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ARTICLE

A Categorization Model for Educational Values of the History of Mathematics

An Empirical Study

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Abstract There is not a clear consensus on the categorization framework of the educational values of the history of mathematics. By analyzing 20 Chinese teaching cases on integrating the history of mathematics into mathematics teaching based on the relevant literature, this study examined a new categorization framework of the educational values of the history of mathematics by combining the objectives of high school mathematics curriculum in China. This framework includes six dimensions: the harmony of knowledge, the beauty of ideas or methods, the pleasure of inquiries, the improvement of capabilities, the charm of cultures, and the availability of moral education. The results show that this framework better explained the all-educational values of the history of mathematics that all teaching cases showed. Therefore, the framework can guide teachers to better integrate the history of mathematics into teaching.

Keywords History of mathematics · Educational values · Teaching cases · Standards of senior high school mathematics curriculum

1 Introduction

Studies on the relations between the history and pedagogy of mathematics (hereafter referred to as HPM) are very important in the field of mathematics education. In the early history of HPM, many studies and discussions focused on the benefits or educational values of the

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history of mathematics (e.g. Cajori 1899, 1911; Jones 1957). F. Cajori (1859–1930), a pioneer of HPM, believed that the knowledge of the history of science and mathematics may increase students' interest and lead teachers to a deeper appreciation of the difficulties that students encounter, thus regarding it as an effectual aid for teaching (Cajori 1899, 1911, 1917). P. S. Jones (1912–2002), who made great contribution to the birth of the International Study Group of HPM, regarded the history of mathematics as a teaching tool that can be used to make students appreciate and love mathematics, while also deepening their understanding of it (Jones 1957). From the 1990s on, more extensive discussions and deeper studies focused on why the history of mathematics should be integrated into teaching (e.g. Arcavi 1991; Barbin 1991; Bidwell 1993; Ernest 1998; Furinghetti 2000; Furinghetti and Somaglia 1998; Marshall and Rich 2000), of which Fauvel (1991), Tzanakis and Arcavi (2000), Gulikers and Blom (2001), and Jankvist (2009) generalized or classified the educational values of the history of mathematics in succession (see Section 2).

In mainland China, one of the fundamental ideas of the senior high school mathematics curriculum is the embodiment of the cultural values of mathematics. In the *Standards of Senior High School Mathematics Curriculum*, it is pointed out that:

> Mathematics is an important part of human culture. Mathematics curriculum should properly reflect the history, application and new trends of mathematics, the role mathematics plays in promoting the development of the society, the mathematical requirement of the society, the role the society plays in promoting the development of mathematics, the systems of thought of mathematical science, the aesthetic value of mathematics and the innovative spirit of mathematicians. The mathematics curriculum should help students recognize the role mathematics plays in the progress of human civilization and gradually form a proper conception of mathematics. Therefore, the senior high school mathematics curriculum advocates reflecting the cultural values of mathematics. (Translation from Ministry of Education, 2003, p. 4)

Accordingly, one of the optional modules of the curriculum is "Selected Historical Topics of Mathematics", which consists of eleven historical topics.

Given this, much attention has been paid to the history of mathematics in the community of high school mathematics education in China (e.g. Wang and Wang 2013; Zhang and Wang 2007). A large number of studies (all in Chinese) are focused on the educational values of the history of mathematics (e.g. Kang and Hu 2009; Luo and Liu 2012; Tang 2007; Yang 2009; Zhang and Luo 2006), but no uniform framework has been established so far in China. Moreover, few Chinese researchers connected their arguments for educational values with the objectives of mathematics curriculum or classroom teaching.

After the First National Conference on the History of Mathematics and Mathematics Education held in Xi'an in 2005, greater importance was attached to integrating the history of mathematics into teaching (IHT) and relevant cases (hereafter referred to as IHT cases) were developed. Especially, in recent years, dozens of IHT cases on topics related to both primary and secondary mathematics were published. These promoted the dissemination and communication of the ideas of HPM and attracted more attention from in-service mathematics teachers in mainland China, many of whom became interested in joining the HPM community and practicing IHT. Regrettably, no framework is available for analyzing the educational values of the history of mathematics in IHT lessons.

Today, lectures on HPM are warmly welcomed by both in-service and pre-service teachers, and HPM courses are incorporated into the mathematics education graduate programs in some normal universities. Practice has indicated that it is an effective strategy to give lectures or teach the aforesaid course by means of the published IHT cases to promote teachers' appreciation of the educational values of the history of mathematics. Therefore, it is necessary to identify the characteristics of various IHT cases in terms of some theoretical framework that organizes these educational values.

This study aims to build a categorization model on the educational values of the history of mathematics for students based on the literature and objectives of the newly revised senior high school mathematics curriculum in mainland China, and examine its effectiveness by analyzing IHT cases.

2 Some Well-known Categorizations of Educational Values of the History of Mathematics

Based on literature over the past few decades, Fauvel (1991) listed 15 reasons for using history in mathematics teaching:

(F1) History helps increase motivation for learning;

(F2) History gives mathematics a human face;

(F3) Historical development helps to order the presentation of topics in the curriculum;

(F4) Showing students how concepts have developed helps their understanding;

(F5) History changes students' perceptions of mathematics;

(F6) Comparing ancient and modern establishes values of modern techniques;

(F7) History helps to develop a multicultural approach;

(F8) History provides opportunities for investigations;

(F9) Past obstacles to [the] development [of mathematical knowledge] help to explain the difficulties that today's students find;

(F10) Students derive comfort from realizing that they are not the only ones with problems;

(F11) History encourages quicker learners to look further;

(F12) History helps to explain the role of mathematics in society;

(F13) History makes mathematics less frightening;

(F14) Exploring history helps students/teachers sustain their interest and excitement in mathematics;

(F15) History provides opportunity for cross-curricular work with other teachers or subjects. (p. 4)

Among these 15 reasons, 10 relate to students (F1, F2, F4–6, F8, F10–13), 4 relate to teachers (F3, F7, F9, F15) and 1 relates to for both (F14). However, the author presents these 15 reasons without classifying them using any categorizations.

Tzanakis and Arcavi (2000) classified the educational values of the history of mathematics into five dimensions, which are called "the ICMI whys" in Jankvist (2009) or "The ICMI study whys" in Tzanakis and Thomaidis (2011). Four of the five dimensions focus on students (Table 1).

Gulikers and Blom (2001) put forth a three-dimension framework of the educational values of the history of mathematics for both students and teachers: conceptual arguments, cultural arguments, and motivational arguments. In addition, each dimension includes several educational values of the history of mathematics for teachers and students. Table 2 only summarizes the values for students, which are the primary concern of this study.

Jankvist's (2009) argument for using history in the classroom comprised two dimensions: history as a tool and history as a goal. History as a tool is classified into motivation arguments and cognition arguments. Table 3 summarizes this two-dimension framework.

