Adaptive optimal control approximation for solving a fourth-order elliptic variational inequality

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\textbf{A B S T R A C T}

An optimal control approach is proposed to solve the fourth-order elliptic variational inequality with curvature obstacle. It is proved that the variational inequality is equivalent to the constrained optimal control problem. The finite element approximation of the optimal control problem is constructed and the a priori error estimates and the equivalent a posteriori error estimators are derived. Some numerical experiments are performed to confirm a priori error estimates and demonstrate the effectiveness of the a posteriori estimators.

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\section{1. Introduction}

Fourth-order elliptic variational inequalities with obstacles have been widely studied, such as variational inequalities with curvature obstacle, displacement obstacle, and so on. There have been many researches for solving these variational inequalities. For example, Noor and Al-Said used the finite difference method in [1,2], and they developed a new cubic spline method for computing the approximate solution of a system of fourth-order boundary value problems associated with an obstacle in [3]. Glowinski, Lions and Tremolieres used penalty and relaxation methods in [4,5], Momania, Moadia and Noor used the decomposition method in [6], Shi and Wang used non-conforming finite element methods in [7–11] to solve these fourth order variational inequalities with obstacles. Also, Brézis and Stampacchia studied the regularity of a fourth order variational inequality with curvature obstacle with two type of boundary conditions in [12]. Up to now, it is still a difficult and interesting problem to solve a fourth-order variational inequality effectively.

The purpose of this article is to develop a new approach to solve a fourth-order elliptic variational inequality with curvature obstacle. The idea is to use the adaptive optimal control approach to obtain the solution indirectly. To this end, we translate the fourth-order variational inequality with curvature obstacle into the equivalent form, which contains two Poisson equations and a 2nd order variational inequality. The optimality condition we obtained in this paper is equivalent to the mixed variational form deduced by Deng and Shen in [13]. In [13], they have the priori error estimates for $\|y - y_h\|_{H^1(\Omega)}$ and $\|u - u_h\|_{L^2(\Omega)}$ with the order $h^{1-\varepsilon}$, $\forall \varepsilon > 0$, but in our paper, we obtained the $h^1$ order convergence for the above two error estimates.

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The paper is organized as follows: The notations used throughout the paper are introduced in Section 2, together with the equivalence proof between the fourth-order variational inequality and the constrained optimal control problem, and also the optimality conditions are given. In Section 3, we present the finite element approximation of the control problem. In Section 4, we give the priori error estimate of \( y \). Equivalent a posteriori error estimates with \( H^1 \) norm and \( L^2 \) norm are deduced in Section 5. In Section 6, we give two numerical experiments to demonstrate our error estimates developed in Sections 4 and 5.

2. Fourth-order variational inequality and its equivalent form

Let \( \Omega \) be a convex domain in \( \mathbb{R}^2 \) with the Lipschitz boundary \( \partial \Omega \). In this paper we adopt the standard notation \( W^{m,q}(\Omega) \) for the Sobolev spaces on \( \Omega \) with norm \( \| \cdot \|_{W^{m,q}(\Omega)} \) and seminorm \( | \cdot |_{W^{m,q}(\Omega)} \). We set \( W_0^{m,q}(\Omega) \equiv \{ w \in W^{m,q}(\Omega) : w|_{\partial \Omega} = 0 \} \) and denote \( W^{m,2}(\Omega) \) \( (W_0^{m,2}(\Omega)) \) by \( H^m(\Omega) \) \( (H_0^m(\Omega)) \). In addition, \( c \) or \( C \) denotes a general positive constant independent of \( h \).

2.1. Fourth-order variational inequality

Let \( K = \{ y \in H^2(\Omega) \cap H^1_0(\Omega) : \Delta y \leq 0 \text{ in } \Omega \} \). Here, the condition satisfied in \( K \) is called the curvature obstacle. We consider the following fourth-order variational inequality problem: Find \( y \) in \( K \) such that

\[
\int_{\Omega} \Delta y \Delta (w - y) \geq \int_{\Omega} f(w - y), \quad \forall w \in K
\]

where \( f \in L^2(\Omega) \) is a given function.

In engineering, the variational inequality problem (2.1) describes the curvature obstacle problem. In general it is very difficult to solve the fourth-order variational inequality problem. In the next subsection, we translate the variational inequality problem (2.1) into an equivalent optimal control problem, which is governed by the second order PDE, so that it can be solved easily by using known-well adaptive finite element methods.

2.2. Equivalent optimal control problem

Define a convex set \( K' \) of the form:

\[
K' = \{ u \in L^2(\Omega) : u \geq 0 \text{ (a.e.) in } \Omega \}.
\]

It is clear that \( K' \) is closed in \( L^2(\Omega) \). We formulate the optimal control problem equivalent to the fourth-order variational inequality problem (2.1) in the following equivalent theorem.

Theorem 2.1. The problem (2.1) is equivalent to the following optimal control problem:

\[
\begin{align*}
\min_{u \in K'} \left\{ J(y, u) = \frac{1}{2} \int_{\Omega} u^2 - \int_{\Omega} fy \right\}, \\
\text{s.t. } -\Delta y = u \quad \text{in } \Omega, \quad y = 0 \text{ on } \partial \Omega.
\end{align*}
\]

Proof. It is clear that the minimized problem (2.2) may be represented by

\[
\min_{y \in K} \left\{ J(y) = \frac{1}{2} \int_{\Omega} (\Delta y)^2 - \int_{\Omega} fy \right\}.
\]

Obviously, \( J \) is convex. Suppose \( y \) is the solution of the minimized problem (2.3). From [5], we know the following conclusion:

\[
J(y) = \min_{y \in K} J(y) \iff J'(y)(w - y) \geq 0, \quad \forall w \in K.
\]

Since

\[
J'(y)(w - y) = \int_{\Omega} \Delta y \Delta (w - y) - \int_{\Omega} f(w - y) \geq 0, \quad \forall w \in K,
\]

we deduce that \( J'(y)(w - y) \geq 0 \) for each \( w \in K \) is equivalent to (2.1). Then Theorem 2.1 is proved. \( \square \)

Theorem 2.1 shows that one could solve the optimal control problem (2.2) instead of solving the variational inequality problem (2.1). In the following parts we study how to solve the optimal control problem (2.2). To this end, we give a weak formula for the state equation. Let the state space \( V = H^1_0(\Omega) \) and the control space \( U = L^2(\Omega) \). Set

\[
a(y, w) = \int_{\Omega} \nabla y \cdot \nabla w \quad \forall y, w \in V; \quad (f, g) = \int_{\Omega} fg \quad \forall f, g \in U.
\]
Therefore, the control problem (2.2) can be restated as the following:

\[
\text{(OCP)} : \begin{cases}
\min_{u \in K} \left\{ f(y, u) = \frac{1}{2} \int_{\Omega} u^2 - \int_{\Omega} fy \right\}, \\
a(y, w) = (u, w), \quad \forall w \in H_0^1(\Omega).
\end{cases}
\]  

(2.4)

It is clear that the control problem (OCP) has a unique solution \((y, u) \in V \times U\).

