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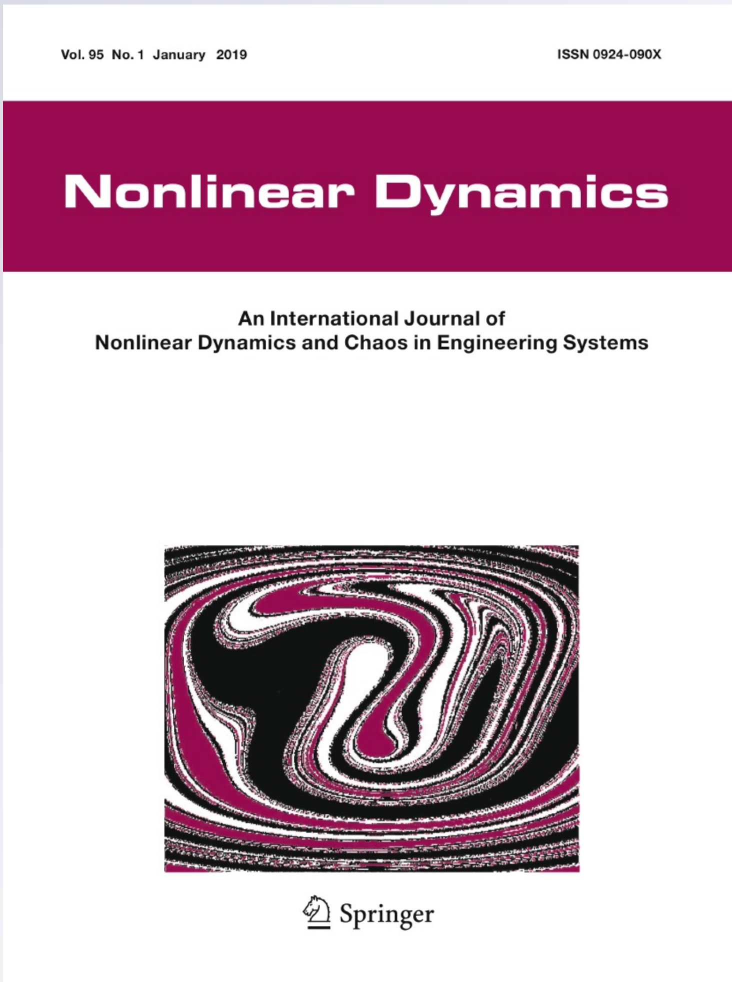

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ORIGINAL PAPER

Semirational solutions to the coupled Fokas–Lenells equations

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Abstract The Darboux transformation for the coupled Fokas–Lenells equations and the special vector solution of the corresponding Lax pair are constructed. Utilizing a limiting process, some novel high-order semirational solutions of the coupled system are given. They include high-order rogue waves interacting with multi-bright or dark solitons, and high-order rogue waves interacting with multi-breathers. Also, the dynamic structures of the first- and second-order semirational solutions are discussed. Furthermore, it is shown that the free parameter γ in the special vector solution can influence the interactional processes (fusion or separation) among different nonlinear waves. Compared to the uncoupled systems, there may exist more abundant and interesting solutions in the coupled ones.

Keywords Coupled Fokas–Lenells equations · Darboux transformation · Semirational solutions

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1 Introduction

It is well known that the nonlinear localized waves have been widely researched in a lot of documents, which usually include rogue waves (RWs) [1,2], breathers [3–5], solitons [6,7] and lump solutions [8,9]. RWs are commonly defined as the gigantic waves owning extreme amplitudes and seem to appear from nowhere and disappear without a trace [10]. Moreover, breathers are periodic in space (Kuznetsov-Ma breathers) [3] or time (Akhmediev breathers) [4,5] or both. Owing to the balance between nonlinearity and dispersion in the nonlinear models, solitons can be generated [7]. Additionally, they always keep their amplitudes and speeds unchanged during propagation. Furthermore, lump solutions are special rational localization solutions, which propagate in all directions both in time and in space [8]. In reality, actual wave dynamics is a superposition of various types of nonlinear waves [11–13].

