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Valuation of equity-indexed annuity under stochastic mortality and interest rate☆

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ABSTRACT

An equity-indexed annuity (EIA) contract offers a proportional participation in the return on a specified equity index, in addition to a guaranteed return on the single premium. In this paper, we discuss the valuation of equity-indexed annuities under stochastic mortality and interest rate which are assumed to be dependent on each other. Employing the method of change of measure, we present the pricing formulas in closed form for the most common product designs: the point-to-point and the annual reset. Finally, we conduct several numerical experiments, in which we analyze the relationship between some parameters and the pricing of EIAs.

1. Introduction

An equity-indexed annuity (EIA) is a hybrid between a fixed and variable annuity that earns a minimum rate of interest and offers a potential gain that is tied to the performance of a stock index, typically S&P 500, i.e. when the stock index goes up, EIA provides policyholders with a rate of return connected to the index return; when the index goes down, EIA provides policyholders with a minimum guaranteed return. Because of their "best of both worlds" design, EIAs are extremely popular. Sales have grown dramatically since their introduction in 1995. Indeed, EIA sales for 2007 were nearly $25.2 billion, a 380% increase over their 2000 level of $5.25 billion (see Marrion (2007)).

EIAs have received considerable attention in the actuarial literature; see, for example, Tiong (2000), Gerber and Shiu (2003), Lin and Tan (2003), Hardy (2003, 2004), Lee (2003), Jaimungal (2004), Kijima and Wong (2007), Boyle and Tian (2008) and Moore (2009). These authors studied the pricing, hedging and risk management of various features of EIAs. It is very important to consider stochastic interest rates when pricing EIA. Lin and Tan (2003) considered the model of stochastic interest rates by postulating a Vasicek model which is correlated to the geometric Brownian motion of the risky asset. They argued that the effects of stochastic interest rates are crucial in EIA pricing by simulation results. Kijima and Wong (2007) adopted the ordinary arbitrage-free pricing principle to price simple and compound annual reset EIAs when the short rate follows the extended Vasicek model.

In all the above papers, mortality risk is considered to be deterministic, or even not included. However, the expected life length has increased considerably in many countries during the past decades with the advances made in the health sciences and medicine, life insurance and annuities are exposed to unanticipated changes over time in the mortality rates of the appropriate reference population, which has forced life insurers to use a stochastic model to describe mortality laws. There exists a vast literature on this topic, e.g. Dahl (2004), Biffis (2005); Biffis et al. (2005), Luciano and Vigna (2005), Biffis and Denuit (2006), Dahl and Moller (2006), Cairns et al. (2006), Dowda et al. (2006), Schrager (2006), Hainaut and Devolder (2008) and Jalen and Mamon (2009).

In this paper, we incorporate the stochastic mortality model and the stochastic interest rate model into the valuation of two types of designs of equity-indexed annuity: the point-to-point and the annual reset. We construct a general model, in which the stochastic mortality and the stochastic interest rate are dependent on each other. In fact, the model incorporates three specific cases: (1) The stochastic interest rate model, when some parameters of the stochastic mortality model equal zero. (2) The stochastic mortality model, when some parameters of the stochastic interest rate model equal zero. (3) The model where the stochastic mortality and the stochastic interest rate are not dependent, when the correlation coefficient between the mortality and the interest rate equals zero. In this paper, we use the method of change of measure together with the Bayes' rule to derive the pricing formulas in closed form for the two EIAs products, and give the results for the above three special cases. In Case 1, the pricing of EIAs is similar to that of

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Lin and Tan (2003) and Kijima and Wong (2007). In order to make the paper be applicable, we introduce some methods for estimating the parameters of the mortality and interest rate models, and conduct several numerical experiments, in which we analyze the relationship between some parameters and the pricing of EIAs.

The rest of this paper is organized as follows. The financial model and mortality model are introduced in Section 2. In Section 3, we price EIAs and get the formulas in closed form. Section 4 gives some numerical experiments to show the relationship between some parameters and the pricing of EIAs. Finally, we conclude the paper in Section 5.

2. The model

2.1. The financial model

Let $(\Omega, \mathcal{F}, \mathsf{Q})$ be a complete probability space, and T is a fixed time horizon. Consider a financial market consisting of two traded assets only: a risky asset with price process $S = (S_t)_{0\le t\le T}$ and a locally risk-free asset with price process $B = (B_t)_{0\le t\le T}$. The risky asset is usually referred to as a stock and the locally risk-free asset as a savings account. Furthermore, it is assumed that Q is a risk-neutral measure for the market so that the discounted price process of S is a Q-martingale, and the price processes of two assets are given by the Q-dynamics

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sigma_1 S_t \mathrm{d}W_1(t) + \sigma_2 S_t \mathrm{d}W_2(t), \tag{2.1}$$

$$\mathrm{d}B_t = r_t B_t \mathrm{d}t, \tag{2.2}$$

where r_t denotes the instantaneous short rate, $W_1(t)$ and $W_2(t)$ are two independent standard Brownian motions, σ_1 and σ_2 are constant coefficients. In addition, we assume that the short rate satisfy the extended Vasicek interest model

$$\mathrm{d}r_t = k(\theta - r_t)\mathrm{d}t + \sigma_3 \mathrm{d}W_1(t), \tag{2.3}$$

where k, θ and σ_3 are positive constant coefficients.

By Itô's formula, r_t and S_t can be rewritten as follows

$$S_t = S_0 \mathrm{e}^{\int_0^t r_s \mathrm{d}s - \frac{1}{2}\sigma_1^2 t - \frac{1}{2}\sigma_2^2 t + \sigma_1 W_1(t) + \sigma_2 W_2(t)}, \tag{2.4}$$

$$r_t = r_0 \mathrm{e}^{-kt} + \theta(1 - \mathrm{e}^{-kt}) + \sigma_3 \int_0^t \mathrm{e}^{-k(t-u)} \mathrm{d}W_1(u). \tag{2.5}$$

In order to simplify the notation, we denote $Y(t) = \ln \frac{S_t}{S_0}$. Then $S_t = S_0 \mathrm{e}^{Y(t)}$.