**Theorem 2.2** (Optimality Condition). A pair \((y, u)\) is the solution of (OCP) if and only if there is a co-state \(p \in V\) such that the triplet \((y, p, u) \in V \times V \times U\) satisfies the following optimality conditions:

\[
\text{(OCP–OPT)} : \begin{cases}
\theta((y, u), \tau) = 0, \\
a(y, w) = (u, w), \\
a(q, p) = (f, q), \\
\forall q, \tau \in V.
\end{cases}
\]  

(2.5)

From (OCP–OPT) and [14] we know that

\[
u = \max \{0, p\}.
\]  

(2.6)

The optimality condition (OCP–OPT) provides another way to approach the fourth-order variational inequality (2.1). The (OCP–OPT) has at least three advantages: one is that it is the second-order problem so as to be easily solved by using usual conforming finite elements; another is that the variational inequality in (OCP–OPT) is easily solved directly by (2.6). But it is very difficult to bring a curvature obstacle to finite element approximation. The third advantage, also most important in computation, is the system is separated, i.e.,

\[
\text{(OCP–OPT)′} : \begin{cases}
(a) \ a(q, p) = (f, q), \quad \forall q \in V, \\
(b) \ u = \max \{0, p\}, \\
(c) \ a(y, w) = (u, w), \quad \forall w \in V,
\end{cases}
\]  

(2.7)

in which \(p\) is independent of \(y\) so that it can be solved independently from (2.7)(a), then \(u\) is obtained from (2.7)(b) and finally \(y\) is solved by (2.7)(c).

By use of the regularity theory of the partial differential equations of the second order elliptic type, we can directly obtain the following the regularity theorem.

**Theorem 2.3** (Regularity [15]). Assume that \(\Omega\) is a convex domain or with smooth boundary. Then \(u \in W^{1,\infty}(\Omega), y \in H^2(\Omega) \cap H_0^1(\Omega)\) and \(p \in H^2(\Omega) \cap H_0^1(\Omega)\). There holds a priori estimate

\[
\begin{align*}
(a) \ |p|_{L^2(\Omega)} & \leq C \|f\|_{L^2(\Omega)}, \\
(b) \ |u|_{W^{1,1}(\Omega)} & \leq |p|_{W^{1,1}(\Omega)}, \\
(c) \ |y|_{W^{2,1}(\Omega)} & \leq C \|u\|_{L^2(\Omega)},
\end{align*}
\]

(2.8)

3. Finite element approximation

Let us consider the finite element approximation of the control problem (OCP). Here we consider only n-simplex elements, as are widely used in engineering applications. Also we consider only conforming finite elements. For simplicity, we assume that \(\Omega\) is a polygonal domain.

Let \(T^h = \bigcup \tau\) be a partitioning of \(\Omega\) into disjoint regular n-simplices \(\tau\), in which each element has at most one face on \(\partial \Omega\), and \(\tilde{\tau}\) and \(\tilde{\tau}'\) have either only one common vertex or a whole edge or face if \(\tau\) and \(\tau'\) \in \(T^h\). Associated with \(T^h\) is a finite dimensional subspace \(S^h(\Omega)\) such that \(\chi_{\tau}\), are linear functions for each \(\chi \in S^h\) and each \(\tau \in T^h\). Let \(V^h = S^h \cap H_0^1(\Omega)\). It is easy to see that \(V^h \subset V\).

Let \(T^h_{\tau} = \bigcup \tau_{\tau}\) be another partition of \(\Omega\) such that \(\tilde{\Omega} = \bigcup_{\tau_{\tau} \in T^h_{\tau}} \tilde{\tau}_{\tau}\). Assume that \(\tau_{\tau}\) and \(\tilde{\tau}_{\tau}\) have at most either one common vertex or a whole edge or face if \(\tau_{\tau}\) and \(\tau_{\tau}'\) \in \(T^h_{\tau}\). Associated with \(T^h_{\tau}\) is another finite dimensional subspace \(W^h_{\tau}\) of \(L^2(\Omega)\), such that \(\chi_{\tau_{\tau}}\) are constants for each \(\chi \in W^h_{\tau}\) and each \(\tau_{\tau} \in T^h_{\tau}\). Let \(U^h = W^h_{\tau}\) and \(K^h = (U^h \cap K')\).

Let \(h_{\tau_{\tau}}\) denote the maximum diameter of the element \((\tau_{\tau})\) in \(T^h(T^h_{\tau})\).

Then a possible finite element approximation of (OCP), which we shall label (OCP)\(^h\), is as follows:

\[
\text{(OCP)\(^h\)} : \begin{cases}
\min_{u_h \in K^h} \frac{1}{2} \int_{\Omega} u_h^2 - \int_{\Omega} fy_h \\
a(y_h, w_h) = (u_h, w_h), \quad \forall w_h \in V^h.
\end{cases}
\]  

(3.1)

The discretized control problem (OCP)\(^h\) has a unique solution \((y_h, u_h) \in V^h \times U^h\). Similarly as given in [16] or [14], we have the following **Theorem 3.1**.


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Theorem 4.1. A pair \((y_h, u_h) \in V^h \times U^h\) is the solution of \((OCP)^h\) if and only if there is a co-state \(p_h \in V^h\) such that the triplet \((y_h, p_h, u_h)\) satisfies the following optimality conditions labeled as \((OCP^\text{OPT})^h\):

\[
(OCP^\text{OPT})^h : \begin{cases} 
  a(y_h, u_h) = (u_h, w_h), \quad \forall w_h \in V_h, \\
  a(q_h, p_h) = (f, q_h), \quad \forall q_h \in V_h, \quad \text{(3.2)} \\
  (u_h - p_h, v_h - u_h) \geq 0, \quad \forall v_h \in K^h.
\end{cases}
\]

Since \(U^h\) is the space of piecewise constant elements, hence

\[
u_h = \max\{0, \mathcal{P}_h p_h\}, \quad \text{(3.3)}
\]

where \(\mathcal{P}_h\) is the \(L^2\) project from \(L^2(\Omega)\) onto \(U^h\) such that

\[
\mathcal{P}_h v|_{\tau_u} = \bar{v}|_{\tau} = \frac{1}{|\tau_u|} \int_{\tau_u} v, \quad \forall \tau_u \in \mathcal{T}_h, \quad v \in L^2(\Omega).
\]

Similarly, (3.2) may be rewritten as

\[
(OCP^\text{OPT'})^h : \begin{cases} 
  a(q_h, p_h) = (f, q_h), \quad \forall q_h \in V^h, \\
  u_h = \max\{0, \mathcal{P}_h p_h\}, \quad \text{(3.4)} \\
  a(y_h, w_h) = (u_h, w_h), \quad \forall w_h \in V^h.
\end{cases}
\]

which are two the standard finite element systems.

In the next sections, we will discuss the a priori error estimates and a posteriori estimator of the finite element approximation \((OCT^\text{OPT})^h\).

4. A priori error estimate

We analyze a priori error estimates of the finite element approximation \((OCT^\text{OPT})^h\). Let \((y, p, u)\) and \((y_h, p_h, u_h)\) be the solutions of \((OCP^\text{OPT})\) and \((OCP^\text{OPT'})^h\), respectively. We have the following error estimates.

Theorem 4.1. There holds the a priori error estimate

\[
\|p - p_h\|_{L^2(\Omega)} + h_r \|\nabla (p - p_h)\|_{L^2(\Omega)} \leq C h_r^2 \|p\|_{H^1(\Omega)} \leq C h_r^2 \|f\|_{L^2(\Omega)}, \quad \text{(4.1)}
\]

Proof. Noting that

\[
a(p - p_h, q_h) = 0, \quad \forall q_h \in V^h
\]

and by using the standard finite element methods, we have the error estimate (4.1).