The above categorization frameworks of educational values of the history of mathematics are elaborate and valuable, but they do not build any relationships between the educational values and mathematics curriculum standards. Therefore, a new framework focusing on such relationships is needed. On the other hand, the educational values included in these

Table 1 Tzanakis and Arcavi's classification of the educational values of the history of mathematics for students

Dimension	Educational values of the history of mathematics
The learning of mathematics	(TA1) History uncovers why and how mathematical concepts, structures and ideas were created and supplies motivation for learning; (TA2) History provides a vast reservoir of relevant questions, problems, and expositions which can motivate, interest and engage students; (TA3) History exposes interrelations between mathematics and other subjects; (TA4) History promotes students' general skills, such as reading, writing, looking for resources, documenting, discussing, analyzing, and communicating mathematically.
Views on the nature of mathematics and mathematical activity	(TA5) History indicates that mistakes, heuristic arguments, uncertainties, doubts, intuitive arguments, blind alleys, controversies and alternative approaches to problems are not only legitimate but also an integral part of mathematics in the making; (TA6) History helps students understand the evolution of mathematical language, notation, terminology, computational methods, modes of expression and representations;
The affective predisposition towards mathematics	(TA7) History informs students that mathematics is an evolving and human subject rather than a system of rigid truths; (TA8) History makes known the value of persisting with ideas, of attempting to undertake lines of inquiry, of posing questions, and of attempting to develop creative ways of thought; (TA9) History informs students that they should not get discouraged by failure, mistakes, uncertainties, or misunderstandings.
The appreciation of mathematics as a cultural endeavor	(TA10) History tells students that mathematics is driven not only by utilitarian reasons, but also developed for its own sake, motivated by aesthetic criteria, intellectual curiosity, challenge and pleasure, recreational purposes, etc.; (TA11) History provides examples of how the internal development of mathematics has been influenced by social and cultural factors; (TA12) History makes students aware of the diversity of mathematical culture and is conducive to developing tolerance and respect among fellow students.

frameworks, though partly confirmed by teaching practice and empirical studies, are to be supported by more evidence.

3 A New Framework for Categorization

Mainland China is now witnessing a new reform of high school mathematics curricula. In the recently revised version of the *Standards of Senior High School Mathematics Curriculum*, which is anticipated to come out soon, the objectives of mathematics curricula include four aspects, which are listed below (Ministry of Education 2017).

- *Four* "*Basics*". Students should obtain basic knowledge, basic skills, basic ideas and basic experience through mathematics activities, which are necessary for future study and personal growth.
- *Four Abilities.* Students should enhance their abilities to both find and pose problems mathematically, and also to analyze and solve mathematical problems.

Table 2 Gulikers and Blom's classification of the educational values of the history of mathematics for students

Dimension	Educational values of the history of mathematics
Conceptual arguments	(GB1) History makes mathematics more concrete and gives students more insight; (GB2) Students derive comfort from realizing that they are not the only ones who have misunderstandings or make mistakes; (GB3) By comparing ancient and modern techniques students become aware of the advantages of the latter; (GB4) History helps students to learn in a non-linear way; (GB5) Historical problems provide alternative methods of solution and make students pursue creative thinking.
Cultural arguments	(GB6) History informs students that mathematics is a human and dynamic activity influenced by social and cultural factors; (GB7) History presents the development of mathematics as a human activity and not only as a system of rigid truths, thus presenting mathematics with a human face; (GB8) Knowing about female mathematicians in history, girls may be stimulated to learn mathematics.
Motivational arguments	(GB9) History increases students' interest for learning; (GB10) History motivates students; (GB11) History makes mathematics lessons less frightening, more enjoyable and exciting; (GB12) History enables brighter learners to look further.

- *Core Competencies.* Students should develop their core competencies, i.e., those of mathematical abstraction, logical reasoning, mathematical modeling, visual imagination, mathematical operation, and data analysis.

Table 3 Jankvist's classification of the educational values of the history of mathematics for students

Dimension	Educational values of the history of mathematics	
History as a tool	History is a motivating factor for students in their learning of mathematics	(J1) History sustains the students' interest and excitement in mathematics; (J2) A historical approach gives mathematics a more human face; (J3) History makes mathematics less frightening; (J4) Students derive comfort from history.
	History is a cognitive tool in supporting learning of mathematics	(J5) History improves learning by providing a different point of view or mode of presentation; (J6) A historical phenomenology prepares the development of a hypothetical learning trajectory; (J7) History helps both identify students' obstacles and overcome them; (J8) The recapitulation argument or historical parallelism.
History as a goal	Learning aspects of the history of mathematics serves a purpose in it of itself	The history of mathematics informs that (J9) Mathematics is a discipline that has undergone an evolution and not something that has arisen out of thin air; (J10) Human beings have taken part in the evolution of mathematics; (J11) Mathematics has evolved through many different cultures throughout history; (J12) Different cultures have had an influence on the shaping of mathematics and vice versa; (J13) The evolution of mathematics is driven by internal and external forces.

- *Affect and Beliefs.* Students should increase their interest in mathematics, strengthen their self-confidence in performing well at mathematics and develop a good study habit; they should nurture a scientific spirit of questioning, thinking and truth-seeking; they should also appreciate the scientific, practical, cultural, and aesthetic values of mathematics.

In view of the potential roles the history of mathematics can play in the four aspects from *Standards of Senior High School Mathematics Curriculum*, the educational values of the history of mathematics proposed in the literature can be reclassified into six dimensions: (a) the harmony of knowledge; (b) the beauty of ideas or methods; (c) the pleasure of inquiries; (d) the improvement of capabilities; (e) the charm of culture; and (f) the availability of moral education. We define the six dimensions as follows.

(a) The *harmony of knowledge.* The ancient Roman philosopher Cicero said, "If we are guided by nature, then nature will never let us go astray" (Comenius 1907). Likewise, if we are guided by history, then history will never let us go astray. The history of mathematics tells us that any mathematical concepts, formulas, theorems, and ideas have not arisen out of thin air, but underwent processes of genesis and evolution in natural ways. Learning from history ensures that new knowledge can also be generated in a natural way in the classroom, which is in accordance with students' cognitive basis. Meanwhile, history tells a teacher how to bridge the gap between the known and the unknown, and promotes students' understanding of new knowledge. This naturalness and fluidity of knowledge throughout history is what we call the harmony of knowledge. It is through this dimension that history facilitates students' acquisition of "basic knowledge".

(b) The *beauty of ideas or methods*. Teaching of theorems or formulas is not just for their application. The ideas or methods behind the theorems or formulas are also learning goals. However, textbooks only provide us with one or two proofs or derivation methods for a specific theorem or formula. Uncovering the veil of history across time, countless sages left us a huge treasure of brilliant mathematical ideas and methods, of which some can be integrated into teaching. The cleverness, ingenuity, diversity and flexibility of ideas or methods that come from different time and space makeup what we call the *beauty of ideas or methods*. It is through this dimension that history helps students grasp "basic ideas".

(c) The *pleasure of inquiries*. Mathematics teaching from the perspective of IHT focuses on the natural way in which knowledge is generated. Thus, student's inquiries are indispensable for a good IHT case. By using history-based mathematics problems and drawing lessons from the development of mathematical concepts, teachers can design proper classroom activities for students to think, investigate and explore. In this way, students can experience the genesis and evolution of a new concept, formula, theorem, or idea, thereby accumulating experience in mathematical activities and gaining confidence in exploring new knowledge. This is the main idea expressed in the *pleasure of inquiries*. It is through this dimension that history provides students with opportunities to gain "basic experiences of activity".