2.2. The mortality model

All processes mentioned hereafter are defined on the above introduced probability space $(\Omega, \mathcal{F}, \mathsf{Q})$. We denote by τ_x the random variable which represents the remaining lifetime of the policy holder on the age x. $(\mathcal{H}_t)_{t\ge 0}$ is the smallest σ-algebra with respect to which τ_x is a stopping time. Let μ_{x+t} be the intensity of τ_x at time t, and the Q-dynamics of the mortality intensity is given by

$$\mathrm{d}\mu_t = a\mu_t \mathrm{d}t + b\mathrm{d}W_1(t) + c\mathrm{d}W_3(t), \tag{2.6}$$

where a, b, c are constant coefficients, $W_3(t)$ is a standard Brownian motion, and independent with $W_1(t)$, $W_2(t)$.

It is obvious that the solution of μ_t is given by

$$\mu_t = \mu_0 \mathrm{e}^{at} + \int_0^t \mathrm{e}^{a(t-s)} b\mathrm{d}W_1(s) + \int_0^t \mathrm{e}^{a(t-s)} c\mathrm{d}W_3(s). \tag{2.7}$$

The t-year survival probability of an x year old is given by

$${}_tp_x = \mathsf{Q}(\tau_x > t) = \mathsf{E}\left[\exp\left(-\int_0^t \mu_{x+s}\mathrm{d}s\right)\right], \tag{2.8}$$

where $\mathsf{E}[\cdot]$ denotes the expectation under the measure Q.

Let

$$\mathcal{G}_t = \sigma(r_u, 0 \le u \le T) \vee \sigma(S_u, 0 \le u \le T) \vee \sigma(\mu_u, 0 \le u \le T) \vee \mathcal{H}_t.$$

Then $\mathcal{G}_0$ contains complete information on the interest rate process, the risky asset price process and the intensity of τ_x, all the way up to time T. We have

$$\mathsf{Q}(\tau_x > t|\mathcal{G}_0) = \exp\left(-\int_0^t \mu(x+s)\mathrm{d}s\right). \tag{2.9}$$

2.3. The combined model

Let a filtration $(\mathcal{F}_t)_{0\le t\le T}$ be $\mathcal{F}_T = \mathcal{F}$ and

$$\mathcal{F}_t = \sigma(r_u, 0 \le u \le t) \vee \sigma(S_u, 0 \le u \le t) \vee \sigma(\mu_u, 0 \le u \le t) \vee \mathcal{H}_t.$$

We let the correlation coefficient of $\sigma_3 \mathrm{d}W_1(t)$ and $b\mathrm{d}W_1(t) + c\mathrm{d}W_3(t)$ be ρ, then ρ can describe the correlation of the mortality and the interest rate, and

$$\rho = \mathrm{corr}(\sigma_3 \mathrm{d}W_1(t), b\mathrm{d}W_1(t) + c\mathrm{d}W_3(t)) = \frac{b}{\sqrt{b^2 + c^2}}. \tag{2.10}$$

Similarly, $\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}$ can describe the correlation of interest rate and the risky asset price.

3. Pricing EIA

In this section, we will discuss how to evaluate EIAs using the models in Section 2.

3.1. The point-to-point design

Let α be the participation rate, which is usually less than or equal to 1 in practice, g be a guaranteed minimum (continuously compounded) return and an initial premium be 1. At maturity time T, if the insured is survival the policy will pay $\mathrm{e}^{\alpha Y(T)}$ or e^{gT}, whichever is higher; or the insured died in $(t, t+1]$, the insurer will pay $\max(\mathrm{e}^{\alpha Y(t+1)}, \mathrm{e}^{g.(t+1)})$ at time $t+1$. Therefore, at the policy end time T, this policy earns a percentage of the realized return on the risky asset over T periods, with the provision of a minimum guaranteed return of g compounded continuously over time. Hence, we can express the value of this policy under measure Q by

$$P_{pp} = \mathsf{E}\left[\max(\mathrm{e}^{\alpha Y(T)}, \mathrm{e}^{gT})\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} I(\tau_x > T) + \sum_{t=0}^{T-1} \max(\mathrm{e}^{\alpha Y(t+1)}, \mathrm{e}^{g.(t+1)})\mathrm{e}^{-\int_0^{t+1} r_s \mathrm{d}s} I(t < \tau_x \le t+1)\right], \tag{3.1}$$

where $I(\cdot)$ is the indicator function (i.e., $I(A) = 1$ if A is true and 0 otherwise).

In order to obtain the present value of the EIA P_{pp}, we need to introduce two new measures, Q_r and Q_μ.

Proposition 3.1. *Let $\xi(T)$ denote the Radon–Nikodym derivative*

$$\xi(T) = \frac{\mathrm{d}\mathsf{Q}_r}{\mathrm{d}\mathsf{Q}} = \frac{\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}}{\mathsf{E}\left[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}\right]} = \exp\left(-\int_0^T \frac{\sigma_3(1-\mathrm{e}^{-k(T-t)})}{k}\mathrm{d}W_1(t) - \frac{1}{2}\int_0^T \left(\frac{\sigma_3(1-\mathrm{e}^{-k(T-t)})}{k}\right)^2 \mathrm{d}t\right). \tag{3.2}$$

Then $\tilde{W}_1(t)$ is a Q_r-Wiener process, where

$$\tilde{W}_1(t) = W_1(t) + \int_0^t \frac{\sigma_3(1-\mathrm{e}^{-k(t-s)})}{k}\mathrm{d}s. \tag{3.3}$$

Proof. According to (2.5), we have

$$-\int_0^T r_t\mathrm{d}t = -\int_0^T \left(r_0\mathrm{e}^{-kt} + \theta(1-\mathrm{e}^{-kt})\right)\mathrm{d}t - \int_0^T \sigma_3\frac{1-\mathrm{e}^{-k(T-t)}}{k}\mathrm{d}W_1(t). \tag{3.4}$$

Since

$$-\int_0^T r_t\mathrm{d}t \sim N\left(-\int_0^T \left(r_0\mathrm{e}^{-kt} + \theta(1-\mathrm{e}^{-kt})\right)\mathrm{d}t, \int_0^T \left(\sigma_3\frac{1-\mathrm{e}^{-k(T-t)}}{k}\right)^2\mathrm{d}t\right),$$

we can easily obtain (3.2). The result comes immediately from Girsanov's theorem. □

Remark 3.2. $W_2(t)$ and $W_3(t)$ are still two Wiener processes and independent with $\tilde{W}_1(t)$ under measure Q.