Theorem 4.2. There holds the a priori error estimate

\[
\|u - u_h\|_{L^2(\Omega)} \leq C \{h_{tu} + h_r^2\}, \quad \text{(4.2)}
\]

Proof. Since

\[
|u - u_h| = \max\{0, p\} - \max\{0, \mathcal{P}_h p_h\} \leq |p - \mathcal{P}_h p_h|,
\]

then it follows from the boundedness of \(L^2\)-projection (see in [17]) that

\[
\|u - u_h\|_{L^2(\Omega)} \leq \|p - \mathcal{P}_h p\|_{L^2(\Omega)} + \|\mathcal{P}_h (p - p_h)\|_{L^2(\Omega)} \leq C \{h_{tu} \|p\|_{H^1(\Omega)} + h_r^2 \|p\|_{H^2(\Omega)}\}.
\]

This ends the proof of Theorem 4.2.

Lemma 4.1. There holds the a priori error estimate

\[
\|y - y_h\|_{H^1(\Omega)} \leq C \{h_{tu} + h_r\} \quad \text{(4.3)}
\]

\[
\|y - y_h\|_{L^2(\Omega)} \leq C \{h_{tu} + h_r^2\}. \quad \text{(4.4)}
\]

Proof. Let \(\bar{u} = \max\{0, \mathcal{P}_h p\}\). Introduce the auxiliary problem: \(y_h(\bar{u}) \in V^h\) such that

\[
a(y_h(\bar{u}), w_h) = (\bar{u}, w_h), \quad \forall w_h \in V^h.
\]

Noting that

\[
a(y_h(\bar{u}) - y_h, w_h) = (\bar{u} - u_h, w_h), \quad \forall w_h \in V^h,
\]

\[
\|y - y_h\|_{H^1(\Omega)} \leq C \{h_{tu} + h_r\} \quad \text{(4.5)}
\]

\[
\|y - y_h\|_{L^2(\Omega)} \leq C \{h_{tu} + h_r^2\}. \quad \text{(4.6)}
\]

Proof. Let \(\bar{u} = \max\{0, \mathcal{P}_h p\}\). Introduce the auxiliary problem: \(y_h(\bar{u}) \in V^h\) such that

\[
a(y_h(\bar{u}), w_h) = (\bar{u}, w_h), \quad \forall w_h \in V^h.
\]

Noting that

\[
a(y_h(\bar{u}) - y_h, w_h) = (\bar{u} - u_h, w_h), \quad \forall w_h \in V^h,
\]

\[
\|y - y_h\|_{H^1(\Omega)} \leq C \{h_{tu} + h_r\} \quad \text{(4.7)}
\]

\[
\|y - y_h\|_{L^2(\Omega)} \leq C \{h_{tu} + h_r^2\}. \quad \text{(4.8)}
\]
we have
\[ \| \nabla (y_h(u_h) - y_h) \|_{L^2(\Omega)} \leq C \| u_h - u \|_{L^2(\Omega)}. \]

It is clear that
\[ \| u_h - u \|_{L^2(\Omega)} = \| \max\{0, \mathcal{P}p\} - \max\{0, \mathcal{P}p_h\} \|_{L^2(\Omega)} \]
\[ \leq \| \mathcal{P}p - p_h \|_{L^2(\Omega)} \leq \| p - p_h \|_{L^2(\Omega)} \leq C h^2 \| p \|_{H^2(\Omega)}. \]

Thus we get
\[ \| \nabla (y_h(u_h) - y_h) \|_{L^2(\Omega)} \leq C h^2 \| p \|_{H^2(\Omega)}. \tag{4.5} \]

Let \( \tilde{y} \) be defined by
\[ \Delta \tilde{y} = u \quad \text{in} \quad \Omega, \quad \tilde{y} = 0 \quad \text{on} \quad \partial \Omega. \tag{4.6} \]

Since \( y_h(\tilde{u}) \) is the solution of the standard finite element scheme of the boundary value problem (4.6), hence we have the a priori error estimate
\[ \| \tilde{y} - y_h(\tilde{u}) \|_{L^2(\Omega)} + h \| \nabla (\tilde{y} - y_h(\tilde{u})) \|_{L^2(\Omega)} \leq C h^2 \| \tilde{y} \|_{H^2(\Omega)} \leq C h^2 \| p \|_{L^2(\Omega)}. \tag{4.7} \]

On the other hand, we have
\[ a(\tilde{y} - y, w) = (u - u, w), \quad \forall w \in H^1_0(\Omega) \]

such that
\[ \| \nabla(\tilde{y} - y) \|_{L^2(\Omega)}^2 = (u - u, \tilde{y} - y). \]

Let \( \Omega_0^+ = \bigcup \{ \tau_U \in T^h_0: p \geq 0 \} \), and \( \Omega_0^+ = \Omega \setminus \Omega_0^+ \). Then
\[ (u - u, \tilde{y} - y) = (\mathcal{P}p - p, \tilde{y} - y)_{\Omega_0^+} + (u - u, \tilde{y} - y)_{\Omega_0^+}. \]

Noting that
\[ |(\mathcal{P}p - p, \tilde{y} - y)_{\Omega_0^+}| = |(\mathcal{P}p - p, \tilde{y} - y - \mathcal{P}p(\tilde{y} - y))_{\Omega_0^+}| \leq C h^2 \| p \|_{H^1(\Omega)} \| \nabla(\tilde{y} - y) \|_{L^2(\Omega)} \]
and
\[ |(\mathcal{P}p - p, \tilde{y} - y)_{\Omega_0^+}| \leq C h^2 \| p \|_{H^1(\Omega)} \| \tilde{y} - y \|_{L^2(\Omega_0^+)}. \]

we have
\[ \| \nabla(\tilde{y} - y) \|_{L^2(\Omega)} \leq C h^2. \tag{4.8} \]

Noting that
\[ \| y - y_h \|_{H^1(\Omega)} \leq C \| y - \tilde{y} \|_{H^1(\Omega)} + \| \tilde{y} - y_h(\tilde{u}) \|_{H^1(\Omega)} + \| y_h(\tilde{u}) - y_h \|_{H^1(\Omega)} \]
\[ \| y - y_h \|_{L^2(\Omega)} \leq C \| y - \tilde{y} \|_{L^2(\Omega)} + \| \tilde{y} - y_h(\tilde{u}) \|_{L^2(\Omega)} + \| y_h(\tilde{u}) - y_h \|_{L^2(\Omega)} \]

and combined (4.5), (4.7) and (4.8), we proved (4.4). \( \square \)

Furthermore, if we decompose \( \Omega \) into three parts: \( \Omega_1 = \{ \tau_U : u|_{\tau_U} > 0 \} \), \( \Omega_2 = \{ \tau_U : u|_{\tau_U} = 0 \} \), \( \Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2) \), there exists the following superconvergence result (see [18], for details):

With the hypothesis that \( \text{meas}(\Omega_3) \leq C h^2 \), we have the optimal error estimate in \( L^2 \) norm:
\[ \| y - y_h \|_{L^2(\Omega)} \leq C \{ h^2_{\tau_U} + h^2_r \}. \tag{4.9} \]

5. Equivalent a posteriori error estimators

In this section we derive the a posteriori error estimators equivalent to the error between the solution of the discrete problem and the solution of the infinite dimensional problem. We can see that \( y \) and \( p \) in the optimal control problem are smooth enough so that we do not need to use the adaptive finite element method to compute \( y \), but sometimes it is useful to compute the curvature term \( \Delta y \), which in our optimal control problem above is \( u \) indeed, and then it is necessary to introduce the adaptive finite element method to our analysis and computation. The numerical experiment in the next section may give us a good explanation.
The following lemmas are important in deriving a posteriori error estimates of residual type.

