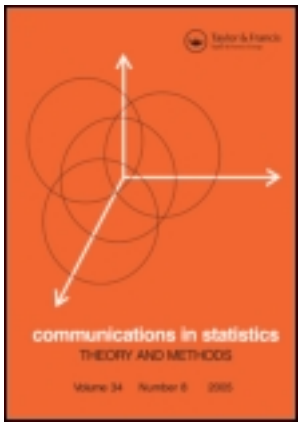

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Optimum Design for Type-I Step-stress Accelerated Life Tests of Two-parameter Weibull Distributions

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In this article, we focus on the general k-step step-stress accelerated life tests with Type-I censoring for two-parameter Weibull distributions based on the tampered failure rate (TFR) model. We get the optimum design for the tests under the criterion of the minimization of the asymptotic variance of the maximum likelihood estimate of the pth percentile of the lifetime under the normal operating conditions. Optimum test plans for the simple step-stress accelerated life tests under Type-I censoring are developed for the Weibull distribution and the exponential distribution in particular. Finally, an example is provided to illustrate the proposed design and a sensitivity analysis is conducted to investigate the robustness of the design.

Keywords *k*-step step-stress accelerated life test; Maximum likelihood estimation; TFR model; Type-I censoring; Weibull distribution.

Mathematics Subject Classification 62N05.

1. Introduction

Products are growing more and more reliable and, consequently, it has become a heavy burden in both testing time and expenditure for enterprises to assess the reliability of the products using traditional life testing methods, which on many occasions, are completely infeasible. As a result, accelerated life tests (ALTs) appeared in the 1960s, including the commonly used constant stress ALTs (CSALTs) and step-stress ALTS (SSALTs); see Nelson (2004) and Meeker and Escobar (1998) for the statistical analysis of these two types of ALTs. More general time-varying-stress ALTs were considered in Hong et al. (2010), and Benavides (2011). Turner (2010) and Fan and Wang (2011) discussed the statistical analysis of system ALTs.

In an ALT, products are tested under stresses that are higher than usual so that more failure data can be collected within a shorter time. Now consider a

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test in which n products undergo a stress level S_1 at time $\tau_0 = 0$, and from time $\tau_1 > 0$ the survivors of the n products will be confronted with a higher stress level $S_2(> S_1)$. In general, from time $\tau_l > \tau_{l-1}$, $l = 1, 2, \ldots, k-1$, the survivors under the previous stress level S_l will go on to be tested under a much higher stress level $S_{l+1} > S_l$. Such kind of a test is usually called k-step SSALT. In an SSALT, $S_1 < S_2 < \cdots < S_k$ are called step stress levels which are usually higher than S_0, the stress level under normal operating conditions. If $k = 2$, the test is called a simple step-stress accelerated life test. For a k-step SSALT, the test can be terminated at a predetermined time τ, which can be infinity to get a complete sample.

Up until now, the existing literature on the analysis of SSALTs uses three types of accelerated tampered models: (1) the tampered random variable (TRV) model proposed by DeGroot and Goel (1979); (2) the cumulative exposure (CE) model given by Nelson (1980); and (3) the tampered failure rate (TFR) model proposed by Bhattacharyya and Soejoeti (1989). These models show the effect of the rise of stress levels on the lifetime of the products. Wang and Fei (2004) gave the condition for the coincidence of them. There are a great number of results of ALTs based on the CE model; see, for example, Nelson (2004). The TFR model is no so popular as the CE model in reliability engineering. However, it is closely related to the proportional hazard (PH) model often used in biostattistics; the hazard rate function in the TFR model under the normal working conditions is similar to the underlying hazard function in the PH model (see Sec. 2) and the stresses in the the TFR model are similar to the covariates in the PH model. It is easy to see that the experiments under the PH model in biostatistics are actually the same as the constant stress ALTs under the TFR model in reliability. Khamis and Higgins (1998) proposed a so-called KH model for Weibull distributions based on a time transformation of the exponential CE model. Xu and Tang (2003) pointed out that the KH model is just a two-step TFR model for Weibull distributions. The maximum likelihood estimates for the parameters in the SSALT for Weibull distributions and log-linear accelerated functions under the TFR model were given in Khamis and Higgins (1998). A special discussion under the inverse power law was given by Tang (2006).

During the past two decades, the problem of optimal scheduling of the ALT has attracted great attention in the reliability literature. The optimum design for SSALT under CE model with Weibull distributions have been studied by many authors, such as Bai and Chung (1992) and Bai and Kim (1993). Recently, Hong et al. (2010) obtained a test plan based on a log-location-scale distribution with censoring and time-varying stress. Ma and Meeker (2010) gave a strategy for planning ALTs with small sample sizes. Yang (2010) considered ALT plans for predicting warranty cost. Liu and Tang (2010) proposed a Bayesian approach to planning an ALT for repairable systems. Liu and Qiu presented a method for planning multiple-step step-stress ALT with statistically independent competing risks. All these results are all discussed under the CE model. However, there is little discussion about the optimum design under the TFR model. The optimum design for simple SSALTs under TFR model with complete sample from Weibull distributions was presented by Alhadeed and Yang (2002). We will use the same optimum criterion and same procedure as in Alhadeed and Yang (2002) but in a more general way; we consider planning the general Type-I k-step SSALTs for Weibull distributions, with not only the times of changing stress levels but also the changing stress levels as the optimum variables.

The rest of this article is organized as follows. Section 2 gives some basic assumptions and some fundamental results. Section 3 gives the optimum design

for the Type-I k-step SSALT for Weibull distributions. Section 4 discusses the special case for $k = 2$, i.e. optimum design for simple SSALT. The result for the optimum time of changing stress in the simple SSALT for the exponential case is also given. In Section 5, an example is provided to illustrate the proposed design with a sensitivity analysis to investigate the robustness of the design. Finally, some concluding remarks are made in Section 6.

2. Basic Assumptions and Related Conclusions

For the Weibull SSALT, we need four basic assumptions.

Assumption 1 (Stress Levels)**.** The SSALT is done under the ordered stress levels, $S_1 < S_2 < \cdots < S_k$.