(d) The *improvement of capabilities***.** Inquiries into historical or history-based problems help students foster their basic skills, "four abilities" and core competencies. Investigations into history or examinations of historical materials can improve students' other capabilities, such as reading, writing, documenting, and communicating mathematically. History also provides various representations of concepts or ideas that help foster students' capability to represent mathematical content.

(e) The *charm of culture*. The history of mathematics is an integral part of mathematical culture. Therefore, when history is integrated into teaching, mathematical culture itself is also presented in the classroom. Additionally, the history of mathematics is a bridge between mathematics and other disciplines. Kline (1958) proposed the cultural principle as one of his four principles of mathematics curriculum, advocating relating mathematics to history, science, philosophy, social science, art, music, literature, and logic. Mathematics is not the patent of any one nation, but rather mathematicians from different times and places have all made their own, unique contributions to its development. Thus, the history of mathematics shows the diversity of mathematical culture. It is through making this connection that history helps students "appreciate the scientific, practical, cultural and aesthetic values of mathematics." (Translation from Ministry of Education, 2017, p. 2)

(f) The *availability of moral education*. Moral education and talent cultivation is the fundamental task of education in mainland China, to which the objectives of all subject curricula are subservient, mathematics being no exception. History reveals that mathematics is a human cultural endeavor and that mathematicians, with their industry, perseverance, truth-seeking, etc., left us a precious spiritual treasure, which deserves to be passed down to future students. Due to this connection, the history of mathematics is indispensable for moral education.

The detailed indications of each dimension and the corresponding arguments from literature are shown in Table 4.

4 Method

4.1 Samples

We selected twenty published IHT cases[1] on high school mathematics as samples, which were published during the past decade (2007–2016). We selected these cases because the first author of this paper took part in discussions or acted as an advisor in the course of instructional design and lesson studies. The information on these cases is given in Table 5.

4.2 Sample Production and Structure

The IHT cases were produced after several practices and discussions. They are the results of cooperation among university researchers and in-service high school mathematics teachers, who makeup an HPM community. Figure 1 shows the procedure of the development of IHT cases (Wang 2017).

Of the twenty cases, cases 4, 6, 8, 9 and 13 were developed by a senior high school mathematics studio[2] that plays an important role in the HPM community.

Take case 8 as an example. First, the senior teacher, Jin, encountered a problem in her prior teaching: many students merely memorized the formula such as $\log_a M + \log_a N = \log_a (MN)$ (M,

[1] An IHT case refers to a developed paper on a completed IHT teaching.

[2] A mathematics studio is a learning community whose aim is to help its members develop their profession in teaching. One expert and qualified teacher is the leader of this community. All members from various schools are experienced teachers, who are often highly motivated with the desire for promotion. The mathematics studios are common in Mainland China.

Table 4 A categorical framework for the educational values of the history of mathematics for students

Dimension	Instruction	Categories from literature
Harmony of knowledge	(1) History shows the natural genesis and evolution of mathematical concepts, formulas, theorems and ideas and contributes to students' understanding; (2) History makes mathematics more concrete and gives students greater insight; (3) History uncovers why and how mathematical concepts, formulas, theorems and ideas were invented or discovered and supplies motivation for learning; (4) Teaching inspired by history is aligned with students' cognitive development; (5) History helps students identify and overcome the obstacles or difficulties they encounter in learning; (6) History helps students understand the evolution of mathematical language, symbols, terminology, representations, etc.	F1, F4, F9, TA1, TA6, GB1, J5–8
Beauty of ideas or methods	(1) Historical problems provide alternative methods of solution and make students pursue creative thinking; (2) By comparing ancient and modern techniques, students appreciate the advantages or values of the latter; (3) Historical ideas or methods, which cannot be found in today's textbooks, broaden students' view or thinking.	F6, GB3, GB5
Pleasure of inquiries	(1) History provides students with opportunities for investigations; (2) History provides a vast reservoir of questions and problems which can motivate, interest, and engage students.	F8, TA2
Improvement of capabilities	(1) History helps students learn in a non-linear way; (2) Historical problems help improve competencies in abstraction, operation, logical reasoning, visual imagination and modeling; (3) History can help improve the ability to use various representations (e.g. algebraic and geometrical ones) to present the same ideas; (4) History promotes students' general skills, such as reading, writing, looking for resources, documenting, discussing, analyzing, and communicating mathematically; (5) History encourages quicker learners to look further.	F11, TA4, GB4, GB12
Charm of culture	(1) History makes mathematics less frightening, more enjoyable, and exciting; (2) History gives mathematics a human face; (3) History helps to explain the role of mathematics in society; (4) History changes students' perceptions of mathematics; (5) History shows the multi-culture of mathematics; (6) History exposes interrelations between mathematics and other subjects; (7) History reveals that mathematics is an evolving and human subject rather than a system of rigid truths; (8) History informs that mathematics is influenced by social and cultural factors; (9) History indicates that mistakes, heuristic arguments, uncertainties, doubts, intuitive arguments, blind alleys, controversies and alternative approaches to problems are not only legitimate but also an integral part of mathematics in the making.	F2, F5, F7, F12, F13, F15, TA3, TA5, TA7, TA10–12, GB 6–7, J2–3, J 9–13

Table 4 (continued)

Dimension	Instruction	Categories from literature
Availability of moral education	(1) History stimulates students' interest in mathematics; (2) Students derive comfort from realizing that they are not the only ones with problems; (3) History tells students of the values of persisting with ideas, of attempting to undertake lines of inquiry, of posing questions, and of attempting to develop creative ways of thought; (4) History informs students that they should not get discouraged by failure, mistakes, uncertainties, or misunderstandings; (5) "Dialogue with mathematicians through time and space" establishes students' confidence in learning mathematics. (6) History develops tolerance and respect among fellow students. (7) Knowing about female mathematicians in history, girls may be stimulated in their study of mathematics.	F10, F14, TA8, TA9, TA12, GB2, GB8–11, J1, J4

$N > 0$, $a > 0$, $a \neq 1$) which would be mistaken as $\log_a (M + N) = \log_a M \times \log_a N$ after several weeks. In addition, many students did not understand why they should learn logarithms. As Fauvel (1995) mentioned, "What are logarithms for, nowadays? Why are they still on the syllabus? This is where ideas drawn from the historical development can fruitfully be employed." (p. 45) So, Jin and the researchers (the authors) cooperated to address students' difficulties with