**Lemma 5.1.** Let \( \hat{\pi}_h \) be the average interpolation operator defined in [19]. For \( m = 0 \) or \( 1 \), \( 1 \leq q \leq \infty \) and \( v \in W^{1,q}(\Omega) \),

\[
|v - \hat{\pi}_h v|_{W^{m,q}(\tau)} \leq \sum_{\tau \cap \Omega \neq \emptyset} C \tau^{-1+m} |v|_{W^{1,q}(\tau')}, \quad \forall \tau \in T^h. \tag{5.1}
\]

**Lemma 5.2 ([20]).** For each \( v \in W^{1,q}(\Omega) \), \( 1 \leq q < \infty \),

\[
\|v\|_{W^{0,q}(\Omega)} \leq C \left( h^{-1} \|v\|_{W^{0,q}(\Omega)} + h^{-1} \|v\|_{W^{1,q}(\Omega)} \right), \quad \forall \tau \in T^h \tag{5.2}
\]

**Lemma 5.3 ([21]).** Let \( \pi_h \) be the standard Lagrange interpolation operator. For \( m = 0 \) or \( 1 \), \( q > \frac{n}{2} \) and \( v \in W^{2,q}(\Omega) \),

\[
|v - \pi_h v|_{W^{m,q}(\Omega)} \leq C h^{2-m} |v|_{W^{2,q}(\Omega)}. \tag{5.3}
\]

In order to derive sharp a posteriori error estimates, we divide \( \Omega \) into some subsets:

\[
\begin{align*}
\Omega_0^+ &= \{ x \in \Omega : p_h(x) < 0, u_h > 0 \}, \\
\Omega_0^- &= \{ x \in \Omega : p_h(x) > 0, u_h < 0 \}, \\
\Omega_0 &= \{ x \in \Omega : p_h(x) = 0, u_h = 0 \}.
\end{align*}
\]

Then, it is easy to see that the above three subsets are not intersected each other, and

\[
\hat{\Omega} = \hat{\Omega}_0^+ \cup \hat{\Omega}_0^- \cup \hat{\Omega}_0.
\]

**5.1. Equivalent a posteriori error estimators with \( H^1 \) norm**

In the first part of this section we give the equivalent a posteriori error estimates in \( H^1 \) norms. Of course, here “in \( H^1 \) norms” actually means the estimators of \( \|y - y_h\|_{H^1(\tau)} \) and \( \|p - p_h\|_{H^1(\tau)} \).

**Lemma 5.4.** There holds the estimate

\[
\|u - u_h\|_{L^2(\Omega)}^2 \leq C (\eta_1^2 + \|p_h - p\|_{L^2(\Omega)}^2), \tag{5.4}
\]

where

\[
\eta_1^2 = \int_{\hat{\Omega}_0^- \cup \hat{\Omega}_0^+} (u_h - p_h)^2.
\]

**Proof.** It is clear that

\[
\begin{align*}
\|u - u_h\|_{L^2(\Omega)}^2 &= (u - p, u - u_h) - (u_h - p, u - u_h) \\
&\leq (u_h - p, u_h - u) \\
&= (u_h - p - u_h + p_h, u_h - u) + (u_h - p_h, u_h - u) \\
&\leq \frac{1}{2} \|p_h - p\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u_h - u\|_{L^2(\Omega)}^2 + (u_h - p_h, u_h - u),
\end{align*}
\]

where we used \( (u - p, u - u_h) \leq 0 \), and we have

\[
\|u - u_h\|_{L^2(\Omega)}^2 \leq \|p_h - p\|_{L^2(\Omega)}^2 + 2(u_h - p_h, u_h - u). \tag{5.6}
\]

It follows from the definition of \( \hat{\Omega}_0 \) that

\[
\int_{\hat{\Omega}_0} (u_h - p_h)(u_h - u) \leq 0.
\]

Then we have

\[
\begin{align*}
\|u - u_h\|_{L^2(\Omega)}^2 &\leq \|p_h - p\|_{L^2(\Omega)}^2 + 2 \int_{\hat{\Omega}_0^- \cup \hat{\Omega}_0^+} (u_h - p_h)(u_h - u) \\
&\leq \|p_h - p\|_{L^2(\Omega)}^2 + 2 \int_{\hat{\Omega}_0^- \cup \hat{\Omega}_0^+} (u_h - p_h)^2 + \frac{1}{2} \|u - u_h\|_{L^2(\Omega)}^2.
\end{align*}
\]
Hence,
\[ \|u - u_h\|_{t_2}^2 \leq C \left\{ \|p_h - p\|_{t_2}^2 + \eta_1^2 \right\}. \tag{5.7} \]
This ends the proof of Lemma 5.4. □

**Theorem 5.1.** Let \((y, p, u)\) and \((y_h, p_h, u_h)\) be the solutions of \((\text{OCP–OPT})\) and \((\text{OCP–OPT})\), Then there holds the estimate
\[ \|u - u_h\|_{t_2}^2 + \|y - y_h\|_{t_1}^2 + \|p - p_h\|_{t_1}^2 \leq C \sum_{i=1}^{3} \eta_i^2, \tag{5.8} \]
where
\[ \eta_2 = \sum_{\tau \in t} \int_{\tau} h_\tau^2 f^2 + \sum_{l \in \partial \Omega = 0} h_l [\nabla p_h \cdot n]^2, \]
\[ \eta_3 = \sum_{\tau \in t} \int_{\tau} h_\tau^2 u_h^2 + \sum_{l \in \partial \Omega = 0} h_l [\nabla y_h \cdot n]^2, \]
where \(l\) is a face of an element \(\tau\), \(n\) is the unit normal vector on \(l = \overline{\tau_1} \cap \overline{\tau_2}\) outwards \(\tau_1\), \(h_l\) is the maximum diameter of the face \(l\), and \([\nabla p_h \cdot n]\) and \([\nabla y_h \cdot n]\) are the normal derivative jumps over the interior face \(l\), defined by
\[ [\nabla p_h \cdot n] = (\nabla p_h |_{\overline{\tau_1}} - \nabla p_h |_{\overline{\tau_2}}) \cdot n, \]
\[ [\nabla y_h \cdot n] = (\nabla y_h |_{\overline{\tau_1}} - \nabla y_h |_{\overline{\tau_2}}) \cdot n. \]

**Proof.** It follows from Lemma 5.4 that we need to estimate \(\|p_h - p\|_{t_2}^2\). With the standard analysis of deriving a posteriori error estimator for a Poisson equation, such as in Chapter 1 in [22], we can easily obtain
\[ \|p_h - p\|_{t_2}^2 \leq \|p - p_h\|_{t_1}^2 \leq C \eta_2^2. \]
It follows from Lemma 5.4 that
\[ \|u - u_h\|_{t_2}^2 \leq C \sum_{i=1}^{2} \eta_i^2. \]

Similarly, we can prove
\[ \|y(u_h) - y_h\|_{t_1}^2 \leq C \eta_3^2, \tag{5.9} \]
where \(y(u)\) is the solution of
\[ -\Delta y(u) = u_h, \quad y|_{\partial \Omega} = 0. \tag{5.10} \]

Note that
\[ \|y_h - y\|_{t_1}^2 \leq \|y_h - y(u_h)\|_{t_1}^2 + \|y(u_h) - y\|_{t_1}^2, \tag{5.11} \]
and
\[ \|y(u_h)\|_{t_1}^2 \leq C \|u - u_h\|_{t_2}^2, \tag{5.12} \]
Then
\[ \|u - u_h\|_{t_2}^2 + \|y - y_h\|_{t_1}^2 + \|p - p_h\|_{t_1}^2 \leq C \sum_{i=1}^{3} \eta_i^2. \]
Then we proved Theorem 5.1. □

Using the standard bubble function technique (see [22,23], for example), and with the similar proof in [24], we derive the a posteriori lower bound and then the following equivalent a posteriori error estimates.