Assumption 2 (Lifetime Distribution)**.** The lifetime under a constant stress S_1 follows the Weibull distribution $Wei(\theta_1, \beta)$ survival function (Sf)

$$\overline{F}(t) = \exp\left\{-\left(\frac{t}{\theta_1}\right)^{\beta}\right\}, \quad t > 0, \tag{1}$$

where $\theta_1 > 0$ and $\beta > 0$ are the scale parameter(or characteristic life) and the shape parameter respectively.

Assumption 3 (Accelerated Tampered Model)**.** The hazard rate function(HRf) at a higher stress level is the HRf at a lower stress level multiplied by a unknown factor; that is, the HRf for the two-step SSALT is assumed to be

$$\lambda^*(t) = \begin{cases} \lambda(t), & 0 \le t \le \tau_1 \\ \alpha_1 \lambda(t), & t > \tau_1 \end{cases}, \tag{2}$$

where $\lambda(t)$ is the HRf under S_1, τ_1 is the time at which the stress is changed to S_2. $\alpha_1 = \alpha(S_2, S_1)$ is called the accumulated tampered factor (ATF). It is determined by S_1 and $S_2(> S_1)$, and probably is related to τ_1. This assumption is referred to as the TFR model. The TFR model (2) can be generalized from two-step SSALT to multiple-step SSALT (Madi, 1993). The HRf for the k-step SSALT is assumed to be

$$\lambda^*(t) = \alpha_{j-1}\lambda(t), \quad \tau_{j-1} \le t < \tau_j \quad j = 1, \ldots, k, \tag{3}$$

where $\tau_0 = 0, \tau_k = \infty, \alpha_0 = 1$, $\lambda(t)$ is the HRf under S_1, τ_{j-1} is the stress changing time point at which the stress is changed to S_j, and $\alpha_{j-1} = \alpha_{j-1}(S_j, S_1), j = 1, 2, \ldots, k$ is ATF, which is determined by S_1 and $S_j(> S_1)$, and independent of $S_2, S_3, \ldots, S_{j-1}$, but probably related to the time point τ_{j-1}.

Assumption 4 (Accelerated Function)**.** The characteristic life θ under a constant stress level S has a log-linear form

$$\ln \theta = a + b\varphi(S), \tag{4}$$

where $\varphi(\cdot)$ is a given increasing function of S and $b < 0$.

Based on the assumptions above we have the following propositions from Xu and Tang (2003) and Tang (2006).

Proposition 2.1. *Under Assumptions* 1–3:

1. *the lifetime under a constant stress* $S(> S_1)$ *follows the Weibull distribution with Sf*

$$\overline{F}(t) = \exp\left\{-\left(\frac{t}{\theta(S)}\right)^{\beta}\right\}, \tag{5}$$

where $\theta(S) = \theta_1/(\alpha(S, S_1))^{1/\beta}$;

2. *the lifetime under constant stress* $S_j (j = 1, 2, \ldots, k)$ *follows the Weibull distribution with Sf*

$$\overline{F_j}(t) = \exp\left\{-\left(\frac{t}{\theta_j}\right)^{\beta}\right\}, \quad j = 1, 2, \ldots, k, \tag{6}$$

where $\theta_j = \theta_1/(\alpha_{j-1})^{1/\beta}$, $j = 1, 2, \ldots, k$.

Proposition 2.2. *Under Assumptions* 1–3, *the Sf for the k-step SSALT is*

$$\overline{F}^*(t) = \exp\left\{-\sum_{i=-1}^{j-2}\alpha_i\left[\left(\frac{\tau_{i+1}}{\theta_1}\right)^{\beta} - \left(\frac{\tau_i}{\theta_1}\right)^{\beta}\right] - \alpha_{j-1}\left[\left(\frac{t}{\theta_1}\right)^{\beta} - \left(\frac{\tau_{j-1}}{\theta_1}\right)^{\beta}\right]\right\}, \tag{7}$$
$$\tau_{j-1} \le t < \tau_j, j = 1, 2, \ldots, k,$$

where $\alpha_{-1} = \alpha_0 = 0$, $\tau_{-1} = \tau_0 = 0$, $\tau_k = \infty$. *Thus, its probability density function is*

$$f^*(t) = \sum_{j=1}^{k} f_j^*(t) I_{[\tau_{j-1}, \tau_j)}(t), \tag{8}$$

where

$$f_j^*(t) = \frac{\beta}{\theta_j^{\beta}} t^{\beta-1} \exp\left\{-\sum_{i=-1}^{j-2}\alpha_i\left[\left(\frac{\tau_{i+1}}{\theta_1}\right)^{\beta} - \left(\frac{\tau_i}{\theta_1}\right)^{\beta}\right] - \alpha_{j-1}\left[\left(\frac{t}{\theta_1}\right)^{\beta} - \left(\frac{\tau_{j-1}}{\theta_1}\right)^{\beta}\right]\right\}. \tag{9}$$

Proposition 2.3. *Under Assumptions* 1–4,

1. *the ATF of constant stress S relevant to stress* $S_1 (S > S_1)$, $\alpha(S, S_1)$, *satisfies the log-linear relationship*

$$\alpha(S, S_1) = \exp\{b\beta[\varphi(S_1) - \varphi(S)]\}; \tag{10}$$

2. *for the k-step SSALT, the ATF* $\alpha_j, j = 1, 2, \ldots, k-1$ *satisfies the log-linear relationship*

$$\alpha_j = \exp\left\{b\beta[\varphi(S_1) - \varphi(S_{j+1})]\right\}. \tag{11}$$

3. Optimum Test Plan for k-step SSALT under Type I Censoring

3.1. *Optimality Criterion*

There are a lot of methods to develop optimum SSALT plans by suitably choosing test length, allocation of test units and stress levels. The fractile of the failure time distribution is an important characteristic and indispensable in reliability analysis. In step-tress setting, we need to estimate the fractile of lifetime at the use stress with maximum precision. Thus, the criterion for choosing a SSALT plan is often to find a plan that gives the minimum asymptotic variance of the the maximum likelihood estimate (mle) of a particular fractile. The criterion is widely used in the planing of ALT literature; see Bai and Kim (1993), Nelson (2004), and Liu and Qiu (2011). In general, the principle of optimum plan for the k-step SSALT under Type I censoring is that given the range of stress $[S_L, S_U]$, the higher stress S_k, the censoring time τ and sample size n, we should decide $k-1$ lower stress levels and $k-1$ stress changing time points. In practice, however, it would be too dificult to determine those $2(k-1)$ quantities simultaneously. Instead, the simpler alternative is to preassign the ordered stress levels, $S_1 < S_2 < \cdots < S_k$, and then decide how to allocate the time under each stress so that mle of the log of fractile p lifetime, $\ln \hat{t}_p$, has the highest accuracy under the use stress level S_0. In other words, we should decide the stress changing time points $\tau_j, j = 1, 2, \ldots, k-1 (0 \le \tau_1 \le \tau_2 \le \cdots \le \tau_{k-1} \le \tau)$ so that $AVar(\ln \hat{t}_p)$, the asymptotic variance of $\ln \hat{t}_p$, reaches the minimum. If not stated otherwise, we mean this simplified test plan in the later part of this section, though the procedure to get the objective function keeps the same as for the general test plan.

