ldots, E_{N_h}\}$ denote a regular quasi-uniform subdivision of Ω. Let $h_j = diam(E_j)$ be the element diameter and $h = \max\{h_j, j = 1, \ldots, N_h\}$. The regularity means that there exists a constant $\rho > 0$ such that each E_j contains a ball of radius ρh_j. This regular assumption is used for deriving error estimates in terms of h (i.e. for the h-version). And the quasi-uniformity requirement is that there is a constant $\gamma > 0$ such that $h/h_j \le \gamma$ for all $j \in 1, \ldots, N_h$. This quasi-uniformity assumption is used for deriving error estimates in terms of the degree of polynomials (i.e. for the p-version).

For an integer $s \ge 0$ and a domain $E \subset IR^d$, the usual norm of Sobolev space $H^s(E)$ is denoted by $\|\cdot\|_{s,E}$, the L^2-norm corresponding to $\|\cdot\|_{0,E}$. If $E = \Omega$ and if there is no ambiguity, we simply write $\|\cdot\|_s$. In view of the subdivision $\mathcal{E}_h$, we define a space

$$H^s(\mathcal{E}_h) = \{v \in L^2(\Omega) : v|_{E_j} \in H^s(E_j), j = 1, \ldots, N_h\}.$$

The norms associated with this space are the following "broken" norms ([20]):

$$|||\phi|||_s^2 = \sum_{j=1}^{N_h} \|\phi\|_{s,E_j}^2, \quad |||\phi|||_{L^\infty} = \max_{1\le j\le N_h} ||\phi||_{L^\infty(E_j)},$$

$$|||\phi|||^2_{L^2((\alpha,\beta);H^s)} = \int_\alpha^\beta |||\phi(\cdot,t)|||_s^2 dt, \quad |||\phi|||_{L^\infty((\alpha,\beta);H^s)} = \sup_{\alpha\le t\le\beta} |||\phi(\cdot,t)|||_s$$

$$|||\phi|||_{L^\infty((\alpha,\beta);L^\infty)} = \sup_{\alpha\le t\le\beta} |||\phi|||_{L^\infty}.$$

Let r be a positive integer. The finite element space used in this paper is taken to be

$$\mathcal{D}_r(\mathcal{E}_h) = \{v \in L^2(\Omega) : v|_{E_j} \in P_r(E_j), j = 1, \ldots, N_h\},$$

where $P_r(E_j)$ denotes the set of polynomials of degree less than or equal to r on E_j.

To present approximate schemes, we need some notations on edges (or faces) between two elements. Let the edges (resp. faces for $d = 3$) of $\mathcal{E}_h$ be denoted by $\{e_1, e_2, \ldots, e_{P_h}, e_{P_h+1}, \ldots, e_{M_h}\}$, where $e_k \subset \Omega, 1 \le k \le P_h$, and $e_k \subset \partial\Omega, P_h + 1 \le k \le M_h$. With each edge (or face) e_k, we associate a unit outward normal vector ν_k. For $k > P_h$, ν_k is taken to be the unit outward vector

normal to $\partial\Omega$. The average and the jump for $\phi \in H^s(\mathcal{E}_h)(s > \frac{1}{2})$ on e_k are defined as follows. Let $1 \le k \le P_h$, for $e_k = \partial E_i \cap \partial E_j$ with ν_k exterior to E_i, set

$$\{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}, \quad [\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}.$$

Then, we introduce two symmetric bilinear forms: $\forall \rho, \phi, \psi \in H^2(\mathcal{E}_h)$,

$$A(\rho;\phi,\psi) = (a(\rho)\nabla\phi, \nabla\psi) - \sum_{k=1}^{P_h}\int_{e_k}\{a(\rho)\nabla\phi\cdot\nu_k\}[\psi] - \sum_{k=1}^{P_h}\int_{e_k}\{a(\rho)\nabla\psi\cdot\nu_k\}[\phi],$$

$$B(\rho;\phi,\psi) = (b(\rho)\nabla\phi, \nabla\psi) - \sum_{k=1}^{P_h}\int_{e_k}\{b(\rho)\nabla\phi\cdot\nu_k\}[\psi] - \sum_{k=1}^{P_h}\int_{e_k}\{b(\rho)\nabla\psi\cdot\nu_k\}[\phi].$$

The interior penalty term is defined via the form

$$J_0^\sigma(\phi,\psi) = \sum_{k=1}^{P_h}\frac{\sigma_k}{|e_k|}\int_{e_k}[\phi][\psi], \quad \forall \phi, \psi \in \mathcal{D}_r(\mathcal{E}_h),$$

where: $|e_k|$ denotes the measure of e_k and σ is a discrete positive function that takes the constant value σ_k on the edge (or face) e_k and is bounded below by $\sigma_0 > 0$, above by σ^*.

Some approximation properties and inequalities for finite element space will be used later to derive error estimates. We state them as follows.

Lemma 2.1. *Let $E_j \in \mathcal{E}_h$ and $\phi \in H^s(E_j)$. Then there exist a positive constant C depending on s, γ, ρ but independent of ϕ, r, and h and a sequence $z_r^h \in P_r(E_j), r = 1, 2, \ldots$, such that for any $0 \le q \le s$*

$$\left\|\phi - z_r^h\right\|_{q,E_j} \le C\frac{h_j^{\mu-q}}{r^{s-q}}\|\phi\|_{s,E_j}, \quad s \ge 0, \tag{2.2}$$

$$\left\|\phi - z_r^h\right\|_{0,e_j} \le C\frac{h_j^{\mu-\frac{1}{2}}}{r^{s-\frac{1}{2}}}\|\phi\|_{s,E_j}, \quad s > \frac{1}{2}, \tag{2.3}$$

$$\left\|\phi - z_r^h\right\|_{1,e_j} \le C\frac{h_j^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}}\|\phi\|_{s,E_j}, \quad s > \frac{3}{2}, \tag{2.4}$$

where $\mu = \min(r+1, s)$ and e_j is an edge or a face of E_j.

This lemma shows the hp approximation properties, which were proved in [21, 22].

Lemma 2.2. *For each $E_j \in \mathcal{E}_h$, there exists a positive constant C depending only on γ, ρ such that the following two trace inequalities hold* ([7])

$$\|\phi\|_{0,e_j}^2 \le C\left(\frac{1}{h_j}|\phi|_{0,E_j}^2 + h_j|\phi|_{1,E_j}^2\right), \quad \forall \phi \in H^1(E_j), \tag{2.5}$$

$$\left\|\frac{\partial\phi}{\partial\nu_j}\right\|_{0,e_j}^2 \le C\left(\frac{1}{h_j}|\phi|_{1,E_j}^2 + h_j|\phi|_{2,E_j}^2\right), \quad \forall \phi \in H^2(E_j), \tag{2.6}$$

where e_j is an edge or a face of E_j, ν_j is the unit outward vector normal to e_j.

Lemma 2.3. *For each $E_j \in \mathcal{E}_h$ and $\phi \in P_r(E_j)$, the regular quasi-uniform subdivision $\mathcal{E}_h$ of Ω implies there exists a positive constant C depending only on r, γ, ρ such that two local inverse inequalities hold* ([2, 7])

$$\|\nabla\phi\|_{L^\infty(E_j)} \le Ch_j^{-\frac{d}{2}}\|\nabla\phi\|_{0,E_j}, \tag{2.7}$$

$$\left\|\frac{\partial\phi}{\partial\nu_j}\right\|_{0,e_j} \le Ch_j^{-\frac{1}{2}}\|\nabla\phi\|_{0,E_j}, \tag{2.8}$$

where e_j is an edge or a face of E_j, ν_j is the unit outward vector normal to e_j.