**Theorem 5.2.** Let \((y, p, u)\) and \((y_h, p_h, u_h)\) be the solutions of \((\text{OCP–OPT})\) and \((\text{OCP–OPT})\), then
\[ \eta_1^2 + \eta_2^2 + \eta_3^2 \leq C(\|u - u_h\|_{t_2}^2 + \|p - p_h\|_{t_1}^2 + \|y - y_h\|_{t_1}^2) + C \epsilon_1^2 + C \epsilon^2, \tag{5.13} \]
\[ \|u - u_h\|_{t_2}^2 + \|p - p_h\|_{t_1}^2 + \|y - y_h\|_{t_1}^2 + \epsilon^2 \leq C(\eta_1^2 + \eta_2^2 + \eta_3^2), \tag{5.14} \]
where \( \eta_1 \) is defined in Lemma 5.4, \( \eta_2 \) and \( \eta_3 \) are defined in Theorem 5.1, and

\[
e^2 = \int_{\Omega} (u - p - \mathcal{P}_h(u - p))^2, \quad e_i^2 = \sum_{r \in T_h} \int_r h_r^2 (f - f)^2,
\]

where \( \mathcal{P}_h \) is the \( L^2 \)-project operator from \( L^2(\Omega) \) to \( U^h \), \( \Omega^+ = \{ x \in \Omega^+ : u(x) = 0, \ u_0(x) > 0 \} \), and \( \bar{v} |_{\tau} = \frac{\int_{\tau} v}{|\tau|} \).

**Proof.** The error estimates (5.13) for \( \eta_1 \) can be derived from Lemma 5.6 below, and the proof for \( \eta_2 \) and \( \eta_3 \) is similar. So we just give the proof on estimating the contributions of \( \eta_3 \) here.

Using the standard bubble function technique (see [22]), it can be proved that there exist polynomials \( w_\tau \in H^1_0(\tau) \) and \( w_l \in H^1_0(\tau_1 \cup \tau_2) \) such that

\[
\int_{\tau} h_r^2 u_h^2 = \int_{\tau} u_hw_\tau, \tag{5.15}
\]

\[
\int_{l} h_l[(\nabla y_h \cdot n)]^2 = \| \nabla y_h \cdot n \|^2, \tag{5.16}
\]

and

\[
\| w_\tau \|^2_{H^1(\tau)} \leq C \int_{\tau} h_r^2 u_h^2, \tag{5.17}
\]

\[
h_r^{-2} \| w_\tau \|^2_{L^2(\tau)} \leq C \int_{\tau} h_r^2 u_h^2, \tag{5.18}
\]

\[
\| w_l \|^2_{H^1(\tau_1 \cup \tau_2)} \leq C \int_{l} h_l[(\nabla y_h \cdot n)]^2, \tag{5.19}
\]

\[
h_r^{-2} \| w_l \|^2_{L^2(\tau_1 \cup \tau_2)} \leq C \int_{l} h_l[(\nabla y_h \cdot n)]^2. \tag{5.20}
\]

Then it follows from (5.15), (5.17) and (5.18) that

\[
\int_{\tau} h_r^2 u_h^2 = \int_{\tau} u_\omega \omega_\tau = \int_{\tau} (u - \Delta y - u) \omega_\tau + \int_{\tau} (u_h - u) \omega_\tau = \int_{\tau} (\nabla y_h - \nabla y) \omega_\tau + \int_{\tau} (u_h - u) \omega_\tau \leq \| \nabla y_h - \nabla y \|^2_{L^2(\tau)} + \| u_h - u \|^2_{L^2(\tau)},
\]

where \( C_1 \) is a general positive constant independent of \( h \), and \( \delta = \frac{h}{2C_1} \) is a small number. In the above proof, we used the property that \( y_h \) is a linear function in \( \tau \). Then we have

\[
\sum_{r \in T_h} \int_{\tau} h_r^2 u_h^2 \leq \sum_{r \in T_h} C(\| y - y_h \|^2_{H^1(\tau)} + \| u - u_h \|^2_{L^2(\tau)}).
\]
We only need to estimate the error for $\eta - y$. Then we proved (5.12), and Theorem 5.2 follows from Theorem 5.1.

Theorem 5.3. Which deduced that

$$\|\eta - y\|^2_{L^2(\Omega)} + \|u - u_h\|^2_{L^2(\Omega)} + h^2\|u_h\|^2_{L^2(\Omega)} \leq C(\delta)(\|\eta - y\|^2_{H^1(\Omega)} + \|u - u_h\|^2_{L^2(\Omega)} + h^2\|u_h\|^2_{L^2(\Omega)}).$$

which deduced that

$$h_1^2 \int I[(\nabla y_h \cdot n)]^2 \leq C(\delta)(\|\eta - y\|^2_{H^1(\Omega)} + \|u - u_h\|^2_{L^2(\Omega)} + h^2\|u_h\|^2_{L^2(\Omega)}).$$

Combined with (5.21) and (5.22), we have

$$\eta_3 = \sum_{l \leq l \leq l} h_1^2 u_h^2 + \sum_{h \leq h \not= h} h_1^2 \int I[(\nabla y_h \cdot n)]^2 \leq C(\delta)(\|y - y\|^2_{H^1(\Omega)} + \|u - u_h\|^2_{L^2(\Omega)}).$$

Then we proved (5.13), and Theorem 5.2 follows from Theorem 5.1.

5.2. Equivalent a posteriori error estimators with $L^2$ norm

In the rest of this section we give our equivalent a posteriori error estimates in $L^2$ norms. First, we need an a priori regular estimate for the following auxiliary problem:

$$-\Delta \xi = f_1 \text{ in } \Omega, \quad \xi|_{\partial \Omega} = 0,$$

$$-\Delta \zeta = f_2 \text{ in } \Omega, \quad \zeta|_{\partial \Omega} = 0.$$

Lemma 5.5 ([25]). Let $\xi$ and $\zeta$ be the solutions of (5.23) and (5.24) respectively. Assume that $\Omega$ is convex, then

$$\|\xi\|_{H^2(\Omega)} \leq C \|f_1\|_{L^2(\Omega)}, \quad \|\zeta\|_{H^2(\Omega)} \leq C \|f_2\|_{L^2(\Omega)}.$$

Theorem 5.3. Let $(y, p, u)$ and $(y_h, p_h, u_h)$ be solutions of (OCP–OPT) and (OCP–OPT)$^h$, respectively. Then

$$\|u - u_h\|^2_{L^2(\Omega)} + \|y - y_h\|^2_{L^2(\Omega)} + \|p - p_h\|^2_{L^2(\Omega)} \leq C(\eta_1^2 + \eta_2^2)$$

where $\eta_1$ is defined in Lemma 5.4 and

$$\eta_2^2 = \sum_{l \leq l \leq l} h_1^2 \int I(u_h^2) + \sum_{h \leq h \not= h} h_1^2 \int I[y_h \cdot n]^2,$$

$$\eta_3^2 = \sum_{l \leq l \leq l} h_1^2 \int I(u_h^2) + \sum_{h \leq h \not= h} h_1^2 \int I[y_h \cdot n]^2.$$