From Assumption 4, $\ln(\theta_i)$ is a linear function of the stress $\varphi(S_i)$

$$\ln \theta_i = a + b\varphi(S_i), \quad i = 1, 2, \ldots, k,$$

and the log of fractile p of the Weibull distribution at use condition S_0 is

$$\ln t_p = a + b\varphi(S_0) + \Phi^{-1}(p)\sigma,$$

where $\Phi^{-1}(p) = \ln(-\ln(1-p))$ is the fractile p of the standard extreme value distribution and $\sigma = 1/\beta$.

Thus, the optimum test plan that provides the "best" estimate of the model is the one that minimizes, with respect to $\tau_j, j = 1, 2, \ldots, k-1$, the asymptotic variance of the estimate of $\ln t_p$

$$AVar(\ln \hat{t}_p) = AVar(\hat{a} + \hat{b}\varphi(S_0) + \Phi^{-1}(p)\hat{\sigma}), \tag{12}$$

where $(\hat{a}, \hat{b}, \hat{\sigma})$ is the mles of (a, b, σ).

3.2. *The Likelihood Function and Fisher Information Matrix*

For the sake of simplification in deriving the Fisher information matrix, we use τ_k from now on to stand for τ, the termination time of the test. The likelihood function

for a single observation is

$$L(a, b, \beta) = \sum_{j=1}^{k} f_j^*(t) I_{[\tau_{j-1}, \tau_j)}(t) + I_{[\tau, \infty)}(t) \exp\left\{ -\sum_{i=0}^{k-1} \frac{\tau_{i+1}^\beta - \tau_i^\beta}{\theta_{i+1}^\beta} \right\},$$

where $f_j^*(t)$, $j = 1, 2, \ldots, k$ is given in (9). Then the log-likelihood function can be written as

$$\begin{aligned} l(a, b, \beta \mid t) = {} & I_{[0, \tau_1)}(t) \left\{ \ln \beta + (\beta - 1) \ln t - \beta \ln \theta_1 - \frac{t^\beta}{\theta_1^\beta} \right\} \\ & + \sum_{j=2}^{k} I_{[\tau_{j-1}, \tau_j)}(t) \left\{ \ln \beta + (\beta - 1) \ln t - \beta \ln \theta_j \right. \\ & \left. - \sum_{i=0}^{j-2} \frac{\tau_{i+1}^\beta - \tau_i^\beta}{\theta_{i+1}^\beta} - \frac{t^\beta - \tau_{j-1}^\beta}{\theta_j^\beta} \right\} \\ & + I_{[\tau, \infty)}(t) \left\{ -\sum_{i=0}^{k-1} \frac{\tau_{i+1}^\beta - \tau_i^\beta}{\theta_{i+1}^\beta} \right\}. \end{aligned} \tag{13}$$

Using the data from the k-step SSALT, we get the full likelihood function based on (13) and then get the maximum likelihood estimates of a, b, and β, say $\hat{a}$, $\hat{b}$, $\hat{\beta}$, as is done in Alhadeed and Yang (2002) and Tang (2006). To get $AVar(\ln \hat{t}_p)$, we need to find the Fisher information matrix of a, b and β first. We introduce the following functions:

$$\Gamma(x \mid \gamma) = \int_0^x \frac{1}{\Gamma(\gamma)} t^{\gamma - 1} \exp(-t) dt,$$

$$r(x) = \int_0^x u \ln(u) \exp(-u) du,$$

$$q(x) = \int_0^x u (\ln(u))^2 \exp(-u) du.$$

Furthermore, for $j = 1, \ldots, k$, let $\varphi_j = \varphi(S_j)$ and

$$r_j = r\left(\left(\frac{\tau_j}{\theta_j}\right)^\beta\right) - r\left(\left(\frac{\tau_{j-1}}{\theta_j}\right)^\beta\right),$$

$$q_j = q\left(\left(\frac{\tau_j}{\theta_j}\right)^\beta\right) - q\left(\left(\frac{\tau_{j-1}}{\theta_j}\right)^\beta\right),$$

$$g_j = \exp\left[-\sum_{i=0}^{j-2} \frac{\tau_{i+1}^\beta - \tau_i^\beta}{\theta_{i+1}^\beta} + \frac{\tau_{j-1}^\beta}{\theta_j^\beta} \right] \left\{ \Gamma\left(\frac{\tau_j^\beta}{\theta_j^\beta} \middle| 2 \right) - \Gamma\left(\frac{\tau_{j-1}^\beta}{\theta_j^\beta} \middle| 2 \right) \right\},$$

$$h_j = \frac{\theta_j^\beta}{\beta}\left(\frac{f_j^*(\tau_{j-1})}{\tau_{j-1}^{\beta-1}} - \frac{f_j^*(\tau_j)}{\tau_j^{\beta-1}}\right),$$

$$A_j = -1 + \sum_{i=0}^{j-2}\beta\left[\left(\frac{\tau_{i+1}}{\theta_{i+1}}\right)^\beta(\ln(\tau_{i+1}) - \ln(\theta_{i+1})) - \left(\frac{\tau_i}{\theta_{i+1}}\right)^\beta(\ln(\tau_i) - \ln(\theta_{i+1}))\right]$$
$$- \beta\left(\frac{\tau_{j-1}}{\theta_j}\right)^\beta(\ln(\tau_{j-1}) - \ln(\theta_j)),$$