Put $P(t,T) = \mathsf{E}\left[\mathrm{e}^{-\int_t^T r_s\mathrm{d}s}|\mathcal{F}_t\right]$. It is well known that $P(t,T)$ is the price of a T-maturity zero coupon bond in the Vasicek model (2.3) (see Jaimungal and Wang (2006)), and its expression is

$$P(t,T) = \exp\{A(t,T) - B(t,T)r_t\}, \tag{3.5}$$

where

$$A(t,T) = \left(\theta - \frac{\sigma_3^2}{2k^2}\right)(B(t,T) - (T-t)) - \frac{\sigma_3^2B^2(t,T)}{4k},$$
$$B(t,T) = \frac{1-\mathrm{e}^{-k(T-t)}}{k}.$$

Proposition 3.3. *Let $\eta(T)$ stand for another Radon–Nikodym derivative, and*

$$\eta(T) = \frac{\mathrm{d}\mathsf{Q}_\mu}{\mathrm{d}\mathsf{Q}_r} = \frac{\mathrm{e}^{-\int_0^T \mu_{x+s}\mathrm{d}s}}{\mathsf{E}_{\mathsf{Q}_r}\left[\mathrm{e}^{-\int_0^T \mu_{x+s}\mathrm{d}s}\right]}$$
$$= \exp\left\{-\int_0^{x+T}\frac{b}{a}(\mathrm{e}^{a(T+x-t)} - \mathrm{e}^{a\max(0,x-t)})\mathrm{d}\tilde{W}_1(t) - \int_0^{x+T}\frac{c}{a}(\mathrm{e}^{a(T+x-t)} - \mathrm{e}^{a\max(0,x-t)})\mathrm{d}W_3(t) - \frac{1}{2}\int_0^{x+T}\left[(\mathrm{e}^{a(T+x-t)} - \mathrm{e}^{a\max(0,x-t)})^2.\frac{b^2+c^2}{a^2}\right]\mathrm{d}t\right\}, \tag{3.6}$$

where

$$\max(y,z) = \begin{cases} y, & y > z, \\ z, & y \le z. \end{cases}$$

Then $\tilde{\tilde{W}}_1(t), \tilde{W}_3(t)$ are two independent Brownian motions under Q_μ, where

$$\tilde{\tilde{W}}_1(t) = \tilde{W}_1(t) + \int_0^{x+t}\frac{b}{a}(\mathrm{e}^{a(t+x-s)} - \mathrm{e}^{a\max(0,x-s)})\mathrm{d}s,$$
$$\tilde{W}_3(t) = W_3(t) + \int_0^{x+t}\frac{c}{a}(\mathrm{e}^{a(t+x-s)} - \mathrm{e}^{a\max(0,x-s)})\mathrm{d}s.$$

Proof. From (2.7), we can obtain

$$\int_0^T \mu_{x+t}\mathrm{d}t = \int_0^T \mu_0\mathrm{e}^{a(x+t)}\mathrm{d}t + \int_0^T\int_0^{t+x}\mathrm{e}^{a(t+x-s)}b\mathrm{d}W_1(s)\mathrm{d}t + \int_0^T\int_0^{t+x}\mathrm{e}^{a(t+x-s)}c\mathrm{d}W_3(s)\mathrm{d}t$$
$$= \mu_0\frac{\mathrm{e}^{a(x+T)} - \mathrm{e}^{ax}}{a} + \int_0^{x+T}\frac{\mathrm{e}^{a(T+x-s)} - \mathrm{e}^{a\max(0,x-s)}}{a}(b\mathrm{d}W_1(s) + c\mathrm{d}W_3(s)). \tag{3.7}$$

It is obvious by (3.3)

$$\mathrm{d}W_1(t) = \mathrm{d}\tilde{W}_1(t) - \frac{\sigma_3(1-\mathrm{e}^{-kt})}{k}\mathrm{d}t. \tag{3.8}$$

So, (3.7) can be rewritten as

$$\int_0^T \mu_{x+t}\mathrm{d}t = \mu_0\frac{\mathrm{e}^{a(x+T)} - \mathrm{e}^{ax}}{a} + \int_0^{x+T}\frac{\mathrm{e}^{a(T+x-s)} - \mathrm{e}^{a\max(0,x-s)}}{a}(b\mathrm{d}\tilde{W}_1(s) + c\mathrm{d}W_3(s)) - \int_0^{x+T}\frac{(\mathrm{e}^{a(T+x-s)} - \mathrm{e}^{a\max(0,x-s)})b\sigma_3(1-\mathrm{e}^{-ks})}{ak}\mathrm{d}s. \tag{3.9}$$

Let $p(x,0,T) = \mathsf{E}_{\mathsf{Q}_r}\left[\mathrm{e}^{-\int_0^T \mu_{x+s}\mathrm{d}s}\right]$. Then from (3.9) we get

$$p(x,0,T) = \exp\left\{\mu_0\frac{\mathrm{e}^{ax} - \mathrm{e}^{a(x+T)}}{a} + \int_0^{x+T}\frac{(\mathrm{e}^{a(T+x-s)} - \mathrm{e}^{a\max(0,x-s)})b\sigma_3(1-\mathrm{e}^{-ks})}{ak}\mathrm{d}s + \frac{b^2+c^2}{2a^2}\int_0^{x+T}(\mathrm{e}^{a(T+x-t)} - \mathrm{e}^{a\max(0,x-t)})^2\mathrm{d}t\right\}. \tag{3.10}$$

Thus we obtain (3.6). Girsanov's theorem implies that $\tilde{\tilde{W}}_1(t)$ and $\tilde{W}_3(t)$ are two independent Brownian motions under Q_μ. □

Remark 3.4. $W_2(t)$, which is a Brownian motion under Q_μ, is dependent with $\tilde{\tilde{W}}_1(t)$ and $\tilde{W}_3(t)$.