Proof. We only need to estimate the error for $y$ and $p$. Let $\zeta$ be the solution of (5.24) with $f_2 = p - p_h$, let $\zeta_1 = \pi_h \zeta \in V^h$ be the standard Lagrange interpolation of $\zeta$. Then it follows from Lemmas 5.1, 5.5, 5.3 and also Young's inequality that

$$\|p - p_h\|^2_{L^2(\Omega)} = (\nabla \zeta, \nabla (p - p_h))$$

$$= \sum_{l \leq l \leq l} \int f + \Delta p_h)(\zeta - \zeta_1) + \sum_{h \leq h \not= h} \int \nabla p_h \cdot n)(\zeta - \zeta_1)ds$$

$$\leq C \sum_{l \leq l \leq l} h_1^2 \|f\|^2_{L^2(\Omega)} + C \sum_{h \leq h \not= h} h_1^3 \left(\int \nabla p_h \cdot n)^2\right)^{\frac{1}{2}} \|\zeta\|_{H^2(\Omega)}$$

$$\leq C(\delta) \sum_{l \leq l \leq l} h_1^2 \int I^2 + C(\delta) \sum_{h \leq h \not= h} h_1^3 \int \nabla p_h \cdot n)^2 + C_2 \delta \|p - p_h\|^2_{L^2(\Omega)}.
where $C_2$ is a general positive constant independent of $h$, and $\delta = \frac{1}{2C_2}$ is a small number. Then we have

$$
\|p - p_h\|_{L^2(\Omega)}^2 \leq C \sum_{\tau \in T^h} h^4_\tau \left( \int_{\tau} f^2 + C \sum_{\ell \cap \partial \Omega = \emptyset} h^3_\ell \int_{\ell} [\nabla p_h \cdot n]^2 \right).
$$

(5.26)

Similarly, let $\xi$ be the solution of (5.23) with $f_1 = y(u_h) - y_h$, where $y(u_h)$ is the solution of (5.10). Then

$$
\|y(u_h) - y_h\|_{L^2(\Omega)}^2 = (\nabla y(u_h) - y_h, \nabla \xi) \\
\leq C(\delta) \sum_{\tau \in T^h} h^4_\tau \left( \int_{\tau} y_h^2 + C(\delta) \sum_{\ell \cap \partial \Omega = \emptyset} h^3_\ell \int_{\ell} [\nabla y_h \cdot n]^2 + C\delta \|y(u_h) - y_h\|_{L^2(\Omega)}^2 \right).
$$

Hence,

$$
\|y(u_h) - y_h\|_{L^2(\Omega)}^2 \leq C \sum_{\tau \in T^h} h^4_\tau \left( \int_{\tau} (u_h + \Delta y_h)^2 + C \sum_{\ell \cap \partial \Omega = \emptyset} h^3_\ell \int_{\ell} [\nabla y_h \cdot n]^2 \right).
$$

(5.27)

Note that

$$
\|y - y_h\|_{H^1(\Omega)} \leq \|y_h - y(u_h)\|_{H^1(\Omega)} + \|y(u_h) - y\|_{H^1(\Omega)} \\
\leq \|y_h - y(u_h)\|_{H^1(\Omega)} + \|u - u_h\|_{L^2(\Omega)},
$$

(5.28)

then (5.25) follows from (5.26), (5.27), (5.28) and Theorem 5.1. □

Now we are in the position of deriving the a posteriori lower bounds with $L^2$ norm.

**Lemma 5.6.** Let $(y, p, u)$ and $(y_h, p_h, u_h)$ be the solutions of (OCP–OPT) and (OCP–OPT)$_h$, then

$$
\eta^2 \leq C(e^2 + \|u - u_h\|_{L^2(\Omega)}^2 + \|p - p_h\|_{L^2(\Omega)}^2)
$$

where $e^2$ is defined in Theorem 5.2.

**Proof.** To make it clear and complete, we repeat the similar proof in [24]. Let $\Omega_0^d = \{x \in \Omega_0^d : u(x) = 0\}$, and note that

\[
\begin{cases}
u - p = 0, & \text{when } u > 0; \\
u - p \geq 0, & \text{when } u = 0.
\end{cases}
\]

Then we can derive that

$$
\int_{\Omega_0^d} (u_h - p_h)^2 = \int_{\Omega_0^d} (u_h - u)^2 + \int_{\Omega_0^d \setminus \Omega_0^d} (u_h - p_h)^2 \\
= \int_{\Omega_0^d} (u_h - u - p)^2 + \int_{\Omega_0^d \setminus \Omega_0^d} (u_h - p_h - u + p)^2 \\
\leq C \left( \|u - u_h\|_{L^2(\Omega)}^2 + \|p - p_h\|_{L^2(\Omega)}^2 + \int_{\Omega_0^d} p_h^2 \right) \\
\leq C \left( \|u - u_h\|_{L^2(\Omega)}^2 + \|p - p_h\|_{L^2(\Omega)}^2 + \int_{\Omega_0^d} (p_h - p)^2 \right) \\
\leq C(\|u - u_h\|_{L^2(\Omega)}^2 + \|p - p_h\|_{L^2(\Omega)}^2),
$$

where we used $p_h \geq 0$ and $p \leq 0$ on $\Omega_0^d$ in the second last step.

Recall that

$$
\Omega_+ = \{x \in \Omega_0^d : u(x) = 0, \ u_h(x) > 0\},
$$

and note $u > 0$ on $\Omega_0^+ \setminus \Omega_+$, so $u - p = 0$, on $\Omega_0^+ \setminus \Omega_+$. Since $u_h > 0$ on $\Omega_+$, then it follows from (3.4)(b) that $u_h = \max(0, \mathcal{P}_h p_h) = \mathcal{P}_h p_h > 0$ on $\Omega_+$, so $\mathcal{P}_h (u_h - p_h) = u_h - \mathcal{P}_h p_h = 0$ on $\Omega_+$.

$$
\int_{\Omega_0^d} (u_h - p_h)^2 = \int_{\Omega_+} (u_h - p_h)^2 + \int_{\Omega_0^d \setminus \Omega_+} (u_h - p_h)^2 \\
= \int_{\Omega_+} ((u_h - p_h) - \mathcal{P}_h (u_h - p_h))^2 + \int_{\Omega_0^d \setminus \Omega_+} (u_h - p_h - u + p)^2
$$
Lemma 5.7. Here we give some properties of the bubble-functions numbered first. We then define the triangle-bubble function $b_t$ by
\begin{equation}
    b_t = \begin{cases} 
        \lambda_{t,1} \lambda_{t,2} \lambda_{t,3}, & \text{on } \tau, \\
        0, & \text{on } \Omega \setminus \tau.
    \end{cases}
\end{equation}

(5.29)

Given an edge $l = \tau_1 \cap \tau_2$, where $\tau_1, \tau_2 \in \mathcal{T}^n$, we enumerate the vertices of $\tau_1$ and $\tau_2$ such that the vertices of $l$ are numbered first. We then define the edge-bubble function $b_l$ by
\begin{equation}
    b_l = \begin{cases} 
        \lambda_{t_1,1} \lambda_{t_2,2} \lambda_{t_2,3}, & \text{on } \tau_1 \cup \tau_2, \\
        0, & \text{on } \Omega \setminus (\tau_1 \cup \tau_2).
    \end{cases}
\end{equation}

(5.30)

Here we give some properties of the bubble-functions $b_t$ and $b_l$, which were deduced from [22,26]. Its proof immediately follows from Definitions (5.29) and (5.30) and the standard scaling arguments for finite element functions. So we just give the following Lemma 5.7 without proof.