Table 5 Information on the 20 IHT cases on senior high school mathematics

Case	Topic	Area	Literature[1]
1	The extension of number system and introduction of complex numbers (I)	Algebra	Zhang and Wang 2007
2	The addition formula	Trigonometry	Zhang 2007
3	The arithmetic-geometric mean (hereafter referred to as AM-GM) inequality	Algebra	Zhang 2012
4	The concept of ellipse	Analytic geometry	Chen and Wang 2012
5	The geometrical meaning of derivative	Calculus	Wang and Wang 2012
6	The extension of number system and introduction of complex numbers (II)	Algebra	Fang and Wang 2013
7	The application of derivative	Calculus	Wang and Wang 2014
8	The concept of logarithms (I)	Algebra	Jin and Wang 2014
9	The concept of prism	Solid geometry	Chen 2015a
10	The zeros of a function and the roots of an equation	Algebra	Chen 2015b
11	The law of cosines	Trigonometry	Gu and Wang 2015
12	The concept of recursive sequences	Algebra	Li 2015
13	The concept of parabola	Analytic geometry	Xu 2015
14	The concept of logarithms (II)	Algebra	Zhong and Wang 2015
15	The law of sines	Trigonometry	Zhang and Wang 2015
16	The binomial theorem	Algebra	Fang 2016
17	The concept of number sequences	Algebra	Li and Wang 2016
18	The curve and its equation	Analytic geometry	Shi 2016
19	The concept of functions	Algebra	Zhong and Wang 2016
20	The angle of inclination and the slope of a line	Analytic geometry	Yang 2016

[1] The first author of the literature is the teacher who actually gave the corresponding lesson

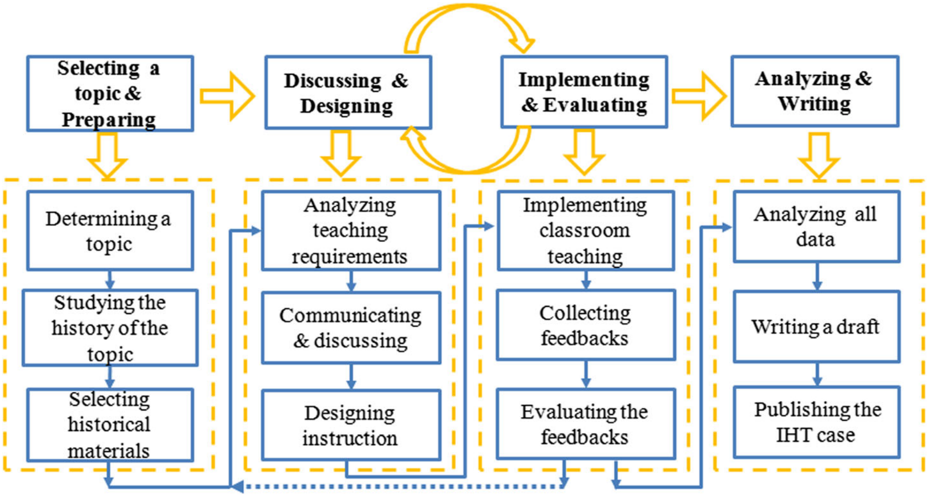


Fig. 1 The procedure of the development of IHT cases

logarithms. With the help of the researchers, Jin began reading historical materials on logarithms and specially selected some to use in class based on her students' situations.

Second, Jin worked together with the members of a senior high school mathematics studio and several researchers to design instruction on the concept of logarithms. In accord with the historical development of logarithms, the first draft of instructional design was completed (see Fig. 2).

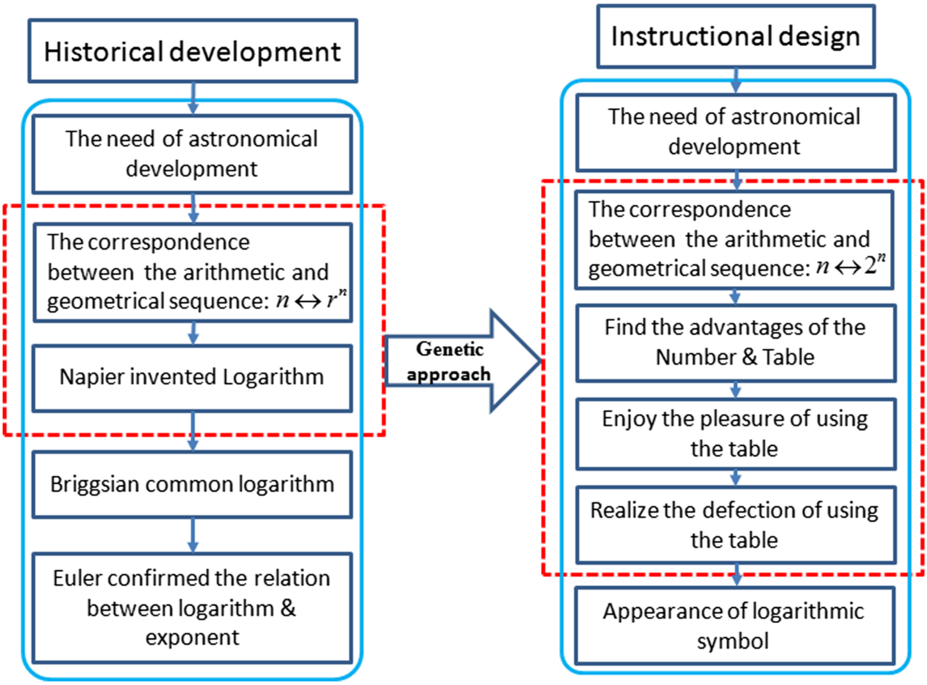


Fig. 2 The roadmap of IHT instructional design on the concept of logarithms

Third, Jin implemented the instruction and others began to observe and evaluate the teaching. Students, teachers and researchers gave their feedback, then, the design was modified according to their feedback. After a third round of this, other teachers and researchers gave more positive feedback, and a relatively satisfactory design was formed. Lastly, based on the feedback from students, teachers, and researchers, Jin modified the instructional design and finished a draft of the IHT case, which was later revised by researchers.

Generally, a published IHT case consists of five sections: the introduction (why the perspective of HPM is adopted), historical materials and their use, the classroom record, the students' feedback collected by means of a questionnaire survey and interviews, and the teacher's reflections.

4.3 Code and Analysis

In the senior high school textbook *Mathematics* (Optional I-2) (People Education Press 2007) and widely used in mainland China, the problem of solving the quadratic equation $x^2 + 1 = 0$ is used to introduce the concept of imaginary numbers: to make the equation solvable, a new number i is introduced so that $i^2 = -1$. The idea of making an equation with no real roots to have roots is consistent with the logical order of the extension of the number system. However, it does not accord with the history of imaginary numbers, nor with students' cognitive basis. Based on this, we take cases 1 and 6 as examples to illustrate how to analyze the IHT cases based on the new framework.

In case 1, the problem of solving the system of quadratic equations $\{ x^2 + y^2 = 2 \ xy = 2$, which baffled G. W. Leibniz (1646–1716), was used to confront students with a dilemma. In case 6, the problem of dividing 10 into two parts whose product is 40, which was solved by G. Cardano (1501–1576), was used to challenge students. In these two problems, the sum of two numbers is a real number, but neither of them are real numbers. In case 6, the cubic equation $x^3 = 15\times + 4$, which was solved by R. Bombelli (1526–1572), was employed to create a strong cognitive conflict on a seemingly impossible equation: $\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121} = 4}$. This situation revealed the necessity of introducing the concept of the imaginary number in order to motivate students to learn the new type of numbers. With the history as a reference, the emergence of imaginary numbers in the class became a natural process. In case 6, the origin of the term "imaginary number" and of the symbol "i" was also explained so that students would not mistake imaginary numbers for "illusory numbers" from the literal meaning of imaginary. Therefore, the history reflects the *harmony of knowledge* in these two cases.