Proposition 3.5. *For any fixed t $(0 \le t \le T)$, $Y(t)$ is normal distribution under measure Q_μ. Denote $\mathsf{E}_{Q_\mu}(Y(t)) = m_1(t)$, $\mathsf{Var}_{Q_\mu}(Y(t)) = m_2(t)$, then*

$$m_1(t) = r_0\frac{1-\mathrm{e}^{-kt}}{k} + \theta\left(t + \frac{\mathrm{e}^{-kt}-1}{k}\right) - \frac{1}{2}(\sigma_1^2+\sigma_2^2)t - \int_0^t\frac{\sigma_3(1-\mathrm{e}^{-k(t-u)}) + k\sigma_1}{k} \times \left(\frac{\sigma_3(1-\mathrm{e}^{-ku})}{k} + \frac{b(\mathrm{e}^{a(u+x)}-1)}{a}\right)\mathrm{d}u,$$
$$m_2(t) = \sigma_2^2t + \int_0^t\left(\frac{\sigma_3(1-\mathrm{e}^{-k(t-u)}) + \sigma_1k}{k}\right)^2\mathrm{d}u.$$

Proof.

$$Y(t) = \int_0^t r_s\mathrm{d}s - \frac{1}{2}\sigma_1^2t - \frac{1}{2}\sigma_2^2t + \sigma_1W_1(t) + \sigma_2W_2(t)$$
$$= r_0\frac{1-\mathrm{e}^{-kt}}{k} + \theta\left(t + \frac{\mathrm{e}^{-kt}-1}{k}\right) - \frac{1}{2}(\sigma_1^2+\sigma_2^2)t$$

$$
\begin{aligned}
&+\int_0^t\left(\frac{(1-\mathrm{e}^{-k(t-u)})\sigma_3}{k}+\sigma_1\right)\mathrm{d}W_1(u)+\sigma_2W_2(t)\\
&=r_0\frac{1-\mathrm{e}^{-kt}}{k}+\theta\left(t+\frac{\mathrm{e}^{-kt}-1}{k}\right)-\frac{1}{2}(\sigma_1^2+\sigma_2^2)t\\
&+\int_0^t\left(\frac{(1-\mathrm{e}^{-k(t-u)})\sigma_3}{k}+\sigma_1\right)\mathrm{d}\tilde{W}_1(u)+\sigma_2W_2(t)\\
&-\int_0^t\frac{\sigma_3(1-\mathrm{e}^{-k(t-u)})+k\sigma_1}{k}\\
&\times\left(\frac{\sigma_3(1-\mathrm{e}^{-ku})}{k}+\frac{b(\mathrm{e}^{a(u+x)}-1)}{a}\right)\mathrm{d}u. \qquad (3.11)
\end{aligned}
$$

Proposition 3.5 comes immediately from (3.11). □

Now we are in a position to calculate the present value of the EIA P_{pp}.

Theorem 3.6. *The present value of the EIA P_{pp} is as follows*

$$
\begin{aligned}
P_{pp}&=P(0,T)p(x,0,T)\left[\mathrm{e}^{gT}\Phi\left(\frac{\frac{gT}{\alpha}-m_1(T)}{\sqrt{m_2(T)}}\right)\right.\\
&\left.+\mathrm{e}^{m_1(T)\alpha+\frac{\alpha^2m_2(T)}{2}}\Phi\left(\frac{m_1(T)+\alpha m_2(T)-\frac{gT}{\alpha}}{\sqrt{m_2(T)}}\right)\right]\\
&+\sum_{t=0}^{T-1}\left\{P(0,t+1)[p(x,0,t)-p(x,0,t+1)]\right.\\
&\times\left[\mathrm{e}^{g.(t+1)}\Phi\left(\frac{\frac{g.(t+1)}{\alpha}-m_1(t+1)}{\sqrt{m_2(t+1)}}\right)\right.\\
&+\mathrm{e}^{m_1(t+1)\alpha+\frac{\alpha^2m_2(t+1)}{2}}\\
&\left.\left.\times\Phi\left(\frac{m_1(t+1)+\alpha m_2(t+1)-\frac{g.(t+1)}{\alpha}}{\sqrt{m_2(t+1)}}\right)\right]\right\}, \qquad (3.12)
\end{aligned}
$$

where $P(0,T)$ and $p(x,0,T)$ are defined by (3.5) and (3.10), respectively.

Proof. It comes from (2.9) that

$$
\begin{aligned}
&\mathrm{E}\left[\max(\mathrm{e}^{\alpha Y(T)},\mathrm{e}^{gT})\mathrm{e}^{-\int_0^T r_s\mathrm{d}s}I(\tau_x>T)\right]\\
&=\mathrm{E}\left\{\mathrm{E}\left[\max(\mathrm{e}^{\alpha Y(T)},\mathrm{e}^{gT})\mathrm{e}^{-\int_0^T r_s\mathrm{d}s}I(\tau_x>T)|\mathcal{G}_0\right]\right\}\\
&=\mathrm{E}\left\{\max(\mathrm{e}^{\alpha Y(T)},\mathrm{e}^{gT})\mathrm{e}^{-\int_0^T r_s\mathrm{d}s}\exp\left(-\int_0^T\mu_{x+t}\mathrm{d}t\right)\right\}. \qquad (3.13)
\end{aligned}
$$