Lemma 5.7. Let $b_t$ and $b_l$ be the bubble-functions defined in (5.29) and (5.30), then
\begin{align*}
    0 &\leq |b_t| \leq 1, \quad 0 \leq |b_l| \leq 1; \\
    \text{supp } b_t &\subset \tau, \quad \text{supp } b_l \subset \tau_1 \cup \tau_2, \quad b_t \in H^n_0(\tau), \quad b_l \in H^n_0(\tau_1 \cup \tau_2); \\
    c_1 h_t^{-2} \|b_t^2\|_{L^2(\tau)} &\leq \|b_t^2\|_{L^2(\tau)} \leq c_2 h_t^{-2} \|b_t^2\|_{L^2(\tau)}; \\
    c_3 h_l^{-2} \|b_l^2\|_{L^2(\tau_1 \cup \tau_2)} &\leq \|b_l^2\|_{L^2(\tau_1 \cup \tau_2)} \leq c_4 h_l^{-2} \|b_l^2\|_{L^2(\tau_1 \cup \tau_2)}.
\end{align*}

Lemma 5.8. Let $(y, p, u)$ and $(y_h, p_h, u_h)$ be the solutions of (OCP–OPT) and (OCP–OPT)$^h$, then
\begin{align*}
    \sum_{t \in \mathcal{T}^n} h_t^4 f^2 &\leq C(\|p - p_h\|^2_{L^2(\Omega)} + \varepsilon^2),
\end{align*}

where
\begin{align*}
    \varepsilon^2 = \sum_{t \in \mathcal{T}^n} \int_{\tau} h_t^4 (f - \tilde{f})^2.
\end{align*}

Proof. Note that
\begin{align*}
    h_t^4 \int_{\tau} \tilde{f}^2 = h_t^4 \int_{\tau} \tilde{f} \omega_t,
\end{align*}

where $\omega_t = c_s h_t^2 b_t^2$, and $c_s = \frac{4}{h_t b_t}$ is a constant. It follows from Lemma 5.7 that $\omega_t \in H_0^2(\tau)$. Then
\begin{align*}
    h_t^4 \int_{\tau} \tilde{f}^2 &\leq \int_{\tau} h_t^4 f \omega_t + \int_{\tau} h_t^4 \tilde{f} (f - \tilde{f}) \omega_t \\
    &\leq \int_{\tau} h_t^4 (f - \Delta p - f) \omega_t + C(\delta) \int_{\tau} h_t^4 (f - \tilde{f})^2 + C \delta h_t^4 \|\omega_t\|^2_{L^2(\tau)} \\
    &= \int_{\tau} h_t^4 (\Delta p_h - \Delta p) \omega_t + C(\delta) \int_{\tau} h_t^4 (f - \tilde{f})^2 + C \delta h_t^4 \|\omega_t\|^2_{L^2(\tau)} \\
    &= \int_{\tau} h_t^4 (\nabla p - \nabla p_h) \nabla \omega_t + C(\delta) \int_{\tau} h_t^4 (f - \tilde{f})^2 + Ch_t^4 \|\omega_t\|^2_{L^2(\tau)}.
\end{align*}
\[ \int_{\tau} h^4_t (p_h - p) \Delta \omega_t + C(\delta) \int_{\tau} h^4_t (f - \bar{f})^2 + C \delta h^4_t \| \omega_t \|_{L^2(\tau)}^2 \]
\[ \leq C(\delta) \left( \| p - p_h \|_{L^2(\tau)}^2 + \int_{\tau} h^4_t (f - \bar{f})^2 \right) + C \delta h^8_t \| \omega_t \|_{H^2(\tau)}^2 \]
\[ \leq C(\delta) \left( \| p - p_h \|_{L^2(\tau)}^2 + \int_{\tau} h^4_t (f - \bar{f})^2 \right) + C \delta h^8_t \| \omega_t \|_{H^2(\tau)}^2 \]
\[ \leq C(\delta) \left( \| p - p_h \|_{L^2(\tau)}^2 + \int_{\tau} h^4_t (f - \bar{f})^2 \right) + C \delta \int_{\tau} h^4_t \bar{f}^2, \]

where \( 0 < \delta \ll 1 \), and we used Lemma 5.7 in the last step. So we have the estimate that
\[ \int_{\tau} h^4_t \bar{f}^2 \leq C \left( \| p - p_h \|_{L^2(\tau)}^2 + \int_{\tau} h^4_t (f - \bar{f})^2 \right). \]

Note that
\[ \int_{\tau} h^4_t \bar{f}^2 \leq C \int_{\tau} h^4_t \bar{f}^2 + C \int_{\tau} h^4_t (f - \bar{f})^2, \]

then Lemma 5.8 is proved. \( \square \)

**Lemma 5.9.** Let \((y, p, u)\) and \((y_h, p_h, u_h)\) be the solutions of (OCP–OPT) and (OCP–OPT)\(h\), then
\[ \sum_{l \in \Omega_2} h^3_l \int_{l} [\nabla p_h \cdot n]^2 \leq C(\| p - p_h \|_{L^2(\Omega)}^2 + \varepsilon^2_b). \]

**Proof.** Note that
\[ h^3_l \int_{l} [\nabla p_h \cdot n]^2 = h^3_l \int_{l} [\nabla p_h \cdot n] \omega_l, \]

where \( \omega_l = c_6 [\nabla p_h \cdot n] b_t^2 \), and \( c_6 = \frac{f_{l^1}}{f_{l^2}} \) is a constant. It follows from Lemma 5.7 that \( \omega_l \in H^2_0(\tau_l \cup \tau_l^1) \). Then
\[ h^3_l \int_{l} [\nabla p_h \cdot n]^2 = \int_{l} h^3_l [\nabla p_h \cdot n] w_l = \int_{l} h^3_l ((\nabla p_h \cdot n) - (\nabla p \cdot n)) w_l \]
\[ = \int_{\tau_l^1 \cup \tau_l^2} h^3_t ((\nabla p_h \cdot n) - (\nabla p \cdot n)) w_l \]
\[ = \int_{\tau_l^1 \cup \tau_l^2} h^3_t (\nabla (p_h - p) \nabla w_l) \]
\[ = \int_{\tau_l^1 \cup \tau_l^2} h^3_t (\Delta (p_h + f) w_l) \]
\[ \leq C(\delta) \| p_h - p \|_{L^2(\tau_l^1 \cup \tau_l^2)}^2 + C(\delta) \int_{\tau_l^1 \cup \tau_l^2} h^4_t \bar{f}^2 + C \delta \left( h^6_t \| w_l \|_{H^2(\tau_l^1 \cup \tau_l^2)}^2 + h^7_t \| w_l \|_{L^2(\tau_l^1 \cup \tau_l^2)}^2 \right) \]
\[ \leq C(\delta) \| p_h - p \|_{L^2(\tau_l^1 \cup \tau_l^2)}^2 + C(\delta) \int_{\tau_l^1 \cup \tau_l^2} h^4_t \bar{f}^2 + C \delta \left( h^6_t \| \nabla p_h \cdot n \|_{H^2(\tau_l^1 \cup \tau_l^2)}^2 \right) \]
\[ \leq C(\delta) \| p_h - p \|_{L^2(\tau_l^1 \cup \tau_l^2)}^2 + C(\delta) \int_{\tau_l^1 \cup \tau_l^2} h^4_t \bar{f}^2 + C \delta \left( h^6_t \| \nabla p_h \cdot n \|_{H^2(\tau_l^1 \cup \tau_l^2)}^2 \right) \]
\[ + h^7_t \| \nabla p_h \cdot n \|^2 \| b_t^2 \|_{L^2(\tau_l^1 \cup \tau_l^2)}^2 \].