$$A_j' = -\varphi_j + \sum_{i=0}^{j-2}\beta\varphi_{i+1}\left[\left(\frac{\tau_{i+1}}{\theta_{i+1}}\right)^\beta(\ln(\tau_{i+1}) - \ln(\theta_{i+1})) - \left(\frac{\tau_i}{\theta_{i+1}}\right)^\beta(\ln(\tau_i) - \ln(\theta_{i+1}))\right]$$
$$- \beta\varphi_j\left(\frac{\tau_{j-1}}{\theta_j}\right)^\beta(\ln(\tau_{j-1}) - \ln(\theta_j)),$$

$$C_j = -\frac{1}{\beta^2} - \sum_{i=0}^{j-2}\left[\left(\frac{\tau_{i+1}}{\theta_{i+1}}\right)^\beta(\ln(\tau_{i+1}) - \ln(\theta_{i+1}))^2 - \left(\frac{\tau_i}{\theta_{i+1}}\right)^\beta(\ln(\tau_i) - \ln(\theta_{i+1}))^2\right]$$
$$+ \left(\frac{\tau_{j-1}}{\theta_j}\right)^\beta(\ln(\tau_{j-1}) - \ln(\theta_j))^2.$$

Then by some algebraic computation, we get the Fisher information matrix of a, b, and β

$$F = n\begin{bmatrix} E\left(-\frac{\partial^2 l}{\partial a^2}\right) & E\left(-\frac{\partial^2 l}{\partial a\partial b}\right) & E\left(-\frac{\partial^2 l}{\partial a\partial \beta}\right) \\ E\left(-\frac{\partial^2 l}{\partial b\partial a}\right) & E\left(-\frac{\partial^2 l}{\partial b^2}\right) & E\left(-\frac{\partial^2 l}{\partial b\partial \beta}\right) \\ E\left(-\frac{\partial^2 l}{\partial \beta\partial a}\right) & E\left(-\frac{\partial^2 l}{\partial \beta\partial b}\right) & E\left(-\frac{\partial^2 l}{\partial \beta^2}\right) \end{bmatrix} = n\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}, \tag{14}$$

where

$$A_{11} = \beta^2\Gamma\left(\frac{\tau_1^\beta}{\theta_1^\beta}\middle| 2\right) + \sum_{j=2}^{k}\beta^2 g_j + \left[\sum_{i=0}^{k-1}\frac{\tau_{i+1}^\beta - \tau_i^\beta}{\theta_{i+1}^\beta}\right]\frac{\beta\theta_k^\beta}{\tau^{\beta-1}}f_k^*(\tau)$$
$$+ \sum_{j=2}^{k}\frac{\beta}{\alpha_{j-1}}\left[\sum_{i=0}^{j-2}\alpha_i(\tau_{i+1}^\beta - \tau_i^\beta) - \alpha_{j-1}\tau_{j-1}^\beta\right]\left[\frac{f_j^*(\tau_{j-1})}{\tau_{j-1}^{\beta-1}} - \frac{f_j^*(\tau_j)}{\tau_j^{\beta-1}}\right],$$

$$A_{12} = \beta^2\varphi_1\Gamma\left(\frac{\tau_1^\beta}{\theta_1^\beta}\middle| 2\right) + \sum_{j=2}^{k}\beta^2\varphi_j g_j + \left[\sum_{i=0}^{k-1}\frac{\varphi_{i+1}(\tau_{i+1}^\beta - \tau_i^\beta)}{\theta_{i+1}^\beta}\right]\frac{\beta\theta_k^\beta}{\tau^{\beta-1}}f_k^*(\tau)$$
$$+ \sum_{j=2}^{k}\frac{\beta}{\alpha_{j-1}}\left(\sum_{i=0}^{j-2}\alpha_i\varphi_{i+1}(\tau_{i+1}^\beta - \tau_i^\beta) - \alpha_{j-1}\varphi_j\tau_{j-1}^\beta\right)\left[\frac{f_j^*(\tau_{j-1})}{\tau_{j-1}^{\beta-1}} - \frac{f_j^*(\tau_j)}{\tau_j^{\beta-1}}\right],$$

$$A_{22} = \beta^2\varphi_1^2\Gamma\left(\frac{\tau_1^\beta}{\theta_1^\beta}\middle| 2\right) + \sum_{j=2}^{k}\beta^2\varphi_j^2 g_j + \left[\sum_{i=0}^{k-1}\frac{\varphi_{i+1}^2(\tau_{i+1}^\beta - \tau_i^\beta)}{\theta_{i+1}^\beta}\right]\frac{\beta\theta_k^\beta}{\tau^{\beta-1}}f_k^*(\tau)$$
$$+ \sum_{j=2}^{k}\frac{\beta}{\alpha_{j-1}}\left(\sum_{i=0}^{j-2}\varphi_{i+1}^2\alpha_i(\tau_{i+1}^\beta - \tau_i^\beta) - \alpha_{j-1}\varphi_j^2\tau_{j-1}^\beta\right)\left[\frac{f_j^*(\tau_{j-1})}{\tau_{j-1}^{\beta-1}} - \frac{f_j^*(\tau_j)}{\tau_j^{\beta-1}}\right],$$