On the other hand, (3.13) can be rewritten by Propositions 3.1 and 3.3 as

$$
\begin{aligned}
&\mathrm{E}\left[\mathrm{e}^{-\int_0^T r_s\mathrm{d}s}\right]\mathrm{E}_{Q_r}\left[\mathrm{e}^{-\int_0^T\mu_{x+s}\mathrm{d}s}\right]\mathrm{E}_{Q_\mu}[\max(\mathrm{e}^{\alpha Y(T)},\mathrm{e}^{gT})]\\
&=\mathrm{E}\left[\mathrm{e}^{-\int_0^T r_s\mathrm{d}s}\right]\mathrm{E}_{Q_r}\left[\mathrm{e}^{-\int_0^T\mu_{x+s}\mathrm{d}s}\right]\\
&\quad\times\mathrm{E}_{Q_\mu}[\mathrm{e}^{\alpha Y(T)}I(\alpha Y(T)>gT)+\mathrm{e}^{gT}I(\alpha Y(T)\le gT)].
\end{aligned}
$$

From Proposition 3.5, we obtain

$$
\begin{aligned}
&\mathrm{E}_{Q_\mu}[\mathrm{e}^{\alpha Y(T)}I(\alpha Y(T)>gT)]\\
&=\int_{\frac{gT}{\alpha}}^{\infty}\mathrm{e}^{\alpha x}\frac{1}{\sqrt{2\pi m_2(T)}}\mathrm{e}^{-\frac{(x-m_1(T))^2}{2m_2(T)}}\mathrm{d}x\\
&=\mathrm{e}^{m_1(T)\alpha+\frac{\alpha^2m_2(T)}{2}}\Phi\left(\frac{m_1(T)+\alpha m_2(T)-\frac{gT}{\alpha}}{\sqrt{m_2(T)}}\right).
\end{aligned}
$$

So, it follows that

$$
\begin{aligned}
&\mathrm{E}\left[\max(\mathrm{e}^{\alpha Y(T)},\mathrm{e}^{gT})\mathrm{e}^{-\int_0^T r_s\mathrm{d}s}I(\tau_x>T)\right]\\
&=P(0,T)p(x,0,T)\left[\mathrm{e}^{gT}\Phi\left(\frac{\frac{gT}{\alpha}-m_1(T)}{\sqrt{m_2(T)}}\right)\right.\\
&\left.+\mathrm{e}^{m_1(T)\alpha+\frac{\alpha^2m_2(T)}{2}}\Phi\left(\frac{m_1(T)+\alpha m_2(T)-\frac{gT}{\alpha}}{\sqrt{m_2(T)}}\right)\right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_{t=0}^{T-1}\mathrm{E}\left[\max(\mathrm{e}^{\alpha Y(t+1)},\mathrm{e}^{g.(t+1)})\mathrm{e}^{-\int_0^{t+1}r_s\mathrm{d}s}I(t<\tau_x\le t+1)\right]\\
&=\sum_{t=0}^{T-1}\mathrm{E}\left\{\max(\mathrm{e}^{\alpha Y(t+1)},\mathrm{e}^{g.(t+1)})\mathrm{e}^{-\int_0^{t+1}r_s\mathrm{d}s}\right.\\
&\left.\times\left(\mathrm{e}^{-\int_0^t\mu_{x+s}\mathrm{d}s}-\mathrm{e}^{-\int_0^{t+1}\mu_{x+s}\mathrm{d}s}\right)\right\}\\
&=\sum_{t=0}^{T-1}\left\{P(0,t+1)[p(x,0,t)-p(x,0,t+1)]\right.\\
&\times\left[\mathrm{e}^{g.(t+1)}\Phi\left(\frac{\frac{g.(t+1)}{\alpha}-m_1(t+1)}{\sqrt{m_2(t+1)}}\right)\right.\\
&+\mathrm{e}^{m_1(t+1)\alpha+\frac{\alpha^2m_2(t+1)}{2}}\\
&\left.\left.\times\Phi\left(\frac{m_1(t+1)+\alpha m_2(t+1)-\frac{g.(t+1)}{\alpha}}{\sqrt{m_2(t+1)}}\right)\right]\right\}.
\end{aligned}
$$

So, we have the final pricing result (3.12). □

3.2. The annual reset design

We assume that, at the end of every period, the annuity will earn the periodic return on an asset $\{S_t\}$ with a participation rate α, or the minimum guaranteed return of g, whichever is higher. Once the interest is credited, the earnings are locked in and never decrease, regardless of the future performance of the market. Thus the value of this EIA policy is

$$
\begin{aligned}
P_{\mathrm{ar}}&=\mathrm{E}\left[\prod_{i=1}^{T}\max(\mathrm{e}^{\alpha Y_i},\mathrm{e}^{g})\mathrm{e}^{-\int_0^T r_s\mathrm{d}s}I(\tau_x>T)\right]\\
&\quad+\sum_{t=0}^{T-1}\mathrm{E}\left[\prod_{i=1}^{t+1}\max(\mathrm{e}^{\alpha Y_i},\mathrm{e}^{g})\mathrm{e}^{-\int_0^{t+1}r_s\mathrm{d}s}I(t<\tau_x\le t+1)\right], \qquad (3.14)
\end{aligned}
$$

where $Y_i=Y(i)-Y(i-1)$.