In recent years, there have been a variety of hybrid solutions among different nonlinear waves researched in many physical models [14–17]. The higher-order semirational solutions including higher-order RWs and higher-order breathers were constructed in the derivative NLS equation [17]. The RW was constructed by the interaction between lump soliton and a pair of resonance kink strip solitons in (2 + 1)-dimensional Korteweg–de Vries (KdV) equation [18]. In [13] and [19], some novel semirational solutions were constructed in local and nonlocal Davey–Stewartson (DS) equations, respectively. Using the Darboux transformation (DT) technique, the hybrid solutions simultaneously including RWs, dark and anti-dark rational traveling waves are exhibited in the nonlocal DS equations [20]. From the mathematical expressions, semirational solution can be defined as a combination of rational and exponential functions [17,19]. In this paper, we focus on semirational solutions of the following coupled Fokas–Lenells (FL) equations [21–24]

$$
\begin{aligned}
&u_{xt}+u+i\left(|u|^2+\frac{1}{2}\sigma|v|^2\right)u_x+\frac{i}{2}\sigma uv^*v_x=0,\\
&v_{xt}+v+i\left(\sigma|v|^2+\frac{1}{2}|u|^2\right)v_x+\frac{i}{2}vu^*u_x=0.
\end{aligned}
\tag{1}
$$

Here, $\sigma = \pm 1$ and the symbol $*$ denotes complex conjugation, and u and v are all complex functions of x and t. Besides, u^* and v^* represent the complex conjugations of u and v, respectively, and "i" is the imaginary unit. The subscripted variables x and t in Eq. (1) denote the corresponding partial differentiation. The symbol "||" is the modulus of a complex function. As the relationship exists between the Camassa–Holm equation and the Korteweg–de Vries (KdV) equation, the FL equation has similar relationship with the nonlinear Schrödinger (NLS) equation [25,26]. Actually, the FL equation is the first negative flow of the hierarchy for the derivative NLS equation [24,27].

In this paragraph, some research history on the FL system will be introduced. The single-component FL equation was first derived in [25] by Fokas. Utilizing the bi-Hamiltonian structure, the Lax pair and conservation laws of the FL system were constructed by Lenells and Fokas [26]. In addition, the high-order RW solutions of the uncoupled FL equation were constructed through the DT technique [27]. In [28], multi-soliton solutions of the single-component FL equation were obtained through DT method. There are many other papers on single-component FL equation, such as dark soliton [29], algebraic geometry solutions [30] and long-time asymptotic behavior of the solution [31]. Similar to the multi-component NLS equations [32], the coupled FL systems are supposed to own more novel and abundant solutions than the ones in the single-component FL equation. With the aid of the spectral gradient method, the coupled FL system was rediscovered. Besides, its Lax pair and conservation laws were also obtained [21]. The coupled FL equations (1) were derived in [23] with $\alpha = -3$, $\beta = \frac{1}{4}$ in the operator L. The general soliton solutions of the coupled system (1) were constructed by DT method [24], such as bright–dark soliton, dark–anti-dark soliton, breather-like soliton and multi-bright (or dark) soliton. In [22], the authors obtained soliton, breathers and RWs for the coupled FL equations. As pointed out in [23], the coupled FL system (1) is equivalent to the coupled FL system in [22] by a gauge transformation. Higher-order RWs of the coupled FL equations were constructed by DT method [33]. Using two integration schemes, optical solitons of the coupled FL equations with differential group delay were given in [34]. The initial boundary value problem for the coupled Fokas–Lenells equations on the half-line was considered by the Riemann–Hilbert approach in [35].

Baronio et al. [11] reported that there existed some semirational solutions in the coupled NLS system, which include the first-order RW interacting with one bright or dark soliton and one breather interacting with the first-order RW. Meanwhile, the experimental conditions to observe these kinds of semirational solutions were given in [11]. RWs on a multi-soliton were obtained in the vector NLS equations by Darboux dressing method [32]. As far as we know, there have not been any reports on semirational solutions of the coupled FL equations (1). It is very necessary to investigate some novel semirational solution of Eq. (1), which includes high-order RWs interacting with multi-bright (or dark) solitons and multi-breathers interacting with high-order RWs.

From the special vector solutions of the Lax pair of the coupled FL equations (1), the concrete expressions of the high-order semirational solutions are given in determinant forms by DT technique [36]. When $\gamma = 0$ (the free parameter in the vector solutions of the Lax pair), the semirational solutions degenerate to the rational ones and they are all RWs. When $\gamma \neq 0$, the semirational solutions can be mainly classified as two types: (1) One component is high-order RWs interacting with multi-bright solitons, and the other one is high-order RWs interacting with multi-dark solitons; (2) two components are all high-order RWs interacting with multi-breathers. By increasing the absolute values of γ, different nonlinear waves can merge with each other.