The following lemma follows directly from Lemma 2.3, which will be used often.

Lemma 2.4. *There exists a positive constant C_1 depending only on Ω, γ, ρ such that*

$$\sum_{k=1}^{P_h}|e_k|\left\|\left\{\frac{\partial\phi}{\partial\nu_k}\right\}\right\|_{0,e_k}^2 \le C_1|||\nabla\phi|||_0^2, \quad \forall\phi \in H^1(\mathcal{E}_h), \tag{2.9}$$

where e_k is an edge or a face of E_k.

Remark. Throughout the article, the letter C will denote a generic positive constant, independent of mesh steps $h, \Delta t$ and changing from appearance to appearance. Occasionally its dependence will be noted explicitly, e.g., $C(\|\nabla u\|_\infty, \|\nabla u_t\|_\infty)$. Other times the dependence will be indicated implicitly. Similar remarks apply to letters $\tilde{C}_i, K_i$ and $\varepsilon_i, i = 1, 2, \ldots$. ε_i will be used to denote generic small positive constant.

C. The Discontinuous Galerkin schemes

The **Semi-discrete time DG scheme** of Eq. (2.1) is to find $U(\cdot, t) \in \mathcal{D}_r(\mathcal{E}_h)$ satisfying

$$(U_t, v) + A(U; U, v) + B(U; U_t, v) + J_0^\sigma(U, v) + J_0^\sigma(U_t, v) = (f(U), v), \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h), \tag{2.10a}$$

$$U(\cdot, 0) = u_{0h}, \tag{2.10b}$$

where $u_{0h} \in \mathcal{D}_r(\mathcal{E}_h)$ is an initial approximation to $u_0(x)$. The existence and uniqueness of the solution for (2.10) will be proved in the following section.

Let $\Delta t = T/N$ and $t_j = j\Delta t, 0 \le j \le N$, where N is a positive integer. We use the following notations:

$$g_j = g(x, t_j), \text{ for } 0 \le j \le N; \quad \partial_t g_j = \frac{g_{j+1} - g_j}{\Delta t}, \quad \text{for } 0 \le j \le N-1,$$

$$g_{j,\theta} = \frac{1+\theta}{2}g_{j+1} + \frac{1-\theta}{2}g_j, \quad g_{t,j,\theta} = \left(\frac{\partial g}{\partial t}\right)_{j,\theta}, \quad \text{for } 0 \le j \le N-1,$$

where $\theta \in [0, 1]$. Define the norms:

$$|||g|||_{l^\infty(L^2)} = \max_{0\le j\le N}|||g_j|||_0, \quad |||g|||_{l^2(H^1)} = \left(\sum_{j=0}^{N-1}|||\nabla g_{j,\theta}|||_0^2\right)^{\frac{1}{2}}.$$

The **fully-discrete time DG scheme** to Eq. (2.1) is: Find $\{U_j\}_{j=0}^N \in \mathcal{D}_r(\mathcal{E}_h)$ satisfying

$$(\partial_t U_j, v) + A(U_{j,\theta}; U_{j,\theta}, v) + B(U_{j,\theta}; \partial_t U_j, v) + J_0^\sigma(U_{j,\theta}, v) + J_0^\sigma(\partial_t U_j, v) = (f(U_{j,\theta}), v), \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h), \quad (2.11a)$$

$$U(\cdot, 0) = u_{0h}, \quad (2.11b)$$

where $\theta \in [0, 1]$. If $\theta = 0$, (2.11a) yields the Crank-Nicolson Discontinuous Galerkin approximation; for $\theta = 1$,(2.11a) is a backward difference Discontinuous Galerkin approximation. By the regularity assumptions (A) and if Δt is sufficiently small, we know that (2.11) has a unique solution ([23]).

From (2.10) and (2.11), we know that the Semi-discrete time DG scheme and fully-discrete time DG scheme are symmetric. And we should pay more attentions to deal with the terms originated from the damping term $\nabla \cdot (b(u)\nabla u_t)$.

III. ERROR ESTIMATE FOR THE SEMI-DISCRETE TIME DG SCHEME

We will first prove the existence and uniqueness of the solution for Semi-discrete time DG scheme, and then derive the optimal $L^\infty(H^1)$ error estimate.

A. Existence and Uniqueness of the Solution

We begin with coercivity results for the bilinear forms A and B.

Lemma 3.1. *There exists a positive constant α such that if σ_0 is sufficiently large, then*

$$A(\rho; \phi, \phi) + \frac{1}{2} J_0^\sigma(\phi, \phi) \geq \alpha \left\{ |||\nabla \phi|||_0^2 + \sum_{k=1}^{P_h} |e_k| \left\| \left\{ \frac{\partial \phi}{\partial \nu_k} \right\} \right\|_{0,e_k}^2 \right\}, \quad (3.1a)$$

and

$$B(\rho; \phi, \phi) + \frac{1}{2} J_0^\sigma(\phi, \phi) \geq \alpha \left\{ |||\nabla \phi|||_0^2 + \sum_{k=1}^{P_h} |e_k| \left\| \left\{ \frac{\partial \phi}{\partial \nu_k} \right\} \right\|_{0,e_k}^2 \right\}, \quad (3.1b)$$

for $\phi \in \mathcal{D}_r(\mathcal{E}_h)$ and $\rho \in H^2(\mathcal{E}_h)$.

Proof. Let C_1 be the constant appearing in Lemma 2.4. Then for arbitrary small constant $\delta > 0$, we have

$$A(\rho; \phi, \phi) + \frac{1}{2} J_0^\sigma(\phi, \phi) \geq a_0 |||\nabla \phi|||_0^2 - 2a^* \sum_{k=1}^{P_h} \left\| \left\{ \frac{\partial \phi}{\partial \nu_k} \right\} \right\|_{0,e_k} \|[\phi]\|_{0,e_k} + \frac{1}{2} J_0^\sigma(\phi, \phi)$$

$$\geq a_0 |||\nabla \phi|||_0^2 - \delta \sum_{k=1}^{P_h} |e_k| \left\| \left\{ \frac{\partial \phi}{\partial \nu_k} \right\} \right\|_{0,e_k}^2 - \frac{(a^*)^2}{\delta} \sum_{k=1}^{P_h} |e_k|^{-1} \|[\phi]\|_{0,e_k}^2 + \frac{1}{2} J_0^\sigma(\phi, \phi)$$

$$\geq \frac{a_0}{2}|||\nabla\phi|||_0^2 + \frac{a_0 - 2C_1\delta}{2C_1}\sum_{k=1}^{P_h}|e_k|\left\|\left\{\frac{\partial\phi}{\partial\nu_k}\right\}\right\|_{0,e_k}^2 + \left(\frac{\sigma_0}{2} - \frac{(a^*)^2}{\delta}\right)\sum_{k=1}^{P_h}|e_k|^{-1}\|[\phi]\|_{0,e_k}^2.$$

For a sufficiently small choice of δ satisfying $\frac{a_0}{2C_1} - \delta > 0$, if assume that $\sigma_0 \geq 2(a^*)^2\delta^{-1}$, then we can obtain

$$A(\rho;\phi,\phi) + \frac{1}{2}J_0^\sigma(\phi,\phi) \geq \alpha\left\{|||\nabla\phi|||_0^2 + \sum_{k=1}^{P_h}|e_k|\left\|\left\{\frac{\partial\phi}{\partial\nu_k}\right\}\right\|_{0,e_k}^2\right\},$$

where $\alpha = min\{\frac{a_0}{2}, \frac{a_0}{2C_1} - \delta\} > 0$.