In history, no mathematicians studied imaginary numbers or extended the number system because of the equation $x^2 + 1 = 0$. Mathematicians originally would neglect an equation if it did not have real roots. However, the three historical problems presented in two cases confronted mathematicians with a "paradox" that could not be neglected. One such case was that the sum of two numbers is a real number, but neither of them are real numbers. What are the numbers then? The other case was that all cubic equations have at least one real root, but so why then does the square root of a negative number appear when the root formula is used? This "contradiction" is the real motivation behind mathematicians probing into and accepting imaginary numbers. These two cases provided students with the opportunity to solve the "contradiction" encountered by mathematicians in history, bringing about the *pleasure of inquiries*.

The historical problems in cases 1 and 6 provided students with opportunities to train them in operation, logical reasoning, and visual imagination. Case 6 used the Gauss complex plane to help students improve their ability to transform algebraic and geometric representations on complex numbers. Worksheets in Case 1 helped students develop their reading, analyzing and communicating skills. These cases show the *improvement of capabilities.*

Both cases impressed upon students a sense of history and helped them better understand the motivation behind creating imaginary numbers. The two cases reconstructed the work of Cardano, Bombelli, Leibniz, Gauss, Euler and other mathematicians so that the content could become more humanized. Ingenious applications of "imaginary numbers" in the history of mathematics made students understand the value of mathematics and the close relationship between mathematics and other disciplines. Euler's formula presented the beauty and magnificence of mathematics. All historical materials in the classroom jointly revealed the *charm of culture* that can be encountered in imaginary numbers.

The birth of imaginary numbers fully embodied mathematicians' adherence to the truth, tireless exploration of the unknown and persistent pursuit of an innovative spirit. By reconstructing in the classroom the production of imaginary numbers in history, these two cases gave students the chance to appreciate the spirit of these mathematicians. The history of mathematics tells students that the birth and development of mathematical concepts are full of twists, turns and hardships, as mathematicians' process of acceptance of imaginary numbers was very slow. The fact that it was an arduous process can be an inspiration to students. Although we stand on the shoulders of giants to study mathematics today, the difficulties, doubts, and frustrations that students face while learning mathematics are the very same ones great mathematicians met in the past. Therefore, students can feel confident and comfortable in their studies by observing the character of those who have come before them. In this way, the value of the *availability of moral education* can be seen in these two cases.

Two of the authors independently coded the 20 IHT cases according to the new categorization framework. They reached 100% agreement in the categories of the *harmony of knowledge,* the *charm of culture,* and the *availability of moral education*. Only 6 codes from the *beauty of ideas or methods* and the *pleasure of inquiries* were different. The discrepancy codes were resolved after they read the students' feedback from all the IHT cases.

5 Results

Table 6 shows the distribution of the six categories of the educational values that all IHT cases present. All cases revealed the *harmony of knowledge*, the *improvement of capabilities*, the *charm of culture,* and the *availability of moral education*. As Table 6 shown, 35% of the cases showed the *beauty of ideas or methods*, and 75% of the cases supported the *pleasure of inquiries*.

5.1 The Harmony of Knowledge

The instructional design of case 19 is based on the history of the concept of functions. A questionnaire administered before teaching revealed that most students' conceptualized function with an analytic expression (see Fig. 3).

A chain of problems was adopted to show the deficiency of old definitions and the necessity of new ones. The relation between the index and time (Fig. 4) shows the deficiency of the "analytic expression" definition. The problem of whether $y = 0$ ($x \in R$) is a function (to which, according to the questionnaire, 40% of subjects gave negative answers) reveals the deficiency

Table 6 Distribution of six types of values that 20 ITH cases present

Area	Case #	Educational values of the history of mathematics					
		HK	BIM	PI	IC	CC	AME
Algebra	1	✓		✓	✓	✓	✓
	3	✓	✓	✓	✓	✓	✓
	6	✓		✓	✓	✓	✓
	8	✓		✓	✓	✓	✓
	10	✓			✓	✓	✓
	12	✓		✓	✓	✓	✓
	14	✓		✓	✓	✓	✓
	16	✓	✓		✓	✓	✓
	17	✓			✓	✓	✓
	19	✓		✓	✓	✓	✓
Solid Geometry	9	✓		✓	✓	✓	✓
Trigonometry	2	✓	✓	✓	✓	✓	✓
	11	✓	✓	✓	✓	✓	✓
	15	✓	✓	✓	✓	✓	✓
Analytical geometry	4	✓		✓	✓	✓	✓
	13	✓		✓	✓	✓	✓
	18	✓	✓		✓	✓	✓
	20	✓			✓	✓	✓
Calculus	5	✓		✓	✓	✓	✓
	7	✓	✓	✓	✓	✓	✓
Total		20	7	15	20	20	20

Note. HK = Harmony of knowledge, BIM = Beautify of ideas or methods, PI = Pleasure of inquiries, IC = Improvement of capabilities, CC = Charm of culture, AME = Availability of moral education

of the "variable dependency" definition given in the mathematics textbook of grade 8 (Shanghai Education Press 2015) and used only in Shanghai. Thus, the students were able to experience the evolution of the concept of functions from an analytic expression to the dependency relation between variables, then to correspondence relation between variables, which shows the *harmony of knowledge*.

In cases 8 and 14, students were first asked to calculate the product of different positive integer powers of 2, and then a table of powers of 2 with corresponding exponents was used to simplify the calculation, converting multiplication to addition. Next, students were asked to find the product of two large numbers 299,792.458 and 31,536,000, which are the speed of light in a vacuum and the number of seconds in one year, respectively. The size of these numbers made the table inutile. So then, the students were led to ask the question of whether the two large numbers can be converted to powers of 2. Finally, the concept of logarithms was introduced. The history of logarithms was implicitly reproduced so that the

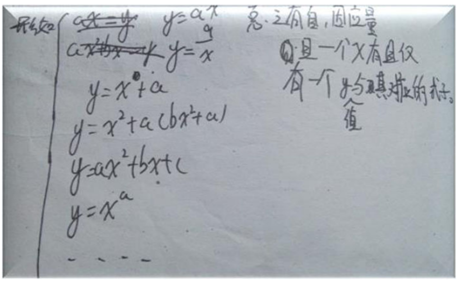


Fig. 3 The "analytic expression" definition of functions given by a 10th grader before learning the new definition

Fig. 4 The graph of Shanghai-Shenzhen 300 index showing the deficiency of the "analytic expression" definition

introduction of the concept of logarithms became natural and also paved a way for logarithm operation later on.

The equation of a straight line appeared in the seventeenth century when analytic geometry was invented, and the concept of slope was coined in the nineteenth century (see Fig. 5). Nowadays, the order of presentation on equation and slope of a straight line in textbooks is reverted. In case 20, based on both history and students' cognitive starting point, teachers naturally lead students to the concept of slope by exploring the geometric meaning of the parameters in the linear function.