Proposition 3.7. *Under the measure Q_μ, $(Y_1,Y_2,\ldots,Y_T)$ is a multivariate normal vector with mean $\mu=(\mu_i)_{1\le i\le T}$ and variance–covariance matrix $\Sigma=(v_{i,j})_{1\le i,j\le T}$, where*

$$
\begin{aligned}
\mu_i&=(r_0-\theta)\frac{\mathrm{e}^{-k(i-1)}-\mathrm{e}^{-ki}}{k}+\theta-\frac{1}{2}(\sigma_1^2+\sigma_2^2)\\
&\quad-\int_0^i\frac{\sigma_3(1-\mathrm{e}^{-k(i-u)})+k\sigma_1}{k}\\
&\quad\times\left(\frac{\sigma_3(1-\mathrm{e}^{-ku})}{k}+\frac{b(\mathrm{e}^{a(u+x)}-1)}{a}\right)\mathrm{d}u\\
&\quad+\int_0^{i-1}\frac{\sigma_3(1-\mathrm{e}^{-k(i-1-u)})+k\sigma_1}{k}\\
&\quad\times\left(\frac{\sigma_3(1-\mathrm{e}^{-ku})}{k}+\frac{b(\mathrm{e}^{a(u+x)}-1)}{a}\right)\mathrm{d}u,
\end{aligned}
$$

$$v_{i,i} = \sigma_2^2 + \int_0^{i-1} \left(\frac{\sigma_3(e^{-k(i-1-u)} - e^{-k(i-u)})}{k} \right)^2 du + \int_{i-1}^{i} \left(\frac{\sigma_3(1 - e^{-k(i-u)}) + \sigma_1 k}{k} \right)^2 du$$

for $i = 1, 2, \ldots, T$ *and*

$$v_{i,j} = \int_0^{i-1} \frac{\sigma_3(e^{-k(i-1-u)} - e^{-k(i-u)})}{k} \frac{\sigma_3(e^{-k(j-1-u)} - e^{-k(j-u)})}{k} du + \int_{i-1}^{i} \frac{\sigma_3(1 - e^{-k(i-u)}) + \sigma_1 k}{k} \frac{\sigma_3(e^{-k(j-1-u)} - e^{-k(j-u)})}{k} du,$$

$$v_{j,i} = v_{i,j}$$

for $1 \le i < j \le T$.

Proof. From (3.11), we have

$$\begin{aligned} Y_i &= (r_0 - \theta)\frac{e^{-k(i-1)} - e^{-ki}}{k} + \theta \\ &\quad - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) + \sigma_2(W_2(i) - W_2(i-1)) \\ &\quad - \int_0^i \frac{\sigma_3(1 - e^{-k(i-u)}) + k\sigma_1}{k} \\ &\quad \times \left(\frac{\sigma_3(1 - e^{-ku})}{k} + \frac{b(e^{a(u+x)} - 1)}{a} \right) du \\ &\quad + \int_0^{i-1} \frac{\sigma_3(1 - e^{-k(i-1-u)}) + k\sigma_1}{k} \\ &\quad \times \left(\frac{\sigma_3(1 - e^{-ku})}{k} + \frac{b(e^{a(u+x)} - 1)}{a} \right) du \\ &\quad + \int_0^{i-1} \frac{\sigma_3(e^{-k(i-1-u)} - e^{-k(i-u)})}{k} d\tilde{\tilde{W}}_1(u) \\ &\quad + \int_{i-1}^{i} \frac{\sigma_3(1 - e^{-k(i-u)}) + \sigma_1 k}{k} d\tilde{\tilde{W}}_1(u). \end{aligned}$$

Then the assertion follows immediately by direct calculation. □

We denote $f(y_1, y_2, \ldots, y_T)$ as the density of $(Y_1, Y_2, \ldots, Y_T)$, and $y = (y_1, y_2, \ldots, y_T)$. Then

$$f(y_1, y_2, \ldots, y_T) = \frac{1}{(2\pi)^{T/2}\sqrt{|\Sigma|}} e^{-\frac{1}{2}(y-\mu)\Sigma^{-1}(y-\mu)'}.$$

Let

$$h_{i1} = e^{\alpha Y_i} I\left(Y_i > \frac{g}{\alpha}\right),$$
$$h_{i2} = e^{g} I\left(Y_i \le \frac{g}{\alpha}\right),$$
$$h(t+1) = E_{Q_\mu}\left[\prod_{i=1}^{t+1} \max(e^{\alpha Y_i}, e^g)\right].$$

Then

$$\begin{aligned} h(t+1) &= E_{Q_\mu}\left[\prod_{i=1}^{t+1}(h_{i1} + h_{i2})\right] \\ &= E_{Q_\mu}\left[\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\cdots\sum_{j_{t+1}=1}^{2} h_{1j_1}h_{2j_2}\cdots h_{t+1j_{t+1}}\right] \\ &= \sum_{j_1=1}^{2}\cdots\sum_{j_{t+1}=1}^{2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_{1j_1}(y_1)\cdots h_{t+1j_{t+1}}(y_{t+1}) \\ &\quad \times f(y_1, y_2, \ldots, y_{t+1}) dy_1 \cdots dy_{t+1}. \end{aligned} \tag{3.15}$$

Theorem 3.8. *The present value of the annual reset EIA is*

$$P_{ar} = P(0,T)p(x,0,T)h(T) + \sum_{t=0}^{T-1}\{P(0,t+1)[p(x,0,t) - p(x,0,t+1)]h(t+1)\} \tag{3.16}$$

where $P(0,T)$, $p(x,0,T)$ *and* $h(T)$ *are defined by* (3.5), (3.10) *and* (3.15), *respectively.*

Proof. It follows immediately from Propositions 3.1 and 3.3 that

$$\begin{aligned} P_{ar} &= P(0,T)p(x,0,T)E_{Q_\mu}\left[\prod_{i=1}^{T}\max(e^{\alpha Y_i}, e^g)\right] \\ &\quad + \sum_{t=0}^{T-1}\left\{P(0,t+1)[p(x,0,t) - p(x,0,t+1)]E_{Q_\mu} \times \left[\prod_{i=1}^{t+1}\max(e^{\alpha Y_i}, e^g)\right]\right\} \\ &= P(0,T)p(x,0,T)h(T) \\ &\quad + \sum_{t=0}^{T-1}\{P(0,t+1)[p(x,0,t) - p(x,0,t+1)]h(t+1)\}. \end{aligned}$$ □