From the standard scaling arguments, it is easy to see that
\[ \| b_t^2 \|_{L^2(\tau_l^1 \cup \tau_l^2)} \leq C h^\frac{1}{2} \| b_t^2 \|_{L^2(\Omega)}, \]

where \( l = \tau_l^1 \cap \tau_l^2 \). Combined with Lemmas 5.7 and 5.8, we have the estimate that
\[ h^3_l \int_{l} [\nabla p_h \cdot n]^2 \leq C(\| p - p_h \|_{L^2(\tau_l^1 \cup \tau_l^2)}^2 + \varepsilon^2_b). \]

Then Lemma 5.9 is proved. \( \square \)
where In this example we consider the variational inequality problem (2.1):

Example 1.

and 5. In computing the solutions, we used the software package AFEpack, see [27] for the details.

6. Numerical experiments

In this section, we carry out some numerical experiments to demonstrate the error estimates developed in Sections 4 and 5. In computing the solutions, we used the software package: AFEpack, see [27] for the details.

Example 1. In this example we consider the variational inequality problem (2.1):

\[
\begin{aligned}
\text{Find } y \in K, \text{ such that } \\
\int_{\Omega} \Delta y \Delta (w - y) \geq \int_{\Omega} f(w - y), \quad \forall w \in K
\end{aligned}
\]

where \( K = \{ y \in H^2(\Omega) \cap H^1_0(\Omega) : \Delta y \leq 0 \text{ in } \Omega \} \).

\[
\begin{aligned}
\Omega &= (0, 1)^2 \\
f &= 2\pi^2 \sin(\pi x_1) \sin(\pi x_2) \\
y &= \frac{1}{2\pi^2} \sin(\pi x_1) \sin(\pi x_2).
\end{aligned}
\]

The numerical results are in Table 1:

<table>
<thead>
<tr>
<th>Nodes</th>
<th>Sides</th>
<th>Elements</th>
<th>( L^2 ) error of ( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>140</td>
<td>377</td>
<td>238</td>
</tr>
<tr>
<td>0.05</td>
<td>507</td>
<td>1438</td>
<td>932</td>
</tr>
<tr>
<td>0.025</td>
<td>1930</td>
<td>5627</td>
<td>3698</td>
</tr>
</tbody>
</table>

From Lemmas 5.8 and 5.9 we obtain the following conclusion.

Lemma 5.10. Let \((y, p, u)\) and \((y_h, p_h, u_h)\) be the solutions of (OCP–OPT) and (OCP–OPT)', then

\[
\hat{\eta}_2^2 \leq C(\|p - p_h\|^2_{L^2(\Omega)} + \epsilon^2).
\]

Similarly, we can prove the following lower bound estimate.

Lemma 5.11. Let \( (y, p, u) \) and \((y_h, p_h, u_h)\) be the solutions of (OCP–OPT) and (OCP–OPT)', then

\[
\eta_2^2 \leq C(\|y - y_h\|^2_{L^2(\Omega)} + \|u - u_h\|^2_{L^2(\Omega)}).
\]

As a corollary of Lemmas 5.10 and 5.11, we have the following theorem.

Theorem 5.4. Let \((y, p, u)\) and \((y_h, p_h, u_h)\) be the solutions of (OCP–OPT) and (OCP–OPT)', respectively. Then

\[
\eta_1^2 + \eta_2^2 + \eta_3^2 \leq C(\|u - u_h\|^2_{L^2(\Omega)} + \|y - y_h\|^2_{L^2(\Omega)} + \|p - p_h\|^2_{L^2(\Omega)} + \epsilon^2 + \epsilon^2),
\]

where \( \eta_1^2, \eta_2^2, \) and \( \eta_3^2 \) are defined in Theorem 5.3, \( \epsilon^2 \) is defined in Lemma 5.8, and \( \epsilon^2 \) is defined in Theorem 5.2.

### Table 1

<table>
<thead>
<tr>
<th>Nodes</th>
<th>Sides</th>
<th>Elements</th>
<th>( L^2 ) error of ( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>140</td>
<td>377</td>
<td>238</td>
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</tr>
</tbody>
</table>
and the co-state equation is \(-\Delta p = f\), in \(\Omega\), \(p|_{\partial\Omega} = 0\). We take
\[
\Omega = (0, 1)^2
\]
\[
p = \sin(2\pi x_1) \sin(2\pi x_2) - \sin(\pi x_1) \sin(\pi x_2)
\]
\[
f = 8\pi^2 \sin(2\pi x_1) \sin(2\pi x_2) - 2\pi^2 \sin(\pi x_1) \sin(\pi x_2)
\]
\[
u = \max(0, p).
\]

We now briefly describe the solution algorithm to be used for solving this example:

**Algorithm**

(i) solve the discretized optimization problem with the projection gradient method on the current meshes and calculate the error estimators \(\eta_1\) or \(\hat{\eta}_1\);

(ii) adjust the meshes using the estimators and update the solution on new meshes, as described. The details on how to adjust the meshes can be seen in [16,27], etc.

In this example, we use \(\eta_1\) as the control mesh refinement indicator, and \(\eta_2 + \eta_3\) as the state's and co-state's for \(H^1\)-norm, and \(\hat{\eta}_2 + \hat{\eta}_3\) as the state's and co-state's for \(L^2\)-norm. In this example, it is very difficult for us to obtain the exact solution of \(y\), so we cannot get the \(L^2\) error for \(y - y_h\). However, we can calculate the FEM solution of \((-\Delta y = u, u|_{\partial\Omega} = 0\) on an deeply refined mesh to be the quasi-exact solution of \(y\), which is denoted as \(jy\) in the following. We use the norm \(\|y_h - jy\|_{L^2(\Omega)}\) to substitute for \(\|y_h - y\|_{L^2(\Omega)}\) where \(y\) is the exact solution of \((-\Delta y = u, u|_{\partial\Omega} = 0\). So in Table 2, the term \(\|y_h - y\|_{L^2(\Omega)}\) is actually \(\|y_h - jy\|_{L^2(\Omega)}\).

The error estimates at initial mesh, adaptive mesh step 1, and the last mesh step are as follows.

With the results shown in Table 2 we can see that we get almost the same accuracy in \(\|u - u_h\|_{L^2(\Omega)}\) on adaptive mesh, with more than 60% of grids saved. As the control problem (OCP) has been translated to the control of \(u\), but not the control of \(y\), so we may obtain same or better results in \(u\) with adaptive finite elements method, and meanwhile save lots of computational work. Here we give the meshes of \(u\) at the refinement steps mentioned above, respectively.

From Fig. 1 we can see that with the indicators’ affect, the grids are refined near the area where the control \(u\) has lack of regularity, and coarsened in other area. With the results in Table 2 and Fig. 1, the convergence of our scheme can be shown.
References