The paper is organized as follows. In Sect. 2, the Lax pair and DT of the coupled FL equations are constructed. In Sect. 3, the special vector solutions of the Lax pair are skillfully given; then, some semirational solutions are obtained by DT method. Besides, some dynamics of the first- and second-order semirational solutions are discussed in detail. The last section includes several conclusions and discussions.

2 Lax pair and Darboux transformation for the coupled FL equations

The Lax pair of the coupled FL equations (1) can be given as [21–24]

$$\Phi_x = (i\lambda^{-2}J + \lambda^{-1}P_x)\Phi = U\Phi, \tag{2}$$

$$\Phi_t = i\left(\frac{1}{4}\lambda^2 J + \frac{1}{2}J\left(P^2 - \lambda P\right)\right)\Phi = V\Phi, \tag{3}$$

with

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & u^* & \sigma v^* \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix},$$

$$U = i\lambda^{-2}J + \lambda^{-1}P_x, \quad V = i(\frac{1}{4}\lambda^2 J + \frac{1}{2}J(P^2 - \lambda P)).$$

Here, "i" is the imaginary unit and the symbol "$*$" indicates the conjugation of a vector or matrix. Besides, $\Phi(x,t) = (\psi(x,t), \chi(x,t), \phi(x,t))^T$ ("T" denotes the transposition of a vector or matrix) and λ is a spectral parameter. The above-mentioned U, V, P and J are all 3×3 matrices. Additionally, we can derive the coupled FL system (1) through the following compatibility relationship $U_t - V_x + UV - VU = 0$.

Based on the DT constructed in [23,24], the first-step fundamental DT of the coupled FL equations (1) can be expressed as follows

$$T[1] = I + \frac{B_1}{\lambda - \lambda_1^*} - \frac{JB_1J}{\lambda + \lambda_1^*}, \tag{4}$$

$$u[1] = u[0] + \frac{2\chi_1\psi_1^*}{\Gamma_1}, \tag{5}$$

$$v[1] = v[0] + \frac{2\phi_1\psi_1^*}{\Gamma_1}, \tag{6}$$

with

$$\Gamma_1 = \frac{2[\lambda_1|\psi_1|^2 - \lambda_1^*(|\chi_1|^2 + |\phi_1|^2)]}{\lambda_1^{*2} - \lambda_1^2},$$

$$B_1 = L_1|y_1\rangle\langle y_1|K,$$

$$L_1 = diag(-\Gamma_1^{*-1}, \Gamma_1^{-1}, \Gamma_1^{-1}),$$

$$K = diag(1, -1, -\sigma),$$

where $\sigma = \pm 1$ and the vector function $\Phi_1(x,t) = (\psi_1(x,t), \chi_1(x,t), \phi_1(x,t))^T$ is the special solution of Lax pair (2)–(3) with $\lambda = \lambda_1$. Here, $u[0]$ and $v[0]$ are the seed solutions of Eq. (1); thus, $u[1]$ and $v[1]$ denote the first-step solutions of Eq. (1) through the above first-step fundamental DT. In the above expressions, I is the 3×3 identity matrix, and B_1 is a 3×3 matrix which is written as $B_1 = L_1|y_1\rangle\langle y_1|K$ and $|y_1\rangle = \Phi_1(x,t) = (\psi_1(x,t), \chi_1(x,t), \phi_1(x,t))^T$. Additionally, $\psi_1^*(x,t)$ and Γ_1^* are the complex conjugations of $\psi(x,t)$ and Γ, respectively. Here, the symbol "(x,t)" is omitted in the expression ϕ. In the whole contents, the symbol "$diag$" denotes 3×3 diagonal matrix and "$|\ \rangle$" represents a column vector; then, "$\langle\ |$" indicates the Hermite conjugation of the corresponding column vector. For example, $|y_1\rangle = \Phi_1 = (\psi_1, \chi_1, \phi_1)^T$ and $\langle y_1| = |y_1\rangle^\dagger = (\psi_1^*, \chi_1^*, \phi_1^*)$; the symbol $\dagger$ denotes Hermite conjugation.