The proof of (3.1b) is similar to that of (3.1a). Hereafter it is assumed that σ_0 is sufficiently large in the sense of Lemma 3.1. This lemma will be used repeatedly in the following estimates. ∎

Note that if $\{\varphi_i(x)\}_{i=1}^m$ is a basis of $\mathcal{D}_r(\mathcal{E}_h)$ and if we write

$$U(x,t) = \sum_{i=1}^m \theta_i(t)\varphi_i(x),$$

then (2.10) reduces to an initial value problem for the system of nonlinear ordinary differential equations

$$\begin{cases} G(\theta)\dfrac{d\theta}{dt} = -D(\theta)\theta + F(\theta), \\ \theta(0) = \theta_0. \end{cases}$$

Here, $\theta = (\theta_1(t), \theta_2(t), \ldots, \theta_m(t))^T$, θ_0 is an initial value vector, $G(\theta) = (G_{ij}(\theta))$ and $D(\theta) = (D_{ij}(\theta))$ are $m \times m$ symmetric matrices respectively, $F(\theta) = (F_1(\theta), F_2(\theta), \ldots, F_m(\theta))^T$ is a vector. They are defined by:

$$\begin{cases} G_{ij} = (\varphi_i, \varphi_j) + B(U;\varphi_i,\varphi_j) + J_0^\sigma(\varphi_i,\varphi_j), \\ D_{ij} = A(U;\varphi_i,\varphi_j) + J_0^\sigma(\varphi_i,\varphi_j), \qquad\qquad \forall i,j = 1,2,\ldots,m. \\ F_j = (f(U),\varphi_j), \end{cases}$$

Take arbitrarily $Y = (y_1, y_2, \ldots, y_m)^T \in IR^m$ and $V = \sum_{i=1}^m y_i\varphi_i$. Then we get

$$Y^T G Y = \sum_{i=1}^m\sum_{j=1}^m G_{ij}y_iy_j = (V,V) + B(U;V,V) + J_0^\sigma(V,V)$$
$$\geq |||V|||_0^2 + \alpha|||\nabla V|||_0^2, \tag{3.2}$$

by using Lemma 3.1.

This suffices to show that the matrix G is positive definite. By the regularity assumptions on a, b, f and the theory of ordinary differential equations, it follows that $\theta(t)$ exists and is unique

for $t > 0$. Then, the existence and uniqueness of the solution for Semi-discrete time DG scheme (2.10) are proved.

B. Optimal $L^\infty(H^1)$ Error Estimate

Theorem 3.1. *Assume Lemma 3.1 hold. For the solution U of the Semi-discrete time DG scheme (2.10), there exists a positive constant C independent of h and r such that*

$$\begin{aligned} &|||U-u|||^2_{L^\infty((0,T);H^1)} + |||(U-u)_t|||^2_{L^2((0,T);H^1)} \\ &\le C\frac{h^{2\mu-2}}{r^{2s-4}}\left\{|||u|||^2_{L^2((0,T);H^s)} + |||u_t|||^2_{L^2((0,T);H^s)}\right\}, \end{aligned} \tag{3.3}$$

where $\mu = min(r+1, s)$, $r \ge 2, s \ge 3$.

Proof. For the solution u of Eq. (2.1), it satisfies $\forall v \in \mathcal{D}_r(\mathcal{E}_h)$

$$(u_t, v) + A(u; u, v) + B(u; u_t, v) + J_0^\sigma(u, v) + J_0^\sigma(u_t, v) = (f(u), v), \tag{3.4}$$

Let $\hat{u} \in \mathcal{D}_r(\mathcal{E}_h)$ be an interpolant of u having optimal hp-approximation errors (2.2)–(2.4). Subtracting (3.4) from (2.10), denoting $U - \hat{u} = \xi, \hat{u} - u = \chi$, we get:

$$\begin{aligned} &(\xi_t, v) + A(U; \xi, v) + B(U; \xi_t, v) + J_0^\sigma(\xi, v) + J_0^\sigma(\xi_t, v) \\ &\quad = (f(U) - f(u) - \chi_t, v) - A(U; \chi, v) - B(U; \chi_t, v) - J_0^\sigma(\chi, v) - J_0^\sigma(\chi_t, v) \\ &\qquad - \{A(U; u, v) - A(u; u, v)\} - \{B(U; u_t, v) - B(u; u_t, v)\} \\ &\qquad\qquad = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} T_1 &= (f(U) - f(u) - \chi_t, v), \\ T_2 &= -(a(U)\nabla\chi + b(U)\nabla\chi_t, \nabla v), \\ T_3 &= \sum_{k=1}^{P_h}\int_{e_k}\{a(U)\nabla\chi\cdot\nu_k\}[v] + \sum_{k=1}^{P_h}\int_{e_k}\{b(U)\nabla\chi_t\cdot\nu_k\}[v], \\ T_4 &= \sum_{k=1}^{P_h}\int_{e_k}\{a(U)\nabla v\cdot\nu_k\}[\chi] + \sum_{k=1}^{P_h}\int_{e_k}\{b(U)\nabla v\cdot\nu_k\}[\chi_t], \\ T_5 &= -J_0^\sigma(\chi, v) - J_0^\sigma(\chi_t, v), \\ T_6 &= -((a(U) - a(u))\nabla u, \nabla v) - ((b(U) - b(u))\nabla u_t, \nabla v), \\ T_7 &= \sum_{k=1}^{P_h}\int_{e_k}\{(a(U) - a(u))\nabla u\cdot\nu_k\}[v] + \sum_{k=1}^{P_h}\int_{e_k}\{(b(U) - b(u))\nabla u_t\cdot\nu_k\}[v]. \end{aligned}$$

To treat the damping term $\nabla\cdot(b(u)\nabla u_t)$ well, we choose $v=\xi+\xi_t$ as a test function in (3.5). The above equation is equal to

$$\begin{aligned}(\xi_t,\xi_t)+A(U;\xi,\xi)+B(U;\xi_t,\xi_t)+J_0^\sigma(\xi,\xi)+J_0^\sigma(\xi_t,\xi_t)&\\ +\frac{1}{2}\frac{d}{dt}\{(\xi,\xi)+2J_0^\sigma(\xi,\xi)+((a(U)+b(U))\nabla\xi,\nabla\xi)\}&\\ =\frac{1}{2}\left(\frac{d}{dt}(a(U)+b(U))\nabla\xi,\nabla\xi\right)+\left\{\sum_{k=1}^{P_h}\int_{e_k}\{(a(U)+b(U))\nabla\xi\cdot\nu_k\}[\xi_t]\right.&\\ \left.+\sum_{k=1}^{P_h}\int_{e_k}\{(a(U)+b(U))\nabla\xi_t\cdot\nu_k\}[\xi]\right\}+\sum_{i=1}^{7}T_i.&\quad(3.6)\end{aligned}$$

To simply, we still denote the last seven terms on the right-hand side of (3.6) by $\{T_i\}_{i=1}^7$.

Now, we turn to analyze the terms of (3.6) to attain our desired estimates.

By Lemma 3.1 and the assumptions (A), it is easy to see that

$$\begin{aligned}\{\text{the left-hand side of (3.6)}\}\geq |||\xi_t|||_0^2+\alpha\left\{|||\nabla\xi|||_0^2+|||\nabla\xi_t|||_0^2\right\}+\frac{1}{2}\left\{J_0^\sigma(\xi,\xi)+J_0^\sigma(\xi_t,\xi_t)\right\}&\\ +\frac{1}{2}\frac{d}{dt}\left\{|||\xi|||_0^2+2a_0|||\nabla\xi|||_0^2+2J_0^\sigma(\xi,\xi)\right\}.&\quad(3.7)\end{aligned}$$

Here, we have omitted some useless positive terms on the left-hand side of (3.6).