Some annual reset EIAs have a cap on the periodic return. Let $d(>g)$ be the cap, this will be the maximum rate of interest that the policy can earn in each period. The value of the EIA policy becomes

$$\begin{aligned} P_{arc} &= E\left[\prod_{i=1}^{T}\max(\min(e^{\alpha Y_i}, e^d), e^g)e^{-\int_0^T r_s ds} I(\tau_x > T)\right] \\ &\quad + \sum_{t=0}^{T-1} E\left[\prod_{i=1}^{t+1}\max(\min(e^{\alpha Y_i}, e^d), e^g)e^{-\int_0^{t+1} r_s ds} \times I(t < \tau_x \le t+1)\right]. \end{aligned} \tag{3.17}$$

Let

$$l_{i1} = e^{\alpha Y_i} I(g < \alpha Y_i \le d),$$
$$l_{i2} = e^{g} I(\alpha Y_i \le g),$$
$$l_{i3} = e^{d} I(\alpha Y_i \ge d),$$
$$l(t+1) = E_{Q_\mu}\left[\prod_{i=1}^{t+1}\max(\min(e^{\alpha Y_i}, e^d), e^g)\right].$$

Then, we have

$$\begin{aligned} l(t+1) &= E_{Q_\mu}\left[\prod_{i=1}^{t+1}(l_{i1} + l_{i2} + l_{i3})\right] \\ &= E_{Q_\mu}\left[\sum_{j_1=1}^{3}\sum_{j_2=1}^{3}\cdots\sum_{j_{t+1}=1}^{3} l_{1j_1}l_{2j_2}\cdots l_{t+1j_{t+1}}\right] \\ &= \sum_{j_1=1}^{3}\cdots\sum_{j_{t+1}=1}^{3}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} l_{1j_1}(y_1)\cdots l_{t+1j_{t+1}}(y_{t+1}) \\ &\quad \times f(y_1, y_2, \ldots, y_{t+1}) dy_1 \cdots dy_{t+1}. \end{aligned} \tag{3.18}$$

According to Propositions 3.1 and 3.3, the proof of the following Theorem 3.9 is completely similar to that of Theorem 3.8, and is omitted here.

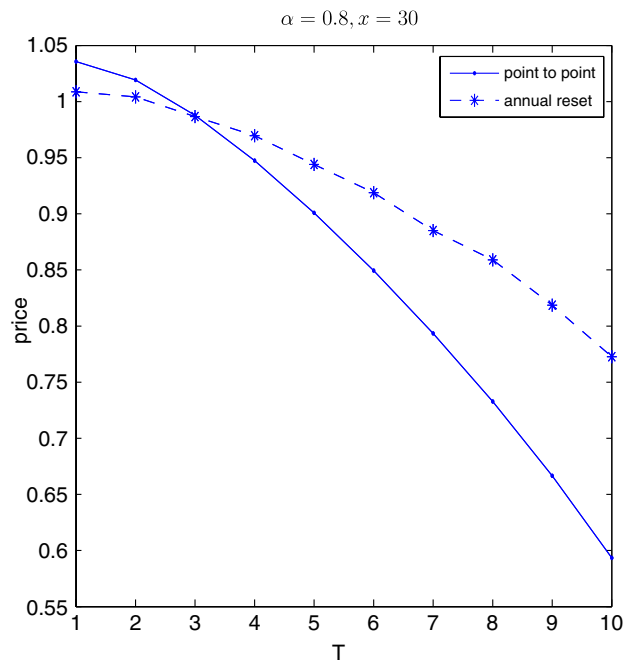

Fig. 1. The effect of T on the prices of point-to-point EIA and annual reset EIA.

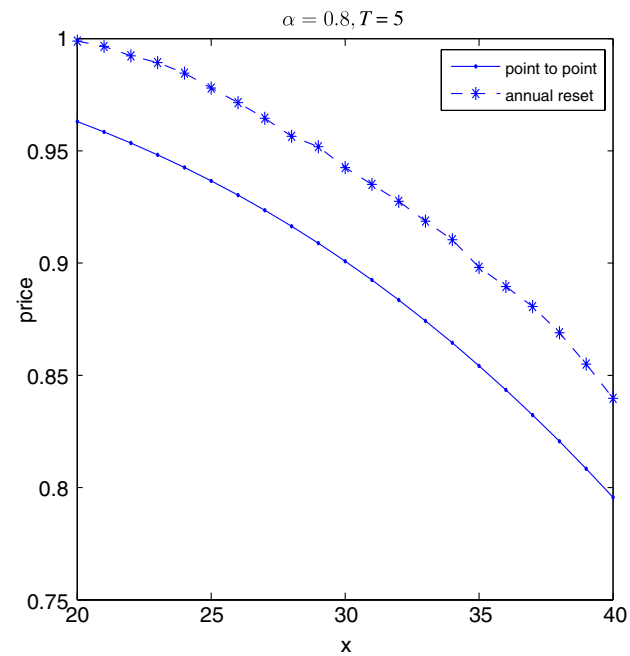

Fig. 2. The effect of x on the prices of point-to-point EIA and annual reset EIA.

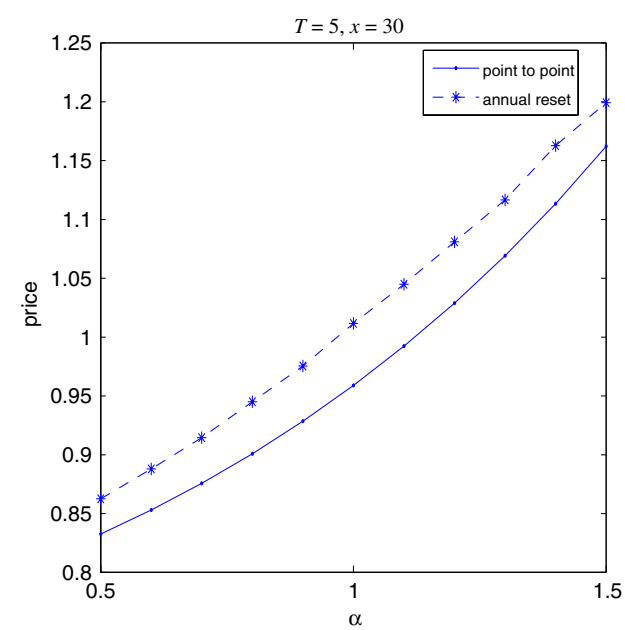

Fig. 3. The effect of α on the prices of point-to-point EIA and annual reset EIA.