Based on the related results received in [23,24], the inverse of Darboux matrix $T[1]$ can be written as

$$T[1]^{-1} = I + \frac{KB_1^\dagger K}{\lambda - \lambda_1} - \frac{JKB_1^\dagger KJ}{\lambda + \lambda_1}, \tag{7}$$

where $B_1^\dagger = (L_1|y_1\rangle\langle y_1|K)^\dagger$ and $\dagger$ represents Hermite conjugation in the whole contents. Enlightened by the method to construct the N-step DT for the derivative NLS equation PROPOSITION 2. and THEOREM 2. in [36], we give the following proposition and theorem.

Proposition 1 *The N-step DT for the coupled FL equations (1) can be written as follows*

$$T_N = T[N]T[N-1]\ldots T[1] = I + \sum_{i=1}^{N}\left(\frac{C_i}{\lambda - \lambda_i^*} - \frac{JC_iJ}{\lambda + \lambda_i^*}\right), \tag{8}$$

and the corresponding N-step inverse of DT for the coupled FL equations (1) can also be written as

$$T_N^{-1} = T[1]^{-1}T[2]^{-1}\ldots T[N]^{-1} = I + \sum_{i=1}^{N}\left(\frac{KD_i^\dagger K}{\lambda - \lambda_i} - \frac{JKD_i^\dagger KJ}{\lambda + \lambda_i}\right). \tag{9}$$

Here, C_i and D_i are all 3×3 undermined matrices,

$$T[i] = I + \frac{B_i}{\lambda - \lambda_i^*} - \frac{JB_iJ}{\lambda + \lambda_i^*},$$

$$T[i]^{-1} = I + \frac{KB_i^\dagger K}{\lambda - \lambda_i} - \frac{JKB_i^\dagger KJ}{\lambda + \lambda_i},$$

$$B_i = L_i|y_i\rangle\langle y_i|K,$$
$$\Gamma_i = \frac{2[\lambda_i|\psi_i|^2 - \lambda_i^*(|\chi_i|^2 + |\phi_i|^2)]}{\lambda_i^{*2} - \lambda_i^2},$$
$$L_i = diag(-\Gamma_i^{*-1}, \Gamma_i^{-1}, \Gamma_i^{-1})\ (1 \le i \le N);$$

the column vector $|y_i\rangle = \Phi_i = (\psi_i, \chi_i, \phi_i)^T$ *is the special solution of Lax pair (2)–(3) with* $\lambda = \lambda_i$ *and* $\langle y_i| = |y_i\rangle^\dagger = (\psi_i^*, \chi_i^*, \phi_i^*)$. *In Proposition 1, the* N*-step DT and the corresponding* N*-step inverse of DT are only written in compact formats using the undermined matrices* C_i *and* D_i. *Utilizing the formulas (8) and (9), these two matrices* C_i *and* D_i *can be determined in the following contents.*

Proof From the N-step iterative formula, the expression of T_N can be directly written as

$$\begin{aligned} T_N &= T[N]T[N-1]\dots T[1] \\ &= \left(I+\frac{B_N}{\lambda-\lambda_N^*}-\frac{JB_NJ}{\lambda+\lambda_N^*}\right)\left(I+\frac{B_{N-1}}{\lambda-\lambda_{N-1}^*}\right. \\ &\quad \left.-\frac{JB_{N-1}J}{\lambda+\lambda_{N-1}^*}\right)\dots\left(I+\frac{B_1}{\lambda-\lambda_1^*}-\frac{JB_1J}{\lambda+\lambda_1^*}\right) \\ &= I + \sum_{i=1}^{N}\left(\frac{C_i}{\lambda-\lambda_i^*} + \frac{F_i}{\lambda+\lambda_i^*}\right), \end{aligned} \tag{10}$$