Before to analyze the right-hand side terms of (3.6), we conclude some inequalities for ξ and χ, which are directly from the inequalities (2.5)–(2.6) and the local inverse inequalities (2.7)–(2.8). They will be used frequently in the following analysis.

$$(B):\begin{cases}\left\|\left\{\frac{\partial\chi}{\partial\nu_k}\right\}\right\|_{0,e_k}\leq Ch^{-\frac{1}{2}}(\|\nabla\chi\|_{0,E_{12}}+h\|\nabla^2\chi\|_{0,E_{12}}),\\ \|[\chi]\|_{0,e_k}\leq Ch^{-\frac{1}{2}}(\|\chi\|_{0,E_{12}}+h\|\nabla\chi\|_{0,E_{12}}),\\ \left\|\left\{\frac{\partial\xi}{\partial\nu_k}\right\}\right\|_{0,e_k}\leq Ch^{-\frac{1}{2}}\|\nabla\xi\|_{0,E_{12}},\\ \|[\xi]\|_{0,e_k}\leq Ch^{-\frac{1}{2}}\|\xi\|_{0,E_{12}},\end{cases}$$

where, we assume that e_k be a given edge by $e_k=\partial E_1\cap\partial E_2$, and denote $E_{12}=E_1\cup E_2$.

Let us make an induction hypothesis that there exists $0<h_0\leq 1$ such that

$$|||\nabla\xi|||_0\leq Ch^{\frac{d}{2}+\varepsilon_0},\quad 0<\varepsilon_0<1,\tag{3.8a}$$

for $h<h_0$ and $\forall t\in[0,T]$. We will check this hypothesis later.

By the induction hypothesis (3.8a) and Lemma 2.3, we have

$$|||\nabla\xi|||_{L^\infty((0,T);L^\infty)}\leq Ch^{-\frac{d}{2}}|||\nabla\xi|||_{L^\infty((0,T);L^2)}\leq Ch^{\varepsilon_0}\leq M,\tag{3.8b}$$

where M is a positive constant.

It follows from (3.8b) that

$$\left|\frac{1}{2}\left(\frac{d}{dt}(a(U)+b(U))\nabla\xi,\nabla\xi\right)\right| \le C\{|||\xi_t+\chi_t|||_0+|||u_t|||_0\}|||\nabla\xi|||_{L^\infty((0,T);L^\infty)}|||\nabla\xi|||_0$$
$$\le \tilde{C}_1\left\{|||\xi_t|||_0^2+|||\chi_t|||_0^2+|||\nabla\xi|||_0^2\right\}. \tag{3.9a}$$

By the definition of $J_0^\sigma(\cdot,\cdot)$ and Lemma 2.4, we can find that

$$\left|\left\{\sum_{k=1}^{P_h}\int_{e_k}\{(a(U)+b(U))\nabla\xi\cdot\nu_k\}[\xi_t]+\sum_{k=1}^{P_h}\int_{e_k}\{(a(U)+b(U))\nabla\xi_t\cdot\nu_k\}[\xi]\right\}\right|$$
$$\le \sum_{k=1}^{P_h}\left\{\varepsilon_1\frac{\sigma_k}{|e_k|}\|[\xi_t]\|_{0,e_k}^2+C|e_k|\left\|\left\{\frac{\partial\xi}{\partial\nu_k}\right\}\right\|_{0,e_k}^2+C\frac{\sigma_k}{|e_k|}\|[\xi]\|_{0,e_k}^2+\varepsilon_1C_1^{-1}|e_k|\left\|\left\{\frac{\partial\xi_t}{\partial\nu_k}\right\}\right\|_{0,e_k}^2\right\}$$
$$\le \varepsilon_1J_0^\sigma(\xi_t,\xi_t)+\varepsilon_1|||\nabla\xi_t|||_0^2+\tilde{C}_2\left\{|||\nabla\xi|||_0^2+J_0^\sigma(\xi,\xi)\right\}. \tag{3.9b}$$

The bounds for T_1 and T_2 are

$$|T_1| \le \varepsilon_2|||\xi_t|||_0^2+\tilde{C}_3\left\{|||\chi|||_0^2+|||\chi_t|||_0^2+|||\xi|||_0^2\right\}, \tag{3.9c}$$

and

$$|T_2| \le \varepsilon_3|||\nabla\xi_t|||_0^2+\tilde{C}_4\left\{|||\nabla\chi|||_0^2+|||\nabla\chi_t|||_0^2+|||\nabla\xi|||_0^2\right\}, \tag{3.9d}$$

respectively.

Using inequalities (B) and Lemma 2.4, we can obtain

$$|T_3| \le \sum_{k=1}^{P_h}\left\{\varepsilon_4\frac{\sigma_k}{|e_k|}\left(\|[\xi]\|_{0,e_k}^2+\|[\xi_t]\|_{0,e_k}^2\right)+C|e_k|\left(\left\|\left\{\frac{\partial\chi}{\partial\nu_k}\right\}\right\|_{0,e_k}^2+\left\|\left\{\frac{\partial\chi_t}{\partial\nu_k}\right\}\right\|_{0,e_k}^2\right)\right\}$$
$$\le \varepsilon_4\left\{J_0^\sigma(\xi,\xi)+J_0^\sigma(\xi_t,\xi_t)\right\}+\tilde{C}_5\left\{|||\nabla\chi|||_0^2+|||\nabla\chi_t|||_0^2\right.$$
$$\left.+h^2|||\nabla^2\chi|||_0^2+h^2|||\nabla^2\chi_t|||_0^2\right\}. \tag{3.9e}$$

We bound T_4 in a similar manner

$$|T_4| \le C\sum_{k=1}^{P_h}\left\{\left\|\left\{\frac{\partial\xi}{\partial\nu_k}\right\}\right\|_{0,e_k}+\left\|\left\{\frac{\partial\xi_t}{\partial\nu_k}\right\}\right\|_{0,e_k}\right\}\{\|[\chi]\|_{0,e_k}+\|[\chi_t]\|_{0,e_k}\}$$
$$\le \varepsilon_5\left\{|||\nabla\xi|||_0^2+|||\nabla\xi_t|||_0^2\right\}+\tilde{C}_6\left\{h^{-2}|||\chi|||_0^2+h^{-2}|||\chi_t|||_0^2\right.$$
$$\left.+|||\nabla\chi|||_0^2+|||\nabla\chi_t|||_0^2\right\}. \tag{3.9f}$$

Again using inequalities (B) and the definition of $J_0^\sigma(\cdot,\cdot)$, term T_5 is bounded as follows

$$|T_5| \le \sum_{k=1}^{P_h}\frac{\sigma_k}{|e_k|}\{\|[\chi]\|_{0,e_k}+\|[\chi_t]\|_{0,e_k}\}\{\|[\xi]\|_{0,e_k}+\|[\xi_t]\|_{0,e_k}\}$$
$$\le \varepsilon_6\left\{J_0^\sigma(\xi,\xi)+J_0^\sigma(\xi_t,\xi_t)\right\}+\tilde{C}_7\left\{h^{-2}|||\chi|||_0^2+h^{-2}|||\chi_t|||_0^2\right.$$
$$\left.+|||\nabla\chi|||_0^2+|||\nabla\chi_t|||_0^2\right\}. \tag{3.9g}$$

For term T_6, it is easy to see

$$|T_6| \le C(\|\nabla u\|_\infty, \|\nabla u_t\|_\infty)|||u-U|||_0(|||\nabla\xi|||_0 + |||\nabla\xi_t|||_0)$$
$$\le \varepsilon_7\big\{|||\nabla\xi|||_0^2 + |||\nabla\xi_t|||_0^2\big\} + \tilde{C}_8\big\{|||\chi|||_0^2 + |||\xi|||_0^2\big\}. \tag{3.9h}$$