The concept of a tangent line to a curve evolved from a static definition to a dynamic and analytic one. In case 5, the teacher built a bridge between the static and dynamic definitions by applying Liu Hui's Cyclotomic Rule. This Rule was used by Liu Hui to compute the area of a

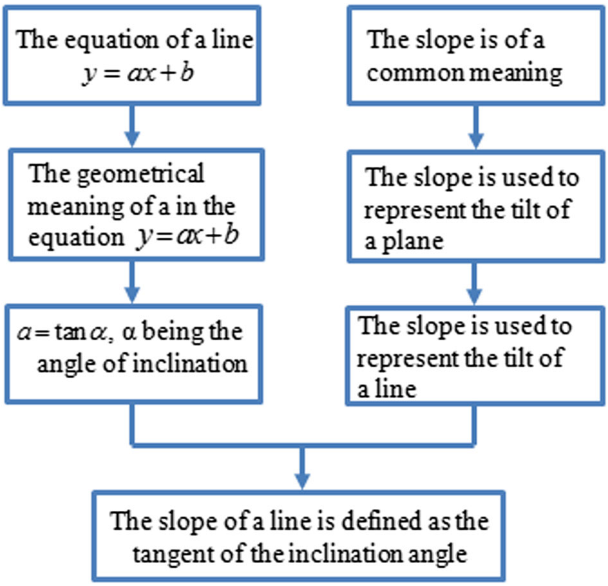


Fig. 5 The evolution of the concept of slope

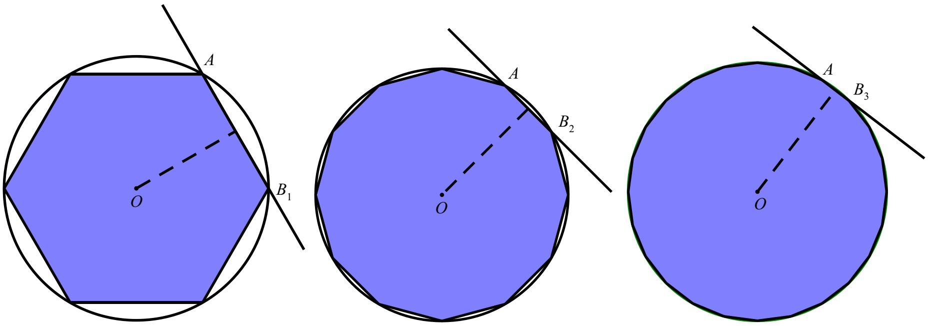


Fig. 6 Liu Hui's Cyclotomic Rule is used to build a bridge between the static and dynamic definitions of the tangent line of a circle

circle by a sequence of inscribed regular polygons with a doubly increasing number of sides (Fig. 6), making the transition natural and harmonious.

As shown in Table 6, the *harmony of knowledge* is incorporated not only in cases of concepts, but also in those on theorems and applications of concepts.

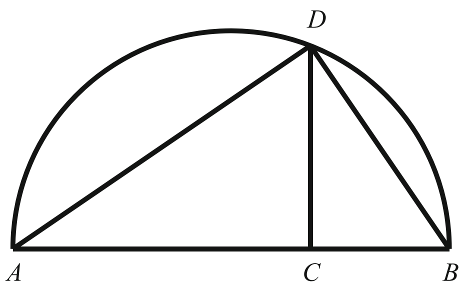
5.2 The Beauty of Ideas or Methods

In case 3, several historical methods were used to derive the AM-GM inequality. First, students were asked to prove Euclid's proposition VI.13 (to find a mean proportional between two given straight lines). In Fig. 7, *AC* and *CB* are two given segments, and *CD* is perpendicular to *AB* and cuts through the semicircle with the diameter of *AB*. *CD*, then, is a geometrical mean between *AC* and *CB*. Students were then asked to construct an arithmetical mean between *AC* and *CB* and compare the two means.

Second, given the methods used by J. Wallis (1616–1703) and N. H. Abel (1802–1829) to prove the proposition on isoperimetric rectangles, students were asked to use these methods to prove the AM-GM inequality.

> Proof 1 (Wallis' Method): Suppose that $a + b = 2A$, $ab = G^2$, then $a\ (2A–a) = G^2$, i.e., $a\ (2A–a) = a^2 - 2A–a + G^2$. From $\Delta\ 4A^2 - 4G^2 \geq 0$, we have $A \geq G$.
> Proof 2 (Abel's Method): Suppose that $a + b = 2A$, $ab = G^2$, then $a = A + x$, $b = A - x$, $A^2 - x^2 = G^2$. Hence $A \geq G$.

Fig. 7 Euclid's Proposition VI.13



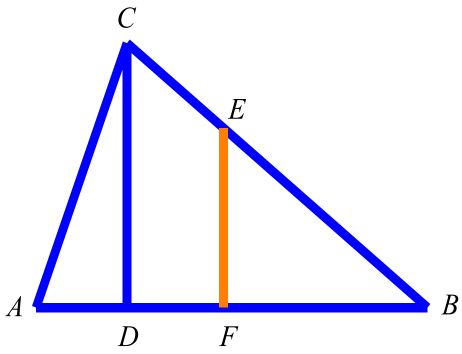


Fig. 8 W. D. Mei's proof of the law of sines

In case 15, the same-diameter method of W. D. Mei (1633–1721), who was a Chinese mathematician of the early Qing Dynasty and the circumscribed circle method of F. Viète (1540–1603) were used to prove the law of sines. In triangle ABC (see Fig. 8), $BC > AC$. Take a point E on the side BC so that $BE = AC$, and draw perpendiculars CD and EF (Mei 1994), then

$$\sin A : \sin B = \frac{CD}{AC} : \frac{EF}{BE} = CD : EF = BC : BE = BC : AC = a : b$$

Mei's method, which helps students visualize the law of sines, is better than that given in the senior high textbook *Required Mathematics* 5 (People Education Press 2007).

F. Viète's proof (see von Braunmühl 1900, pp. 176–177) is shown in Fig. 9. From

$$a = 2BD = 2R \sin BOD = 2R \sin A, b = 2AE = 2R \sin AOE = 2R \sin B,$$

$$a = 2AF = 2R \sin AOF = 2R \sin C, \textit{we have } a : \sin A = b : \sin B = c : \sin C = 2R.$$

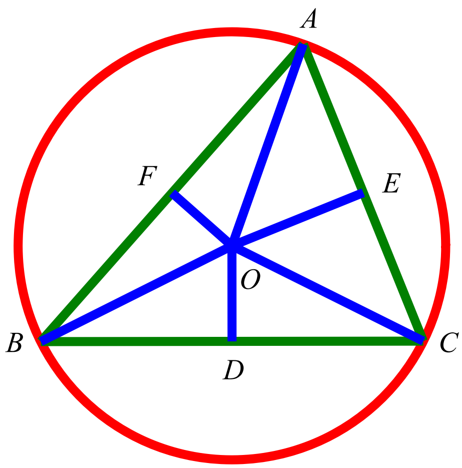


Fig. 9 F. Viète's proof of the law of sines

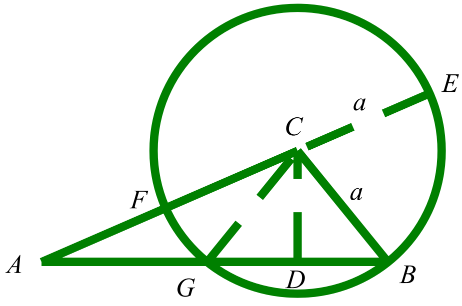


Fig. 10 Viète's geometrical proof of the law of cosines

In case 11, the Euclidean area method, Viète's auxiliary circle method (Fig. 10) was used to prove the law of cosines. Given $AF \times AE = AG \times AB$, we have $(b + a)\,(b - a) = c\,(c - 2a \cos B)$, $b^2 = a^2 + c^2 - 2\,ac \cos B$.