$$
\begin{aligned}
A_{13} = {} & F_1(\tau_1) - r_1 - \sum_{j=2}^{k} A_j h_j - \sum_{j=2}^{k} B_j r_j \\
& - \left[A_k + 1 + \beta \left(\frac{\tau}{\theta_k} \right)^{\beta} (\ln(\tau) - \ln(\theta_k)) + \sum_{i=0}^{k-1} \left(\frac{\tau_{i+1}^{\beta} - \tau_i^{\beta}}{\theta_{i+1}^{\beta}} \right) \right] \frac{\theta_k^{\beta}}{\beta \tau^{\beta-1}} f_k^*(\tau) \\
& - \left[\Gamma \left(\left(\frac{\tau_1}{\theta_1} \right)^{\beta} \middle| 2 \right) + \sum_{j=2}^{k} g_j \right. \\
& \quad \left. + \sum_{j=2}^{k} \left(\sum_{i=0}^{j-2} \frac{\tau_{i+1}^{\beta} - \tau_i^{\beta}}{\theta_{i+1}^{\beta}} - \frac{\tau_{j-1}^{\beta}}{\theta_j^{\beta}} \right) \frac{\theta_j^{\beta}}{\beta} \left(\frac{f_j^*(\tau_{j-1})}{\tau_{j-1}^{\beta-1}} - \frac{f_j^*(\tau_j)}{\tau_j^{\beta-1}} \right) \right], \\
A_{23} = {} & \varphi_1 [F_1(\tau_1) - r_1] - \sum_{j=2}^{k} A_j' h_j - \sum_{j=2}^{k} \varphi_j B_j r_j \\
& - \left[A_k' + \varphi_k + \beta \varphi_k \left(\frac{\tau}{\theta_k} \right)^{\beta} (\ln(\tau) - \ln(\theta_k)) + \sum_{i=0}^{k-1} \frac{(\tau_{i+1}^{\beta} - \tau_i^{\beta}) \varphi_{i+1}}{\theta_{i+1}^{\beta}} \right] \frac{\theta_k^{\beta}}{\beta \tau^{\beta-1}} f_k^*(\tau) \\
& - \left[\varphi_1 \Gamma \left(\left(\frac{\tau_1}{\theta_1} \right)^{\beta} \middle| 2 \right) + \sum_{j=2}^{k} \varphi_j g_j \right. \\
& \quad \left. + \sum_{j=2}^{k} \left(\sum_{i=0}^{j-2} \frac{(\tau_{i+1}^{\beta} - \tau_i^{\beta}) \varphi_{i+1}}{\theta_{i+1}^{\beta}} - \frac{\tau_{j-1}^{\beta} \varphi_j}{\theta_j^{\beta}} \right) \frac{\theta_j^{\beta}}{\beta} \left(\frac{f_j^*(\tau_{j-1})}{\tau_{j-1}^{\beta-1}} - \frac{f_j^*(\tau_j)}{\tau_j^{\beta-1}} \right) \right], \\
A_{33} = {} & \frac{1}{\beta^2} [F_1(\tau) + q_1] - \sum_{j=2}^{k} C_j h_j + \frac{1}{\beta^2} \sum_{j=2}^{k} B_j q_j \\
& - \left[C_k + \frac{1}{\beta^2} - \left(\frac{\tau}{\theta_k} \right)^{\beta} (\ln(\tau) - \ln(\theta_k))^2 \right] \frac{\theta_k^{\beta}}{\beta \tau^{\beta-1}} f_k^*(\tau).
\end{aligned}
$$

Therefore, the asymptotic variance-covariance matrix of $\hat{a}$, $\hat{b}$, and $\hat{\beta}$ is

$$
\Sigma = \begin{bmatrix} Var(\hat{a}) & cov(\hat{a}, \hat{b}) & cov(\hat{a}, \hat{\beta}) \\ cov(\hat{a}, \hat{b}) & Var(\hat{b}) & cov(\hat{b}, \hat{\beta}) \\ cov(\hat{a}, \hat{\beta}) & cov(\hat{b}, \hat{\beta}) & Var(\hat{\beta}) \end{bmatrix} = F^{-1}. \tag{15}
$$

Hence, the asymptotic variance of $\ln \hat{t}_p$ under the use stress level S_0 is

$$
AVar(\ln \hat{t}_p) = \left[1,\ \varphi_0,\ -\tfrac{1}{\beta^2} \Phi^{-1}(p) \right] \Sigma \begin{bmatrix} 1 \\ \varphi_0 \\ -\tfrac{1}{\beta^2} \Phi^{-1}(p) \end{bmatrix}. \tag{16}
$$

Given the empirical value of a, b, and β, we get the optimum stress changing time points τ_j, $j = 1, 2, \ldots, k-1$.

4. Optimum Test Plan for Simple SSALT under Type I Censoring

As we have pointed out in Sec. 3.1, we should determine $2(k-1)$ quantities. In this section, we consider the simple SSALT, i.e., $k = 2$. Suppose that n products are initially placed on test at S_1 and run until time τ_1, then the stress is changed to S_2. At S_2, testing continues until τ. The principle of optimum simple SSALT under Type I censoring is that given the range of the stress level, $[S_L, S_U]$, the higher stress S_2, the censoring time τ and sample size n, we should decide the lower stress $S_1 (S_L \le S_1 \le S_2 \le S_U)$ and the time of changing stress $\tau_1 (0 \le \tau_1 \le \tau)$ so that $AVar(\ln \hat{t}_p)$ reaches the minimum.

Proposition 4.1. *Under Assumptions* 1–4, *for the simple SSALT, we have the results corresponding to Propositions* 2.2 *and* 2.3.

1. *The Sf of the simple SSALT is*

$$\overline{F}^*(t) = \begin{cases} \exp\left[\left(\dfrac{t}{\theta_1}\right)^\beta\right], & t < \tau_1 \\ \exp\left[\left(\dfrac{t}{\theta_2}\right)^\beta\right] \exp\left[\left(\dfrac{\tau_1}{\theta_2}\right)^\beta - \left(\dfrac{\tau_1}{\theta_1}\right)^\beta\right], & t \ge \tau_1 \end{cases}$$

and the probability density function is

$$f^*(t) = \begin{cases} \dfrac{\beta}{\theta_1^\beta} t^{\beta-1} \exp\left[\left(\dfrac{t}{\theta_1}\right)^\beta\right], & t < \tau_1 \\ \dfrac{\beta}{\theta_2^\beta} t^{\beta-1} \exp\left[\left(\dfrac{t}{\theta_2}\right)^\beta\right] \exp\left[\left(\dfrac{\tau_1}{\theta_2}\right)^\beta - \left(\dfrac{\tau_1}{\theta_1}\right)^\beta\right], & t \ge \tau_1 \end{cases}$$

where $\theta_2 = \theta_1/(\alpha_1)^{1/\beta}$.