Noticing the assumption (A) and making similar argument as (3.9b), we achieve

$$|T_7| \le C(\|\nabla u\|_\infty, \|\nabla u_t\|_\infty)\sum_{k=1}^{P_h}\int_{e_k}\|\{u-U\}\cdot\nu_k\|_{0,e_k}(\|[\xi]\|_{0,e_k} + \|[\xi_t]\|_{0,e_k})$$
$$\le \varepsilon_8\big\{J_0^\sigma(\xi,\xi) + J_0^\sigma(\xi_t,\xi_t)\big\} + \tilde{C}_9\big\{|||\nabla\chi|||_0^2 + h^2|||\nabla^2\chi|||_0^2 + |||\xi|||_0^2\big\}. \tag{3.9i}$$

Combining the above inequalities from (3.7) to (3.9i), and choosing $\{\varepsilon_i\}_{i=1}^8$ small enough, we can get

$$|||\xi_t|||_0^2 + \alpha\big\{|||\nabla\xi|||_0^2 + |||\nabla\xi_t|||_0^2\big\} + \frac{1}{2}\big\{J_0^\sigma(\xi,\xi) + J_0^\sigma(\xi_t,\xi_t)\big\}$$
$$+ \frac{1}{2}\frac{d}{dt}\big\{|||\xi|||_0^2 + 2a_0|||\nabla\xi|||_0^2 + 2J_0^\sigma(\xi,\xi)\big\}$$
$$\le K_1|||\xi|||_0^2 + K_2|||\nabla\xi|||_0^2 + K_3J_0^\sigma(\xi,\xi) + \left(K_4 + \frac{K_5}{h^2}\right)\big\{|||\chi|||_0^2 + |||\chi_t|||_0^2\big\}$$
$$+ K_6\big\{|||\nabla\chi|||_0^2 + |||\nabla\chi_t|||_0^2\big\} + K_7h^2\big\{|||\nabla^2\chi|||_0^2 + |||\nabla^2\chi_t|||_0^2\big\}, \tag{3.10}$$

where $\{K_i\}_{i=1}^7$ are positive constants independent of h and r.

Integrate (3.10) with respect to time from 0 to $\tau \le T$ to obtain

$$\frac{1}{2}|||\xi|||_0^2(\tau) + a_0|||\nabla\xi|||_0^2(\tau) + J_0^\sigma(\xi,\xi)(\tau) + \int_0^\tau\big\{|||\xi_t|||_0^2 + \alpha|||\nabla\xi_t|||_0^2 + J_0^\sigma(\xi_t,\xi_t)\big\}dt$$
$$\le \frac{1}{2}|||\xi|||_0^2(0) + a_0|||\nabla\xi|||_0^2(0) + J_0^\sigma(\xi,\xi)(0) + \int_0^\tau\big\{K_1|||\xi|||_0^2 + K_2|||\nabla\xi|||_0^2 + K_3J_0^\sigma(\xi,\xi)$$
$$+ \left(K_4 + \frac{K_5}{h^2}\right)\big\{|||\chi|||_0^2 + |||\chi_t|||_0^2\big\} + K_6\big\{|||\nabla\chi|||_0^2 + |||\nabla\chi_t|||_0^2\big\}$$
$$+ K_7h^2\big\{|||\nabla^2\chi|||_0^2 + |||\nabla^2\chi_t|||_0^2\big\}\big\}dt. \tag{3.11}$$

Choosing appropriate initial approximation conditions, and using Gronwall's inequality and hp approximation properties (2.2)–(2.4), we finally obtain

$$|||\xi|||_0^2(\tau) + 2a_0|||\nabla\xi|||_0^2(\tau) + 2J_0^\sigma(\xi,\xi)(\tau) + 2\int_0^\tau\big\{|||\xi_t|||_0^2 + \alpha|||\nabla\xi_t|||_0^2 + J_0^\sigma(\xi_t,\xi_t)\big\}dt$$
$$\le C\frac{h^{2\mu-2}}{r^{2s-4}}\big\{|||u|||^2_{L^2((0,T);H^s)} + |||u_t|||^2_{L^2((0,T);H^s)}\big\}. \tag{3.12}$$

The result (3.3) follows by triangle inequality, Lemma 2.1 and (3.12).

A verification of the induction hypothesis (3.8a) can be given at this point. First, noting that $\xi = U - \hat{u}$ and $\hat{u}$ is the interpolant of u, (3.8a) holds obviously for $t = 0$ if the initial conditions $U(0) = u_{0h}$ and $\hat{u}(0)$ are chosen appropriately.

Next, we use a proof by contradiction to verify (3.8a). Because $\|\nabla\xi\|$ is continuous with respect to t variable (see **Remark** below for the reason), we suppose that there exists $0 \le t^* < T$ such that (3.8b) holds for $t \in [0, t^*)$, but $|||\nabla\xi|||_0(t^*) > Ch^{\frac{d}{2}+\varepsilon_0}$.

Take a sequence $\{t_n\} \subset [0, t^*)$ such that $t_n \longrightarrow t^*$ as $n \longrightarrow +\infty$. For this sequence $\{t_n\}$, (3.8b) holds so that the similar argument as (3.12) gives

$$|||\xi|||_0^2(t_n) + 2a_0|||\nabla\xi|||_0^2(t_n) + 2J_0^\sigma(\xi,\xi)(t_n) + 2\int_0^{t_n}\left\{|||\xi_t|||_0^2 + \alpha|||\nabla\xi_t|||_0^2 + J_0^\sigma(\xi_t,\xi_t)\right\}dt \le C\frac{h^{2\mu-2}}{r^{2s-4}}\left\{|||u|||^2_{L^2((0,t_n);H^s)} + |||u_t|||^2_{L^2((0,t_n);H^s)}\right\}. \quad (3.13)$$

Thus, it follows for $\mu = min(r+1, s)$, $r \ge 2$ and $s \ge 3$ that

$$|||\nabla\xi|||_0(t_n) \le C\frac{h^{\mu-1}}{r^{s-2}} \le Ch^{\frac{d}{2}+\varepsilon_0}. \quad (3.14)$$

By the continuity of $\|\nabla\xi\|$ with respect to t and (3.14), we know that

$$|||\nabla\xi|||_0(t^*) \le Ch^{\frac{d}{2}+\varepsilon_0}, \quad (3.15)$$

which contradicts the assumption. Hence, the induction hypothesis (3.8a) holds for $\forall t \in [0, T]$. ■

Remark. In the proof of Lemma 3.1, for the solution $U(x,t) = \sum_{i=1}^m \theta_i(t)\varphi_i(x)$ of the Semi-discrete time DG scheme (2.10), we have proven that $\theta(t) = (\theta_1(t), \theta_2(t), \ldots, \theta_m(t))^T$ satisfies the system of nonlinear ordinary differential equations about t and exists uniquely for $t > 0$. This suffices to show that $\theta(t)$ is continuous with respect to t so that $\nabla U(x,t) = \sum_{i=1}^m \theta_i(t)\nabla\varphi_i(x)$ is also.

On the other hand, because $\hat{u}$ is the interpolant of the exact solution u of (2.1) in the finite element space $\mathcal{D}_r(\mathcal{E}_h)$, we can define $\hat{u}(x,t) = \sum_{i=1}^m \lambda_i(t)\varphi_i(x)$, where $\lambda(t) = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t))^T$ has as many t derivatives as $u(x,t)$. By the regularity assumptions **(A)** on u, it is evident that $\nabla\hat{u}$ is continuous with respect to t.