Taking the residues for both sides of Eq. (10) with $\lambda = \lambda_i^*$ and $\lambda = -\lambda_i^*$, respectively, we can get the two expressions of residues as

$$\begin{aligned} \mathrm{Res}|_{\lambda=\lambda_i^*}T_N &= \left(I+\frac{B_N}{\lambda_i^*-\lambda_N^*}-\frac{JB_NJ}{\lambda_i^*+\lambda_N^*}\right)\dots \\ B_i\dots&\left(I+\frac{B_1}{\lambda_i^*-\lambda_1^*}-\frac{JB_1J}{\lambda_i^*+\lambda_1^*}\right) = C_i, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \mathrm{Res}|_{\lambda=-\lambda_i^*}T_N &= -\left(I+\frac{B_N}{-\lambda_i^*-\lambda_N^*}-\frac{JB_NJ}{-\lambda_i^*+\lambda_N^*}\right)\dots \\ JB_iJ\dots&\left(I+\frac{B_1}{-\lambda_i^*-\lambda_1^*}-\frac{JB_1J}{-\lambda_i^*+\lambda_1^*}\right) = F_i. \end{aligned} \tag{12}$$

Here, the symbol "Res" indicates residue of a matrix. From Eq. (12), we can directly calculate that $-J\,\mathrm{Res}|_{\lambda=-\lambda_i^*}T_NJ = \mathrm{Res}|_{\lambda=\lambda_i^*}T_N$; namely, $F_i = -JC_iJ$, and Eq. (8) is proved. Similarly, Eq. (9) can be also proved.

This completes the proof. □

Setting $\Phi_i = (\psi_i, \chi_i, \phi_i)^T$ is the special solutions of Lax pair (2)–(3) with $u = u[0]$, $v = v[0]$ and $\lambda = \lambda_i$ $(1 \le i \le N)$, besides, Φ_i, ψ_i, χ_i and ϕ_i are all complex functions of x and t. Furthermore, the transformations between the new potential functions $u[N]$, $v[N]$ and the seed solutions $u[0]$, $v[0]$ can be constructed by the N-step DT T_N, which is shown in the following theorem.

Theorem 1 *The transformations for the potential functions in the* N*-step DT can be expressed in the following determinant forms*

$$u[N] = u[0] - 2\frac{\begin{vmatrix} M & Y_1^\dagger \\ Y_2 & 0 \end{vmatrix}}{|M|}, \quad v[N] = v[0] - 2\frac{\begin{vmatrix} M & Y_1^\dagger \\ Y_3 & 0 \end{vmatrix}}{|M|}, \tag{13}$$

where $M = (m_{ij})_{N\times N}$,

$$Y_1 = (\psi_1, \psi_2, \dots, \psi_N),\quad Y_1^\dagger = (\psi_1^*, \psi_2^*, \dots, \psi_N^*)^T,$$
$$Y_2 = (\chi_1, \chi_2, \dots, \chi_N),\ Y_3 = (\phi_1, \phi_2, \dots, \phi_N),$$
$$m_{ij} = \frac{\Phi_i^\dagger K \Phi_j}{\lambda_i^* - \lambda_j} - \frac{\Phi_i^\dagger KJ\Phi_j}{\lambda_i^* + \lambda_j}\ (1 \le i, j \le N),$$

where $\Phi_i = (\psi_i, \chi_i, \phi_i)^T$ *and* $\Phi_j = (\psi_j, \chi_j, \phi_j)^T$ *are the corresponding column vector solutions of the Lax pair (2)–(3) with* $\lambda = \lambda_i$ *and* $\lambda = \lambda_j$, *respectively.*

Proof Proposition 1 indicates that $C_i = Res|_{\lambda=-\lambda_i^*}T_N = (I+\frac{B_N}{\lambda_i^*-\lambda_N^*}-\frac{JB_NJ}{\lambda_i^*+\lambda_N^*})\dots B_i\ \dots(I+\frac{B_1}{\lambda_i^*-\lambda_1^*}-\frac{JB_1J}{\lambda_i^*+\lambda_1^*})$, and the ranks of C_i admit the following inequality $1 \le r(C_i) \le min(r(B_i), r(I+\frac{B_j}{\lambda_i^*-\lambda_j^*}-\frac{JB_jJ}{\lambda_i^*+\lambda_j^*}))$ $(j \ne i, 1 \le i, j \le N)$; here the symbol "r" represents the rank of a matrix. Since $B_i = L_i|y_i\rangle\langle y_i|K$, we can directly calculate that $r(B_i) = 1$ and then $r(C_i) = 1$.