2. *The ATF* α_1 *satisfies the log-linear relationship*

$$\alpha_1 = \exp\{b\beta[\varphi_1 - \varphi_2]\}.$$

According to (14), for the special case when $k = 2$, we get the Fisher information matrix of a, b, and β immediately. For simplicity, some of the notations becomes

$$r_1 = r\left(\left(\frac{\tau_1}{\theta_1}\right)^\beta\right), \quad r_2 = r\left(\left(\frac{\tau}{\theta_2}\right)^\beta\right) - r\left(\left(\frac{\tau_1}{\theta_2}\right)^\beta\right),$$

$$q_1 = q\left(\left(\frac{\tau_1}{\theta_1}\right)^\beta\right), \quad q_2 = q\left(\left(\frac{\tau}{\theta_2}\right)^\beta\right) - q\left(\left(\frac{\tau_1}{\theta_2}\right)^\beta\right),$$

$$B_2 = \exp\left[\left(\frac{\tau_1}{\theta_2}\right)^\beta - \left(\frac{\tau_1}{\theta_1}\right)^\beta\right],$$

$$g_2 = B_2\left[\Gamma\left(\frac{\tau^\beta}{\theta_2^\beta}\middle| m\right) - \Gamma\left(\frac{\tau_1^\beta}{\theta_2^\beta}\middle| m\right)\right],$$

$$h_2 = \frac{\theta_2^\beta}{\beta}\left(\frac{f_2^*(\tau_1)}{\tau_1^{\beta-1}} - \frac{f_2^*(\tau)}{\tau^{\beta-1}}\right),$$

$$A_2 = -1 + \beta\left(\frac{\tau_1}{\theta_1}\right)^\beta (\ln(\tau_1) - \ln(\theta_1)) - \beta\left(\frac{\tau_1}{\theta_2}\right)^\beta (\ln(\tau_1) - \ln(\theta_2)),$$

$$A_2' = -\phi_2 + \beta\phi_1\left(\frac{\tau_1}{\theta_1}\right)^\beta (\ln(\tau_1) - \ln(\theta_1)) - \beta\phi_2\left(\frac{\tau_1}{\theta_2}\right)^\beta (\ln(\tau_1) - \ln(\theta_2)),$$

$$C_2 = -\frac{1}{\beta^2} - \left(\frac{\tau_1}{\theta_1}\right)^\beta (\ln(\tau_1) - \ln(\theta_1))^2 + \left(\frac{\tau_1}{\theta_2}\right)^\beta (\ln(\tau_1) - \ln(\theta_2))^2.$$

Then, the Fisher information matrix of a, b, and β is

$$F = n\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{bmatrix}, \tag{17}$$

where

$$B_{11} = \beta^2\Gamma\left(\frac{\tau_1^\beta}{\theta_1^\beta}\middle| 2\right) + \beta^2 g_2 + \beta\tau f_2^*(\tau) + \beta\tau_1\left(\frac{1}{\alpha_1} - 1\right) f_2^*(\tau_1),$$

$$B_{12} = \beta^2\varphi_1\Gamma\left(\frac{\tau_1^\beta}{\theta_1^\beta}\middle| 2\right) + \beta^2\varphi_2 g_2 + \beta\varphi_2\tau f_2^*(\tau) + \beta\tau_1\left(\frac{\varphi_1}{\alpha_1} - \varphi_2\right) f_2^*(\tau_1),$$

$$B_{22} = \beta^2\varphi_1^2\Gamma\left(\frac{\tau_1^\beta}{\theta_1^\beta}\middle| 2\right) + \beta^2\varphi_2^2 g_2 + \beta\varphi_2^2\tau f_2^*(\tau) + \beta\tau_1\left(\frac{\varphi_1^2}{\alpha_1} - \varphi_2^2\right) f_2^*(\tau_1),$$

$$\begin{aligned} B_{13} = {} & F_1(\tau_1) - r_1 - A_2 h_2 - B_2 r_2 \\ & - \left[A_2 + 1 + \beta\left(\frac{\tau}{\theta_2}\right)^\beta (\ln(\tau) - \ln(\theta_2))\right]\frac{\theta_2^\beta}{\beta\tau^{\beta-1}} f_2^*(\tau) \\ & - \left[\Gamma\left(\left(\frac{\tau_1}{\theta_1}\right)^\beta\middle| 2\right) + g_2 + \frac{\tau}{\beta} f_2^*(\tau) + \left(\frac{1}{\theta_1^\beta} - \frac{1}{\theta_2^\beta}\right)\frac{\theta_2^\beta\tau_1}{\beta} f_2^*(\tau_1)\right], \end{aligned}$$

$$\begin{aligned} B_{23} = {} & \varphi_1[F_1(\tau_1) - r_1] - A_2' h_2 - B_2\varphi_2 r_2 \\ & - \left[A_2' + \varphi_2 + \beta\varphi_2\left(\frac{\tau}{\theta_2}\right)^\beta (\ln(\tau) - \ln(\theta_2))\right]\frac{\theta_2^\beta}{\beta\tau^{\beta-1}} f_2^*(\tau) \\ & - \left[\varphi_1\Gamma\left(\left(\frac{\tau_1}{\theta_1}\right)^\beta\middle| 2\right) + \varphi_2 g_2 + \frac{\varphi_2\tau}{\beta} f_2^*(\tau) + \left(\frac{\varphi_1}{\theta_1^\beta} - \frac{\varphi_2}{\theta_2^\beta}\right)\frac{\theta_2^\beta\tau_1}{\beta} f_2^*(\tau_1)\right], \end{aligned}$$

$$\begin{aligned} B_{33} = {} & \frac{1}{\beta^2}[F_1(\tau_1) + q_1] - C_2 h_2 + \frac{1}{\beta^2} B_2 q_2 \\ & - \left[C_2 + \frac{1}{\beta^2} - \left(\frac{\tau}{\theta_2}\right)^\beta (\ln(\tau) - \ln(\theta_2))^2\right]\frac{\theta_2^\beta}{\beta\tau^{\beta-1}} f_2^*(\tau). \end{aligned}$$

In particular, if $\beta = 1$, then the Weibull distribution is the exponential distribution. And if S_1 is given, then we only need to determine the stress changing time point τ_1. Thus, we have

$$AVar(\ln \hat{t}_p) = \frac{1}{n} \frac{p_1(\varphi_0 - \varphi_1)^2 + p_2(1 - p_1)(\varphi_0 - \varphi_2)^2}{p_1 p_2 (1 - p_1)(\varphi_1 - \varphi_2)^2},$$

where $p_1 = 1 - \exp\left(-\frac{\tau_1}{\theta_1}\right)$, $p_2 = 1 - \exp\left(-\frac{\alpha_1(\tau - \tau_1)}{\theta_1}\right) = 1 - \exp\left(-\frac{\tau - \tau_1}{\theta_2}\right)$.