From the reasons above, we know that $\nabla\xi = \nabla U - \nabla\hat{u}$ and $||\nabla\xi||$ are both continuous with respect to t.

IV. ERROR ESTIMATE FOR THE FULLY-DISCRETE TIME DG SCHEME

We now consider the error estimates for the fully-discrete time DG scheme (2.11). Assume that for case $\theta \in (0, 1]$, $u_{tt} \in L^\infty((0,T); H^3(\Omega))$ and for case $\theta = 0$,$u_{ttt} \in L^\infty((0,T); H^3(\Omega))$.

Let $\hat{u} \in \mathcal{D}_r(\mathcal{E}_h)$ be an interpolant of u having optimal hp-approximation errors (2.2)–(2.4). We have the following lemma.

Lemma 4.1. *For $\frac{\partial\hat{u}}{\partial t}$, which is the interpolant of $\frac{\partial u}{\partial t}$, there exists*

$$\partial_t\hat{u}_j = \frac{\hat{u}_{j+1} - \hat{u}_j}{\Delta t} = \hat{u}_t(x, t_{j,\theta}) + \Delta t\,\rho_{j,\theta}, \quad \forall x \in \Omega, \quad (4.1)$$

where

$$|||\rho_{j,\theta}|||_0 \le \bar{C}_1 |||u_{tt}|||_{L^\infty((t_j,t_{j+1});H^1)}.$$

In the particular case $\theta = 0$*, we have*

$$|||\rho_{j,0}|||_0 \le \bar{C}_2 \Delta t |||u_{ttt}|||_{L^\infty((t_j,t_{j+1});H^1)}.$$

$\bar{C}_1$ *and* $\bar{C}_2$ *are two positive constants independent of* $u, \hat{u}$ *and* r.

Proof. The proof is a straightforward application of the Taylor expansions. ∎

We are now ready to prove the following $l^\infty(H^1)$ and $l^2(H^1)$ error estimates.

Theorem 4.1. *Let* $s \ge 2$ *and Lemma 3.1 hold. For the solution* $\{U_j\}_{j=1}^N$ *of the fully-discrete time DG scheme (2.11), if* Δt *is sufficiently small, there exist constants* C^* *and* $\hat{C}$ *independent of* h *and* r *such that for* $\mu = min(r+1, s), r \ge 1$

$$|||U-u|||^2_{l^\infty(H^1)} + \alpha\Delta t |||U-u|||^2_{l^2(H^1)}$$
$$\le C^* \frac{h^{2\mu-2}}{r^{2s-4}} \Delta t \sum_{j=0}^{N} \left\{ |||u_j|||^2_{H^s} + |||(u_t)_j|||^2_{H^s} \right\} + \hat{C}(\Delta t)^2 \sum_{j=0}^{N-1} \Delta t |||u_{tt}|||^2_{L^\infty((t_j,t_{j+1});H^3)}. \quad (4.2a)$$

For $\theta = 0$*, we have*

$$|||U-u|||^2_{L^\infty(H^1)} + \alpha\Delta t |||U-u|||^2_{l^2(H^1)}$$
$$\le C^* \frac{h^{2\mu-2}}{r^{2s-4}} \Delta t \sum_{j=0}^{N} \left\{ |||u_j|||^2_{H^s} + |||(u_t)_j|||^2_{H^s} \right\} + \hat{C}(\Delta t)^4 \sum_{j=0}^{N-1} \Delta t |||u_{ttt}|||^2_{L^\infty((t_j,t_{j+1});H^3)}. \quad (4.2b)$$

Remark. The results of Theorem 4.1 indicate that spatial rates in H^1 and time truncation errors in L^2 are optimal.

Proof. By (3.4), we can get the following equation on time $t = t_{j,\theta} = \frac{t_{j+1}+t_j}{2} + \frac{\theta}{2}\Delta t$, for $0 \le j \le N-1$ and $\theta \in [0,1]$

$$(u_{t,j,\theta}, v) + A(u_{j,\theta}; u_{j,\theta}, v) + B(u_{j,\theta}; u_{t,j,\theta}, v) + J_0^\sigma(u_{j,\theta}, v) + J_0^\sigma(u_{t,j,\theta}, v) = (f(u_{j,\theta}), v).$$

Using Lemma 4.1 and the above equation, denoting $\chi_{j,\theta} = \hat{u}_{j,\theta} - u_{j,\theta}$, we can deduce

$$\begin{aligned}(\partial_t \hat{u}_j, v) &+ A(U_{j,\theta}; \hat{u}_{j,\theta}, v) + B(U_{j,\theta}; \partial_t \hat{u}_j, v) + J_0^\sigma(\hat{u}_{j,\theta}, v) + J_0^\sigma(\partial_t \hat{u}_j, v) \\ &= (f(u_{j,\theta}) + \Delta t \rho_{j,\theta} + \chi_{t,j,\theta}, v) + A(U_{j,\theta}; \chi_{j,\theta}, v) + B(U_{j,\theta}; \chi_{t,j,\theta} + \Delta t \rho_{j,\theta}, v) \\ &\quad + \{A(U_{j,\theta}; u_{j,\theta}, v) - A(u_{j,\theta}; u_{j,\theta}, v)\} + \{B(U_{j,\theta}; u_{t,j,\theta}, v) - B(u_{j,\theta}; u_{t,j,\theta}, v)\} \\ &\quad + J_0^\sigma(\chi_{j,\theta}, v) + J_0^\sigma(\chi_{t,j,\theta} + \Delta t \rho_{j,\theta}, v). \quad (4.3)\end{aligned}$$

Subtracting (4.3) from (2.11), denoting $\xi_{j,\theta} = U_{j,\theta} - \hat{u}_{j,\theta}$ to get:

$$\begin{aligned}(\partial_t \xi_j, v) + A(U_{j,\theta}; \xi_{j,\theta}, v) + B(U_{j,\theta}; \partial_t \xi_j, v) + J_0^\sigma(\xi_{j,\theta}, v) + J_0^\sigma(\partial_t \xi_j, v)\\ = (f(U_{j,\theta}) - f(u_{j,\theta}) - \Delta t \rho_{j,\theta} - \chi_{t,j,\theta}, v) - A(U_{j,\theta}; \chi_{j,\theta}, v)\\ - B(U_{j,\theta}; \chi_{t,j,\theta} + \Delta t \rho_{j,\theta}, v) - \{A(U_{j,\theta}; u_{j,\theta}, v) - A(u_{j,\theta}; u_{j,\theta}, v)\}\\ - \left\{B(U_{j,\theta}; u_{t,j,\theta}, v) - B(u_{j,\theta}; u_{t,j,\theta}, v)\right\} - J_0^\sigma(\chi_{j,\theta}, v) - J_0^\sigma(\chi_{t,j,\theta} + \Delta t \rho_{j,\theta}, v)\\ = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7, \quad (4.4)\end{aligned}$$

where:

$$\begin{aligned}D_1 &= (f(U_{j,\theta}) - f(u_{j,\theta}) - \Delta t \rho_{j,\theta} - \chi_{t,j,\theta}, v),\\ D_2 &= -(a(U_{j,\theta})\nabla \chi_{j,\theta}, \nabla v) - (b(U_{j,\theta})\nabla(\chi_{t,j,\theta} + \Delta t \rho_{j,\theta}), \nabla v),\\ D_3 &= \sum_{k=1}^{P_h} \int_{e_k} \{a(U_{j,\theta})\nabla \chi_{j,\theta} \cdot \nu_k\}[v] + \sum_{k=1}^{P_h} \int_{e_k} \{b(U_{j,\theta})\nabla(\chi_{t,j,\theta} + \Delta t \rho_{j,\theta}) \cdot \nu_k\}[v],\\ D_4 &= \sum_{k=1}^{P_h} \int_{e_k} \{a(U_{j,\theta})\nabla v \cdot \nu_k\}[\chi_{j,\theta}] + \sum_{k=1}^{P_h} \int_{e_k} \{b(U_{j,\theta})\nabla v \cdot \nu_k\}[\chi_{t,j,\theta} + \Delta t \rho_{j,\theta}],\\ D_5 &= -J_0^\sigma(\chi_{j,\theta}, v) - J_0^\sigma(\chi_{t,j,\theta} + \Delta t \rho_{j,\theta}, v),\\ D_6 &= -((a(U_{j,\theta}) - a(u_{j,\theta}))\nabla u_{j,\theta}, \nabla v) - ((b(U_{j,\theta}) - b(u_{j,\theta}))\nabla u_{t,j,\theta}, \nabla v),\\ D_7 &= \sum_{k=1}^{P_h} \int_{e_k} \{(a(U_{j,\theta}) - a(u_{j,\theta}))\nabla u_{j,\theta} \cdot \nu_k\}[v] + \sum_{k=1}^{P_h} \int_{e_k} \{(b(U_{j,\theta}) - b(u_{j,\theta}))\nabla u_{t,j,\theta} \cdot \nu_k\}[v].\end{aligned}$$

We begin to analyze (4.4) by choosing $v = \xi_{j,\theta} + \partial_t \xi_j$. The above equation is equal to

$$\begin{aligned}(\partial_t \xi_j, \xi_{j,\theta} + \partial_t \xi_j) + A(U_{j,\theta}; \xi_{j,\theta}, \xi_{j,\theta}) + B(U_{j,\theta}; \partial_t \xi_j, \partial_t \xi_j) + J_0^\sigma(\xi_{j,\theta}, \xi_{j,\theta})\\ + J_0^\sigma(\partial_t \xi_j, \partial_t \xi_j) + ((a(U_{j,\theta}) + b(U_{j,\theta}))\nabla \xi_{j,\theta}, \nabla \partial_t \xi_j) + 2J_0^\sigma(\xi_{j,\theta}, \partial_t \xi_j)\\ = R_1 + R_2 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7, \quad (4.5)\end{aligned}$$

where

$$\begin{aligned}R_1 &= \sum_{k=1}^{P_h} \int_{e_k} \{(a(U_{j,\theta}) + b(U_{j,\theta}))\nabla \xi_{j,\theta} \cdot \nu_k\}[\partial_t \xi_j],\\ R_2 &= \sum_{k=1}^{P_h} \int_{e_k} \{(a(U_{j,\theta}) + b(U_{j,\theta}))\nabla \partial_t \xi_j \cdot \nu_k\}[\xi_{j,\theta}].\end{aligned}$$

First, using Lemma 3.1, it is easy to see that

$$\begin{aligned}
&\{\text{the left-hand side of (4.5)}\}\\
\geq&\frac{1}{2\Delta t}\big(|||\xi_{j+1}|||_0^2-|||\xi_j|||_0^2\big)+\alpha\big\{|||\nabla\xi_{j,\theta}|||_0^2+|||\nabla\partial_t\xi_j|||_0^2\big\}\\
&+\frac{a_0}{\Delta t}\big(|||\nabla\xi_{j+1}|||_0^2-|||\nabla\xi_j|||_0^2\big)+\frac{1}{2}J_0^\sigma(\xi_{j,\theta},\xi_{j,\theta})\\
&+\frac{1}{2}J_0^\sigma(\partial_t\xi_j,\partial_t\xi_j)+\frac{1}{\Delta t}\big\{J_0^\sigma(\xi_{j+1},\xi_{j+1})-J_0^\sigma(\xi_j,\xi_j)\big\}. \qquad (4.6)
\end{aligned}$$

Here, we also have omitted some useless positive terms.

Now, we turn to analyze the right-hand side terms of (4.5). It is easy to know

$$\begin{aligned}
|R_1+R_2|\leq&\varepsilon_1 J_0^\sigma(\partial_t\xi_j,\partial_t\xi_j)+\tilde{C}_1\big\{|||\nabla\xi_j|||_0^2+|||\nabla\xi_{j+1}|||_0^2\big\}\\
&+\varepsilon_2|||\nabla\partial_t\xi_j|||_0^2+\tilde{C}_2\big\{J_0^\sigma(\xi_{j+1},\xi_{j+1})+J_0^\sigma(\xi_j,\xi_j)\big\}. \qquad (4.7a)
\end{aligned}$$

By using similar arguments as in Section III for terms $\{T_i\}_{i=1}^7$, we have analysis for terms $\{D_i\}_{i=1}^7$.

$$|D_1|\leq\varepsilon_3|||\partial_t\xi_j|||_0^2+\tilde{C}_3\big\{|||\rho_{j,\theta}|||_0^2(\Delta t)^2+|||\chi_{j,\theta}|||_0^2+|||\chi_{t,j,\theta}|||_0^2+|||\xi_{j,\theta}|||_0^2\big\}. \qquad (4.7b)$$

$$|D_2|\leq\varepsilon_4\big\{|||\nabla\xi_{j,\theta}|||_0^2+|||\nabla\partial_t\xi_j|||_0^2\big\}+\tilde{C}_4\big\{|||\nabla\rho_{j,\theta}|||_0^2(\Delta t)^2+|||\nabla\chi_{j,\theta}|||_0^2+|||\nabla\chi_{t,j,\theta}|||_0^2\big\}. \qquad (4.7c)$$

$$\begin{aligned}
|D_3|\leq&\varepsilon_5\big\{J_0^\sigma(\xi_{j,\theta},\xi_{j,\theta})+J_0^\sigma(\partial_t\xi_j,\partial_t\xi_j)\big\}+\tilde{C}_5\big\{|||\xi_{j,\theta}|||_0^2+|||\nabla\chi_{j,\theta}|||_0^2+h^2|||\nabla^2\chi_{j,\theta}|||_0^2\\
&+|||\nabla\chi_{t,j,\theta}|||_0^2+h^2|||\nabla^2\chi_{t,j,\theta}|||_0^2+(\Delta t)^2\big(|||\nabla\rho_{j,\theta}|||_0^2+h^2|||\nabla^2\rho_{j,\theta}|||_0^2\big)\big\}. \qquad (4.7d)
\end{aligned}$$

$$\begin{aligned}
|D_4|\leq&\varepsilon_6\big\{|||\nabla\xi_{j,\theta}|||_0^2+|||\nabla\partial_t\xi_j|||_0^2\big\}+\tilde{C}_6\big\{h^{-2}|||\chi_{j,\theta}|||_0^2+|||\nabla\chi_{j,\theta}|||_0^2\\
&+h^{-2}|||\chi_{t,j,\theta}|||_0^2+|||\nabla\chi_{t,j,\theta}|||_0^2+(\Delta t)^2\big(h^{-2}|||\rho_{j,\theta}|||_0^2+|||\nabla\rho_{j,\theta}|||_0^2\big)\big\}. \qquad (4.7e)
\end{aligned}$$

$$\begin{aligned}
|D_5|\leq&\varepsilon_7\big\{J_0^\sigma(\xi_{j,\theta},\xi_{j,\theta})+J_0^\sigma(\partial_t\xi_j,\partial_t\xi_j)\big\}+\tilde{C}_7\big\{h^{-2}|||\chi_{j,\theta}|||_0^2+|||\nabla\chi_{j,\theta}|||_0^2\\
&+h^{-2}|||\chi_{t,j,\theta}|||_0^2+|||\nabla\chi_{t,j,\theta}|||_0^2+(\Delta t)^2\big(h^{-2}|||\rho_{j,\theta}|||_0^2+|||\nabla\rho_{j,\theta}|||_0^2\big)\big\}. \qquad (4.7f)
\end{aligned}$$