In case 16, the method of De Castillon (1708–1791) was used to prove the binomial theorem. This method made it easier for students to understand the relationship between expansion coefficients and combinatorial numbers.

As seen in Table 6, the history of mathematics best embodies the educational value of the *beauty of ideas or methods* in the teaching of formulas or theorems.

5.3 The Pleasure of Inquiries

In case 18, a series of ancient Greek locus problems were adapted for students to explore. At first, students were asked to identify the locus of a point that is at equal distance from two given lines, which could be solved by using the geometrical method. This problem is a special case of Apollonius's general one: to find the locus of a point whose distance from two given lines is a given ratio. Next, given three lines in a plane, two of which being perpendicular to the third one, and the product of the distances from a point to the first two lines being equal to the square of its distance to the third one, students were asked to find the locus of the point. This problem is a special case of the Greek three-line problem. While the students were at a loss with the geometrical method, the analytic method was introduced (see Fig. 11), which proved to be much more convenient and effective.

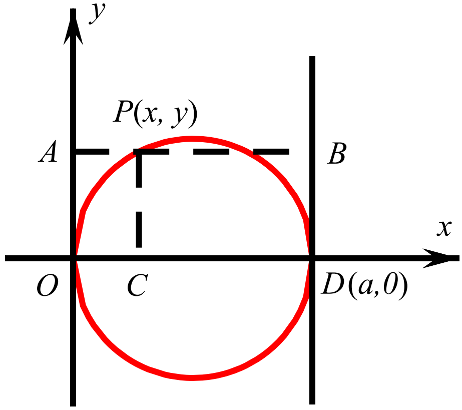


Fig. 11 A special case of the ancient Greek "three-line" problem

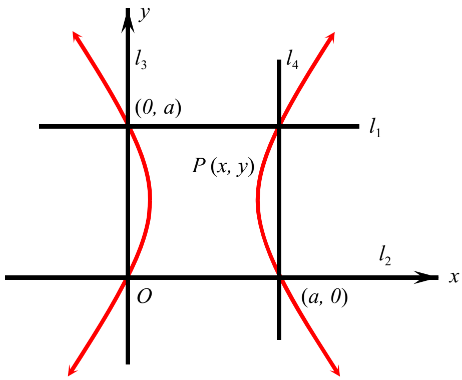


Fig. 12 A special case of the ancient Greek “four-line” problem

At last, a special case of the Greek “four-line” problem was posed for students to apply the analytic method (see Fig. 12).

Three different locus problems form an indivisible whole. Through the analytic method, students can realize the superiority and great value of the combination of algebra and geometry. In this way, students experienced the birth of analytic geometry.

In case 12, students “deduced” a recursive formula through operating the “Tower of Hanoi”. Case 4 started with an experiment that produced the shadow of a ball under torchlight (see Fig. 13), then led students to observe the model of a single sphere inscribed in a cylinder (see Fig. 14).

Next, students were led to observe the model of double spheres inscribed in a cylinder through hands-on activities (see Fig. 15). Finally, students were asked to probe into the property of the focus radius of the ellipse, leading them to discover the locus definition of ellipse that is given in the textbook. It was during the process of exploration that students discovered for themselves what an ellipse is and understood the origin of the locus definition.

In case 13, an optical experiment was designed for students to find the focus of a parabola, then to find the consistency of a parabola with the graph of a quadratic function. The students were then led to find the directrix of the parabola and further derive the definition of a

Fig. 13 The experiment showing the shadow of a ball under the torchlight

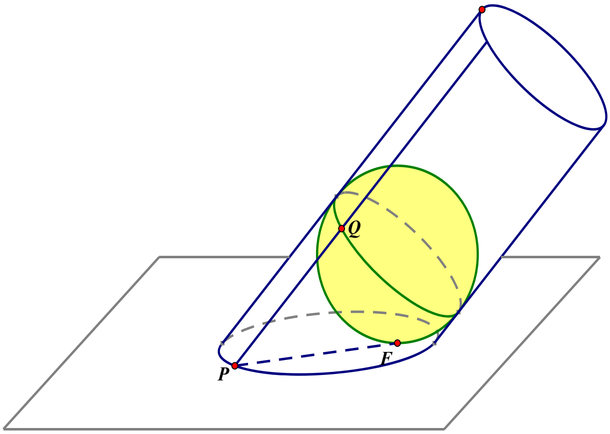


Fig. 14 The model of a single sphere inscribed in a cylinder

parabola. As Table 6 shows, in most IHT cases, the history of mathematics is shown to embody the educational value of the *pleasure of inquiries*.

5.4 The Improvement of Capabilities

In the history of trigonometry, various trigonometric formulas were born out of geometric propositions. Consequently, with the help of geometric figures, students can understand and memorize these formulas. In cases 2, 3, 11 and 15, the geometric method found in the history of mathematics helped develop students' competencies in visual imagination and also their capabilities of translation between trigonometric and geometric representation.

In case 20, three different representations of the concept of slope (geometric, symbolic, and trigonometric) helped cultivate students' ability to transform different representations.

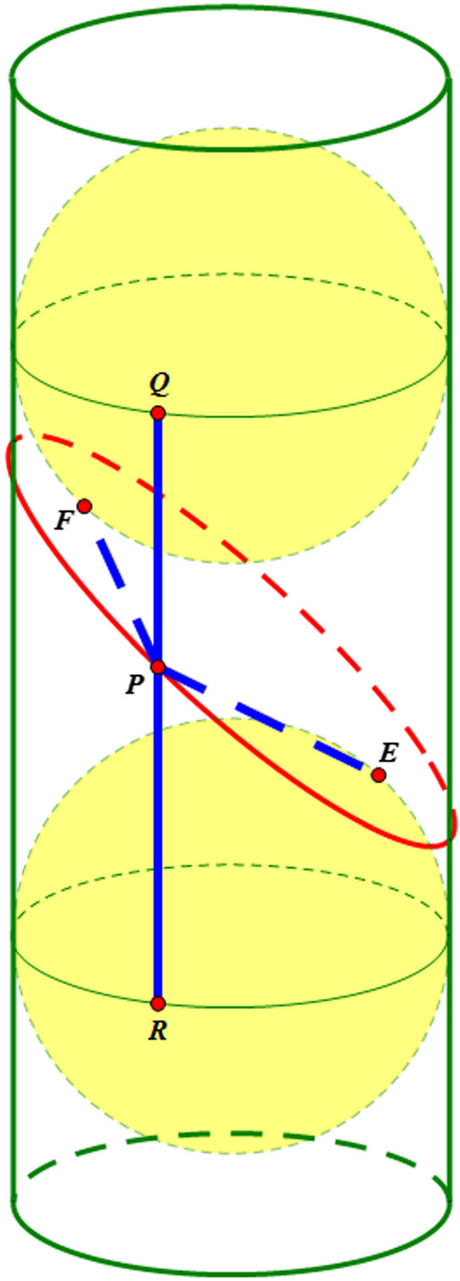
In addition, cases 1–3 used worksheets in the teaching. The worksheets contained a historical overview of knowledge, the mathematicians' ideas, and the discussion of historical problems. Worksheets can help students develop their reading skills.