Let $g(\tau_1) = \frac{(\varphi_0 - \varphi_1)^2}{p_2(1-p_1)} + \frac{(\varphi_0 - \varphi_2)^2}{p_1}$, $g_1(\tau_1) = \frac{1}{p_2(1-p_1)}$, and $g_2(\tau_1) = \frac{1}{p_1}$. Then,

$$\begin{aligned}
\frac{\partial g_1(\tau_1)}{\partial \tau_1} &= \frac{\theta_1(1 - p_2) + \theta_2 p_2}{p_2^2 (1 - p_1)\theta_1 \theta_2}, \\
\frac{\partial^2 g_1(\tau_1)}{\partial \tau_1^2} &= \frac{1}{\theta_1^2 \theta_2^2 p_2^3} \{[\theta_1 + (\theta_2 - \theta_1)p_2]^2 + \theta_1^2(1 - p_2)\} > 0, \\
\frac{\partial g_2(\tau_1)}{\partial \tau_1} &= -\frac{1 - p_1}{\theta_1 p_1^2}, \\
\frac{\partial^2 g_2(\tau_1)}{\partial \tau_1^2} &= \frac{(2 - p_1)(1 - p_1)}{\theta_1^2 p_1^3} > 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial g(\tau_1)}{\partial \tau_1} &= (\varphi_0 - \varphi_1)^2 \frac{\partial g_1(\tau_1)}{\partial \tau_1} + (\varphi_0 - \varphi_2)^2 \frac{\partial g_2(\tau_1)}{\partial \tau_1} \\
&= (\varphi_0 - \varphi_1)^2 \frac{\theta_1(1 - p_2) + \theta_2 p_2}{p_2^2 (1 - p_1)\theta_1 \theta_2} - (\varphi_0 - \varphi_2)^2 \frac{1 - p_1}{\theta_1 p_1^2}, \\
\frac{\partial^2 g(\tau_1)}{\partial \tau_1^2} &= (\varphi_0 - \varphi_1)^2 \frac{\partial^2 g_1(\tau_1)}{\partial \tau_1^2} + (\varphi_0 - \varphi_2)^2 \frac{\partial^2 g_2(\tau_1)}{\partial \tau_1^2} > 0.
\end{aligned}$$

This shows that $\frac{\partial g(\tau_1)}{\partial \tau_1}$ is a strict increasing function of τ_1. Besides, we can easily prove that

$$\lim_{\tau_1 \to 0^+} \frac{\partial g(\tau_1)}{\partial \tau_1} = -\infty, \quad \lim_{\tau_1 \to \tau^-} \frac{\partial g(\tau_1)}{\partial \tau_1} = +\infty.$$

Thus, $g(\tau_1)$ has its unique minimum point in the interval $(0, \tau)$. Therefore, we come to the following conclusion.

Corollary 4.1. *For the exponential distribution, the optimum stress changing time point $\tau_1 \in (0, \tau)$ of the simple SSALT under Type I censoring satisfies*

$$\left(\frac{\varphi_2 - \varphi_0}{\varphi_0 - \varphi_1}\right)^2 = \left[\frac{\theta_1}{\theta_2}(1 - p_2) + p_2\right]\left[\frac{p_1}{p_2(1 - p_1)}\right]^2, \tag{18}$$

and the solution for τ_1 in (18) *is unique.*

This result should be consistent with that of the optimum design for Type-I SSALT for exponential distributions under CE model because of the consistency of CE and TFR models for exponential distributions (Wang and Fei, 2004).

5. Example

Suppose that 100 products are randomly selected for a simple SSALT. The lifetime distribution is the Weibull distribution. And the characteristic life θ and the stress level S satisfy the inverse power law. In practical experiments, the experimental region is between the use stress level S_0 and highest allowable stress level S_M. For simplicity, standardized stress levels $x_i = (\frac{1}{S_0} - \frac{1}{S_i})/(\frac{1}{S_0} - \frac{1}{S_M})$ are used such that the experimental region of accelerating variable x is in the range [0, 1]. Thus, under x_i, the ATF can be expressed as

$$\ln(\theta_i) = \gamma_0 + \gamma_1 x_i, \quad 0 \le x_i \le 1, \tag{19}$$

where (γ_0, γ_1) is a re-parameterization of (a, b).

The TFR model is used as the accelerated tampered model. Let the use stress level $S_0 = 400K$, the higher acceleration temperature level is $S_M = 800K$, and the censoring time $\tau = 4000h$. We want to obtain the optimum stress S_1 and stress changing time point τ_1 so that the mle of $\ln \hat{t}_{0.5}$ has the highest accuracy asymptotically.

5.1. *Preliminary Estimation of Parameters*

Based on experience, some historical data or a preliminary test can be used to get estimates of γ_0, γ_1, and β. For instance, a two-step preliminary SSALT can be done as follows. Put n test units on a life test under a low stress level S_1. As soon as two or three units fail, the survivors of the n products continue to be tested under a higher stress level S_2 until two or three more units fail. Then, the preliminary estimates of γ_0, γ_1, and β can be determined by the maximum likelihood method.

5.2. *Optimum Test Plan*

For illustration, let the preliminary estimates of $(\gamma_0, \gamma_1, \beta)$ be $\gamma_0^\diamond = 9, \gamma_1^\diamond = -2$, and $\beta^\diamond = 2$. Based on (16) and (17), we can solve for the optimum values of S_1 and τ_1, which turn out to be $S_1^* = 437K(x_1^* = 0.1673)$ and $\tau_1^* = 3962$ with $AVar(\ln \hat{t}_{0.5}) = 0.0222366$. We find that they tend to be on or near the boundary, i.e., $S_1^* = 400\,\mathrm{K}(x_1^* = 0)$ and $\tau_1^* = 4000\,\mathrm{h}$. Figure 1 shows the contour plot of $AVar(\ln \hat{t}_{0.5})$ with x_1 and τ_1 near the optimum values of S_1 and τ_1. We may take it as granted as the left-up corner of (S_1, τ_1) tends to provide more information about the life time under the use stress level $S_0 = 400\,\mathrm{K}(x_1 = 0)$.

In practice, we may conduct an SSALT for a fixed stress S_1 to obtain the optimum stress changing time τ_1 as was given in Alhadeed and Yang (2002). For example, given the lower stress $x_1 = 0.75(S_1 = 640\,\mathrm{K})$ and the higher stress $x_2 = 1(S_2 = 800\,\mathrm{K})$, we get the optimum stress changing time point $\tau_1^* = 1961\,\mathrm{h}$ with $AVar(\ln \hat{t}_{0.5}) = 0.2869$. Figure 2 shows the relationship of $AVar(\ln \hat{t}_{0.5})$ with τ_1.