Theorem 3.9. *The price of the annual reset EIA with cap d is*

$$P_{\text{arc}} = P(0,T)p(x,0,T)l(T) + \sum_{t=0}^{T-1}\{P(0,t+1)[p(x,0,t) - p(x,0,t+1)]l(t+1)\} \tag{3.19}$$

where $P(0,T)$, $p(x,0,T)$ and $l(T)$ are defined by (3.5), (3.10) and (3.18), respectively.

Our model is general, we give some special cases as follows.

Case 1. When $b = c = 0$, the mortality rate is deterministic. The result of Theorem 3.8 is similar to that of Lin and Tan (2003) and Kijima and Wong (2007).

Case 2. When $\sigma_3 = 0$, the interest rate is no longer stochastic but deterministic. The combined model can be simplified as the stochastic mortality model.

Case 3. When $\rho = 0$, the mortality and interest rate are not dependent.

4. Numerical experiments

In this section, we analyze numerically the proposed model by considering several types of EIAs. Parameter estimation for the models is a crucial issue in the numerical experiments. Ait-Sahalia (1996) proposed a nonparametric estimation procedure for the parameters of the extended Vasicek interest model. The least squares method and the maximum likelihood technique are two most commonly used methods for estimating the parameters of the stochastic mortality model, see Cairns et al. (2006), Dowda et al. (2006), Jalen and Mamon (2009), etc.

In this paper, we use the same parameter values estimated by Ait-Sahalia (1996) for the interest rate model (2.3), that is, $k = 0.85837$ and $\theta = 0.089102$. This implies that the long-term mean interest rate is around 8.9% and the mean-reverting intensity is 0.85837. We also assume the initial interest rate $r_0 = 0.04$, and the volatility of the interest rate σ_3 takes the value of 8%. The parameters for the mortality model with specification (2.6) are the same as the values estimated by Jalen and Mamon (2009), i.e., $a = 0.078282$, $b^2 + c^2 = 0.002271^2$. The stock index followed (2.1) is governed by a geometric Brownian motion. The volatility of the index is typically constant either 20% or 30% in the empirical analysis, so we can assume $\sigma_1^2 + \sigma_2^2 = 0.2^2$ or $\sigma_1^2 + \sigma_2^2 = 0.3^2$.

We analyze the effect of some parameters on the prices of point-to-point EIA and annual reset EIA. To simplify calculation, we take $\sigma_1 = \sigma_2$, $\sigma_1^2 + \sigma_2^2 = 0.2^2$ and $\rho = \frac{\sqrt{2}}{2}$, then $\sigma_1 = \sigma_2 = 0.1414$, $b = c = 0.001606$.

Figs. 1–3 denote respectively the relation between parameters T, x, α and values of point-to-point EIA and annual reset EIA. From these figures, we can know the values of EIAs, both point-to-point and annual reset, are decreasing functions with respect to variables T and x, whereas they are increasing in α. Fig. 1 shows that the value of point-to-point EIA decreases faster than that of annual reset EIA in T. We can also obtain that the value of point-to-point EIA is always smaller than that of annual reset EIA from Figs. 2 and Fig. 3.

Table 1
Break even participation rate α of point–point EIA.

$g = 0.03, r_0 = 0.04, k = 0.85837,$ $\sigma_3 = 0.08, \theta = 0.089102, \mu_0 = 0, a = 0.078282$							
x	T	$\sigma_1 = \sigma_2 = 0.1414$			$\sigma_1 = \sigma_2 = 0.2121$		
		$\rho = 0$	$\rho = \frac{\sqrt{2}}{2}$	$\rho = 1$	$\rho = 0$	$\rho = \frac{\sqrt{2}}{2}$	$\rho = 1$
	5	0.7563	0.7687	0.7756	0.6738	0.6835	0.6893
20	7	0.8255	0.8440	0.8544	0.7562	0.7715	0.7807
	10	0.8876	0.9152	0.9314	0.8350	0.8592	0.8740
	5	0.7815	0.7795	0.7933	0.6745	0.6928	0.7047
25	7	0.8490	0.8608	0.8821	0.7579	0.7867	0.8056
	10	0.9077	0.9425	0.9767	0.8391	0.8855	0.9170
	5	0.7829	0.8000	0.8286	0.6761	0.7109	0.7359
30	7	0.8519	0.8937	0.9386	0.7617	0.8172	0.8574
	10	0.9140	0.9995	1.0757	0.8483	0.9412	1.0110

Following the existing literature, the pricing analysis for EIA focuses on the break even participation rate, which is defined to be the participation rate at which the initial premium equals its notional principal ($1 in our case). By holding all other parameter values constant, we can get break even participation rate α. In this paper, we only give the break even participation rates α of point-to-point EIA under different scenarios, the results are summarized in Table 1.

It can be shown from Table 1 that the break even participation rate increases in age x, term T and ρ, but decreases in index volatility.

5. Conclusion

In this paper, we make an attempt to incorporate stochastic mortality and interest rate into the valuation of EIAs. Independence between financial and demographic risk factors is not assumed. By using the technique of changing probability measures, we obtain the closed form of the pricing of EIAs, both point-to-point and annual reset. As three special cases: when ρ, the correlation coefficient between the mortality and the interest rate, equals zero, the mortality and the interest rate will become not dependent. When $b = 0$ and $c = 0$, the combined model can be simplified as the stochastic interest rate model, and the result is similar to that of Lin and Tan (2003) and Kijima and Wong (2007). When σ_3, the parameter of interest rate, equals zero, the combined model can be simplified as the stochastic mortality model. Numerical experiments are performed to analyze the effects of some parameters on the pricing of EIAs. The values of EIAs, both point-to-point and annual reset, are decreasing functions with respect to variables T and x, whereas they are increasing in α.

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