5.5 The Charm of Culture

The history of mathematics embodied the educational value of the *charm of culture* in all 20 IHT cases. The interesting problems recorded by The Babylonian Tablets in case 14 and problems adapted from the seventeenth century, which involve the design of cans, wharf position, piping, and communication in case 7, are evidence of the close relationship between mathematics and real life. The problem of instantaneous velocity of curvilinear motion in case 5, which was one of the problems leading to the study of tangent lines of curves in the seventeenth century, and the derivation of the law of refraction in case 7, which was solved by G. W. Leibniz with the derivative, revealed the close connection between mathematics and physics.

The lunar phases table on a Babylonian tablet (Table 7), the story of the discovery of Ceres in case 17, and the problem of measuring the distance of the meteor in case 15 are typical examples of the application of mathematics in astronomy. Our results show that the utilization of these historical events in the cases provided students more opportunities to appreciate the *charm of culture.*

Fig. 15 The model of double spheres inscribed in a cylinder



5.6 The Availability of Moral Education

The stories in some of the cases stimulated students' interest in learning, such as the story about how Fibonacci participated in a mathematics contest at the court in case 10, the cat and mouse problem in the Rhind papyrus and the Joseph's problem in case 17, and the Tower of

Table 7 The Number Sequence Showing the Phase of the Moon in a Babylonian Tablet

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	10	20	40	80	96	112	128	144	160	176	192	208	224	240

Hanoi Game in case 12. Case 9 showed that Euclid's definition of the prism was wrong, informing students that mathematicians in history have also made mistakes. This can show students the right perspective on how to approach the difficulties or setbacks they encounter in learning while also cultivating their critical thinking skills.

In cases 8 and 14, the history of mathematics played important roles in carrying out moral education. Napier spent 20 years seeking the method of simplifying astronomical calculations and finally came to invent logarithms. Napier exhibited a character of strong will and responsibility, and this story thus fully embodies the perseverance of mathematicians. The covenant made between Napier and Briggs led to the birth of common logarithms, which fully reflects the importance of communication.

In cases 6 and 19, the slow, tortuous, and arduous development of the mathematical concepts helped students establish proper conception of mathematics.

In the IHT cases, mathematicians seemed to be "extra" students in the classroom, whereas students themselves seemed to become actual mathematicians doing research work. The dialogues across time and space helped students grow closer to mathematics, love mathematics, and become self-confident in their mathematics learning.

6 Concluding Remarks

The six dimensions of the new categorization framework correspond to the objectives of the senior high school mathematics curriculum as shown in Fig. 16.

Each dimension connects with one of the four aspects in the *Standards of Senior High School Mathematics Curriculum*. Other than contributing to students' acquisition and

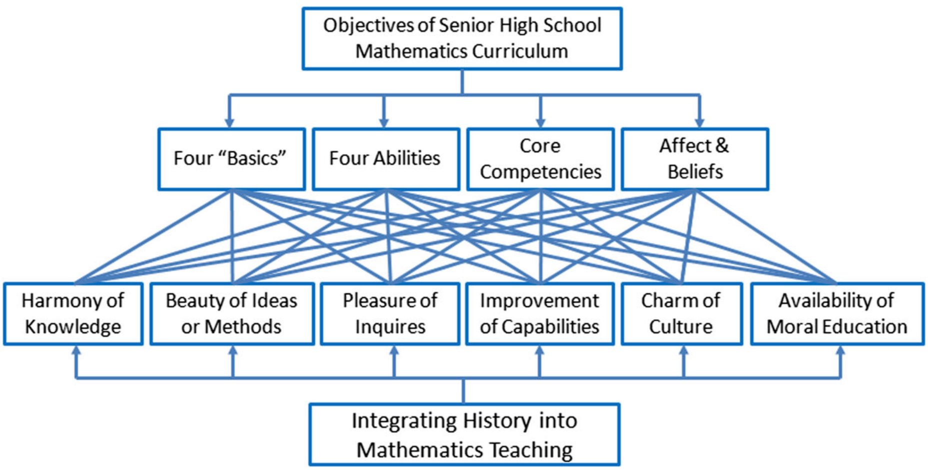


Fig. 16 The relations between the educational values of the history of mathematics and the objectives of the senior high school mathematics curriculum

understanding of basic knowledge, the harmony of knowledge also helps motivate students to discover, pose, analyze and solve problems. This reinforces their capability of abstraction and generalization and also establishes in them a confidence in learning mathematics.

The application of mathematical ideas or methods, which also belong to basic knowledge, helps improve students' capabilities of problem-solving, logical reasoning, and visual imagination, and establishes in them the habit of rational thinking. Inquiries into new knowledge give students the opportunity to apply basic knowledge, skills and ideas, train them to analyze and solve problems, foster their core competencies, allow them to attain feelings of success, and build up their self-confidence.

Improvement of capabilities contributes to the basic skills, the four abilities and the core competencies, and also helps enhance students' self-confidence and leads them to have a positive way of thinking about mathematics learning. According to Jankvist (2009), some aspects of mathematical culture are by themselves goals of learning, and thus the dimension "charm of culture" connects with basic mathematics knowledge. From the perspective of connections between mathematics and the real world or other subjects, students can be provided with the opportunity to improve their four abilities and six core competencies, to arouse or enhance their interest in mathematics, and to acquire a positive view of mathematics.

Moral education connects intimately with the objectives of affect and beliefs. However, it is also shown that positive emotion towards mathematics, good habits of study, and a spirit of persistence, self-confidence, optimism, industry, and truth-seeking can benefit the other three aspects.

Generally, the *harmony of knowledge*, the *charm of culture*, and the *availability of moral education* were supported by all 20 IHT cases. The *beauty of ideas or methods* was mainly observed in IHT cases on formulas or theorems. The *pleasure of inquiries* was mainly seen in cases on concepts, formulas, or theorems. The *improvement of capabilities* was supported by all cases. However, the utilization of the history of mathematics is not the only way to improve students' ability, and it is impossible that the history of mathematics in each case is the only factor that plays a unique role in this regard.

Research on the educational value of the history of mathematics is an important topic of HPM. The establishment of this categorization framework enriches the field of IHT research. It not only provides the theoretical guidance for IHT practices and the development of IHT cases, but also supplies the basis for the evaluation of IHT cases and the integration of the history of mathematics into mathematics textbooks. It is hoped that this framework can be a reference for future studies on the implementation of moral education and talent cultivation in mathematics curriculum.

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Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

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