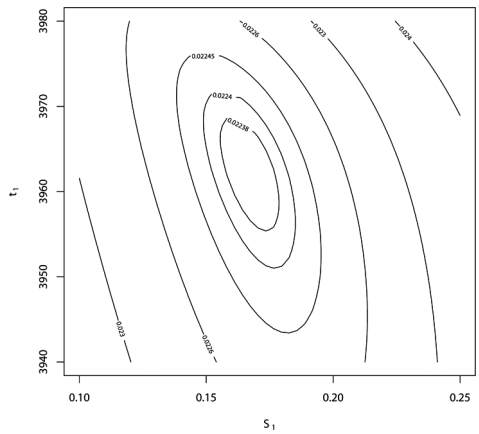

Figure 1. The relationship of $AVar(\ln \hat{t}_p)$ with S_1 and τ_1.

5.3. *Sensitivity Analysis*

In this subsection, we do a sensitivity analysis following the framework of Tseng et al. (2009) and Preeti and Neha (2010). In practice, due to the limited information about the SSALT model, the preliminary estimates of $(\gamma_0, \gamma_1, \beta)$ would be rather rough and may depart from the true value of $(\gamma_0, \gamma_1, \beta)$. Hence, it is important to investigate the effect of the change of these unknown parameters on the optimal test plan. Without loss of generality, we assume that the preliminary estimates $\gamma_0^{\diamond} = 9$, $\gamma_1^{\diamond} = -2$, and $\beta^{\diamond} = 2$ are the true values of the parameters γ_0, γ_1, and β, and ε_1, ε_2, and ε_3 denote the predicted errors for γ_0, γ_1, and β, respectively. Under the same

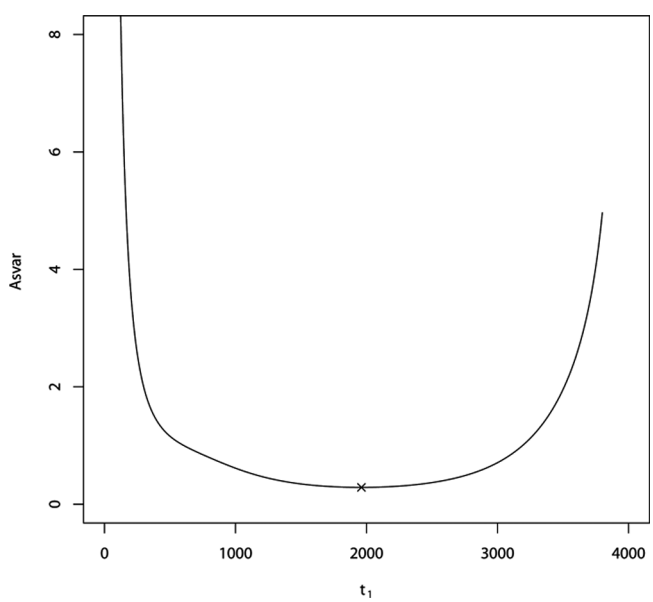

Figure 2. The relationship of $AVar(\ln \hat{t}_{0.5})$ with τ_1 for fixed $S_1 = 640\,\mathrm{K}$.

Table 1
Sensitivity analysis under various combinations of parameters
$((1+\varepsilon_1)\gamma_0, (1+\varepsilon_2)\gamma_1, (1+\varepsilon_3)\beta)$

ε_1	ε_2	ε_3	τ_1*	$AVar(\ln \hat{t}_{0.5})$	Relative bias of τ_1^*	Relative bias of $AVar(\ln \hat{t}_{0.5})$
0	0	0	2106.5	0.2821	0%	0%
−1.5%	−1.5%	−5%	1885.4	0.3059	−10.5%	8.4%
−1.5%	0	0	1848.5	0.2821	−12.2%	0.0%
−1.5%	+1.5%	+5%	1774.7	0.2617	−15.7%	−7.2%
0	−1.5%	0	2143.4	0.2806	1.8%	−0.5%
0	0	+5%	2106.6	0.2602	0.0%	−7.7%
0	+1.5%	−5%	2069.7	0.3088	−1.7%	9.4%
+1.5%	−1.5%	+5%	2438.4	0.2588	15.8%	−8.2%
+1.5%	0	−5%	2438.4	0.3074	15.8%	8.9%
+1.5%	+1.5%	0	2364.6	0.2835	12.3%	0.5%

configuration of $(n, x_1, x_2, \tau) = (100, 0.75, 1, 4000)$, Table 1 presents the optimal plans under various combinations of $((1+\varepsilon_1)\gamma_0, (1+\varepsilon_2)\gamma_1, (1+\varepsilon_3)\beta)$ according to a $L_9(3^{3-1})$ orthogonal array for $\theta = (\gamma_0, \gamma_1, \beta)$. Note that a and b are in log scale of θ, the change of 1.5% in a or b is nearly the same as the variability of 6% to 15% in θ for our example. Thus, from the relative bias of τ_1^* and $AVar(\ln \hat{t}_{0.5})$, we see that the test plans are relatively robust for moderate departures from the assumed values of these parameters.

6. Conclusion and Furthter Area Research

In this article, under TFR model, we obtain an optimum SSALT for the Weibull distribution with Type-I censoring. An example is provided to illustrate our proposed method. The sensitivity analysis carried out reveals that the optimal test plan is relatively robust for moderate departures from the assumed values of these parameters. In practice, the unknown parameters $\theta = (\gamma_0, \gamma_1, \lambda)$ may not be estimated precisely in a pilot study. The optimal designs depend on the preestimates of the parameters. To deal with the uncertainty in the planning values, we introduce some prior to describe this uncertainty. An attractive future research is to follow a Bayesian approach by assigning prior distributions to experimental conditions and deriving the corresponding optimal designs. Liu and Tang (2010) considered this approach to planning an ALT for repairable systems. Consider the optimal design of SSALT subject to a budget constraint is interesting for future research. The constraint may help find the optimum plan with both the stress changing time point(s) and stress(es) not near the boundary.

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