$$|D_6|\leq\varepsilon_7\big\{|||\nabla\xi_{j,\theta}|||_0^2+|||\nabla\partial_t\xi_j|||_0^2\big\}+\tilde{C}_8\big\{|||\xi_{j,\theta}|||_0^2+|||\chi_{j,\theta}|||_0^2\big\}. \qquad (4.7g)$$

$$|D_7|\leq\varepsilon_8\big\{J_0^\sigma(\xi_{j,\theta},\xi_{j,\theta})+J_0^\sigma(\partial_t\xi_j,\partial_t\xi_j)\big\}+\tilde{C}_9\big\{|||\xi_{j,\theta}|||_0^2+|||\nabla\chi_{j,\theta}|||_0^2+h^2|||\nabla^2\chi_{j,\theta}|||_0^2\big\}. \qquad (4.7h)$$

Taking the above inequalities from (4.6) to (4.7h) into (4.5), choosing $\{\varepsilon_i\}_{i=1}^8$ small enough, multiplying both sides of (4.5) by $2\Delta t$ and then summing j from 0 to $N-1$, we get

$$\begin{aligned}
&|||\xi_N|||_0^2+2a_0|||\nabla\xi_N|||_0^2+2J_0^\sigma(\xi_N,\xi_N)+2\alpha\Delta t\sum_{j=0}^{N-1}\big\{|||\nabla\xi_{j,\theta}|||_0^2+|||\nabla\partial_t\xi_j|||_0^2\big\}\\
&\leq|||\xi_0|||_0^2+2a_0|||\nabla\xi_0|||_0^2+2J_0^\sigma(\xi_0,\xi_0)+C\Delta t\sum_{j=1}^{N}\big\{K_1|||\xi_j|||_0^2+K_2|||\nabla\xi_j|||_0^2+K_3J_0^\sigma(\xi_j,\xi_j)\big\}
\end{aligned}$$

$$+ C\Delta t \sum_{j=0}^{N-1} \left\{ \left(K_4 + \frac{K_5}{h^2} \right) \left\{ |||\chi_j|||_0^2 + |||\chi_{t,j}|||_0^2 \right\} + K_6 \left\{ |||\nabla\chi_j|||_0^2 + |||\nabla\chi_{t,j}|||_0^2 \right\} \right.$$

$$+ K_7 h^2 \left\{ |||\nabla^2\chi_j|||_0^2 + |||\nabla^2\chi_{t,j}|||_0^2 \right\} + \left(K_8 + \frac{K_9}{h^2} \right) (\Delta t)^2 \left\{ |||\rho_j|||_0^2 + |||\nabla\rho_j|||_0^2 \right\}$$

$$\left. + K_{10} h^2 (\Delta t)^2 |||\nabla^2\rho_j|||_0^2 \right\}. \quad (4.8)$$

If Δt is sufficiently small, we obtain by Gronwall's lemma

$$|||\xi_N|||_0^2 + 2a_0|||\nabla\xi_N|||_0^2 + 2J_0^\sigma(\xi_N, \xi_N) + 2\alpha\Delta t \sum_{j=0}^{N-1} \left\{ |||\nabla\xi_{j,\theta}|||_0^2 + |||\nabla\partial_t\xi_j|||_0^2 \right\}$$

$$\le |||\xi_0|||_0^2 + 2a_0|||\nabla\xi_0|||_0^2 + 2J_0^\sigma(\xi_0, \xi_0) + C\Delta t \sum_{j=0}^{N-1} \left\{ \left(K_4 + \frac{K_5}{h^2} \right) \left\{ |||\chi_j|||_0^2 + |||\chi_{t,j}|||_0^2 \right\} \right.$$

$$+ K_6 \left\{ |||\nabla\chi_j|||_0^2 + |||\nabla\chi_{t,j}|||_0^2 \right\} + K_7 h^2 \left\{ |||\nabla^2\chi_j|||_0^2 + |||\nabla^2\chi_{t,j}|||_0^2 \right\}$$

$$\left. + \left(K_8 + \frac{K_9}{h^2} \right) (\Delta t)^2 \left\{ |||\rho_j|||_0^2 + |||\nabla\rho_j|||_0^2 \right\} + K_{10} h^2 (\Delta t)^2 |||\nabla^2\rho_j|||_0^2 \right\}. \quad (4.9)$$

Choosing appropriate initial approximation conditions, and using *hp* approximation properties (2.2)–(2.4), we finally derive for $\theta \in (0, 1]$:

$$|||\xi_N|||_0^2 + 2a_0|||\nabla\xi_N|||_0^2 + 2\alpha\Delta t \sum_{j=0}^{N-1} \left\{ |||\nabla\xi_{j,\theta}|||_0^2 + |||\nabla\partial_t\xi_j|||_0^2 \right\}$$

$$\le C^* \frac{h^{2\mu-2}}{r^{2s-4}} \sum_{j=0}^{N} \Delta t \left\{ |||u_j|||_{H^s}^2 + |||(u_t)_j|||_{H^s}^2 \right\} + \hat{C}(\Delta t)^2 \sum_{j=0}^{N-1} \Delta t |||u_{tt}|||_{L^\infty((t_j, t_{j+1}); H^3)}^2. \quad (4.10a)$$

For $\theta = 0$, we have

$$|||\xi_N|||_0^2 + 2a_0|||\nabla\xi_N|||_0^2 + 2\alpha\Delta t \sum_{j=0}^{N-1} \left\{ |||\nabla\xi_{j,\theta}|||_0^2 + |||\nabla\partial_t\xi_j|||_0^2 \right\}$$

$$\le C^* \frac{h^{2\mu-2}}{r^{2s-4}} \sum_{j=0}^{N} \Delta t \left\{ |||u_j|||_{H^s}^2 + |||(u_t)_j|||_{H^s}^2 \right\} + \hat{C}(\Delta t)^4 \sum_{j=0}^{N-1} \Delta t |||u_{ttt}|||_{L^\infty((t_j, t_{j+1}); H^3)}^2. \quad (4.10b)$$

Using triangle inequality, Lemma 2.1 and (4.10), we can easily get the result (4.2). ∎

V. CONCLUSIONS AND PERSPECTIVES

We have presented a Semi-discrete time DG and a family of fully-discrete time DG schemes with interior penalties for nonlinear Sobolev equations. These DG schemes are symmetric. Hp-version

error estimates are analyzed by choosing special test functions to deal with the damping term $\nabla \cdot (b(u)\nabla u_t)$. Prior $L^\infty(H^1)$ error estimate for the Semi-discrete time DG scheme and similarly, $l^\infty(H^1)$ and $l^2(H^1)$ for the fully-discrete time DG scheme are derived. Spatial rates in H^1 and time truncation errors in L^2 are optimal. As far as we know, these are the first optimal estimates established for nonlinear Sobolev equation by using the Discontinuous Galerkin method with interior penalties.

The essential of this article is how to treat the damping term $\nabla \cdot (b(u)\nabla u_t)$, which we have done well by using special test functions. Then, we can easily extend our method to nonlinear Sobolev equations with Dirichlet or nonhomogenous Neumman boundary condition and derive similar error estimates. And also a convection term $d(x,u) \cdot \nabla u$ can be included in, which is a common case in many practical applications [12, 13]. We will present the results for these cases in a forthcoming paper.

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