From $r(C_i) = 1$ and Eq. (11), we can directly calculate that $C_i = |x_i\rangle\langle y_i|$, where $|x_i\rangle$ is an undetermined 3-tuple column vector and $\langle y_i| = |y_i\rangle^\dagger = (\psi_i^*, \chi_i^*, \phi_i^*)$. We consider the conjugate form of the linear system (2)–(3), which can be written as

$$\Psi_x = -\Psi U,\quad \Psi_t = -\Psi V. \tag{14}$$

Here, Ψ is a 3-tuple row vector and U, V own the same forms of the ones in the Lax pair (2)–(3).

It can be directly calculated that U and V in the Lax pair (2)–(3) have the following symmetry:

$$U(\lambda) = -KU(\lambda^*)^\dagger K,\quad V(\lambda) = -KV(\lambda^*)^\dagger K. \tag{15}$$

From Eqs. (14) and (15), we can find that if Φ_i is a solution for the Lax pair(2)–(3) with $\lambda = \lambda_i$, then $\Phi_i^\dagger K$ is a solution for the conjugated system (14) with $\lambda = \lambda_i^*$.

Additionally, the equality holds $T_N T_N^{-1} = I$, then we can have the following residue of $T_N T_N^{-1}$ that $Res|_{\lambda=\lambda_i^*} T_N T_N^{-1} = 0$. It can be rewritten as follows

$$<y_i|T_N^{-1}|_{\lambda=\lambda_i^*} = 0. \tag{16}$$

Besides, $\Phi_i^\dagger K$ is the special solution of (14) with $\lambda = \lambda_i^*$ and admits the following equality

$$\Phi_i^\dagger K T_N^{-1}|_{\lambda=\lambda_i^*} = 0. \tag{17}$$

Comparing Eqs. (16) and (17), we can choose that $<y_i| = \Phi_i^\dagger K$.

In order to calculate the concrete expressions of $C_i = |x_i><y_i| = |x_i>\Phi_i^\dagger K$, we should first derive out the undetermined 3-tuple column vector $|x_i>$. Since $T_N|_{\lambda=\lambda_j}\Phi_j = 0$ $(1 \le j \le N)$, one can get the following equality by N-step DT formula (8)

$$\Phi_j + \sum_{i=1}^{N}\left(\frac{|x_i>\Phi_i^\dagger K\Phi_j}{\lambda_j-\lambda_i^*} - \frac{J|x_i>\Phi_i^\dagger KJ\Phi_j}{\lambda_j+\lambda_i^*}\right) = 0 \quad (j = 1, 2, \ldots, N). \tag{18}$$

Solving (18), we can get

$$(|x_1>, |x_2>, \ldots, |x_N>)_1 = (\psi_1, \psi_2, \ldots, \psi_N)H^{-1},$$
$$(|x_1>, |x_2>, \ldots, |x_N>)_2 = (\chi_1, \chi_2, \ldots, \chi_N)M^{-1},$$
$$(|x_1>, |x_2>, \ldots, |x_N>)_3 = (\phi_1, \phi_2, \ldots, \phi_N)M^{-1},$$

with

$$H = (h_{ij})_{N\times N}, \quad h_{ij} = \frac{\Phi_i^\dagger K\Phi_j}{\lambda_i^*-\lambda_j} + \frac{\Phi_i^\dagger KJ\Phi_j}{\lambda_i^*+\lambda_j},$$
$$M = (m_{ij})_{N\times N}, \quad m_{ij} = \frac{\Phi_i^\dagger K\Phi_j}{\lambda_i^*-\lambda_j} - \frac{\Phi_i^\dagger KJ\Phi_j}{\lambda_i^*+\lambda_j},$$

where these subscripts "$_1$," "$_2$" and "$_3$" stand for the first, second and third rows of the 3-tuple column vector $|x_i>$ $(1 \le i \le N)$, respectively.

Since T_N is the N-step DT for the coupled FL equations (1), we can get

$$T_{N,x} + T_N U = U[N]T_N. \tag{19}$$

Multiplying both sides of Eq. (19) by λ and letting $\lambda \to 0$, we arrive at

$$P[N] = P[0] + \sum_{i=1}^{N}(C_i - JC_iJ), \tag{20}$$

and here $C_i = |x_i><y_i| = |x_i>\Phi_i^\dagger K$, the capital letters $U[N]$ and $V[N]$ denote the N-step transformed matrices of U and V in the Lax pair (2)–(3) through the N-step DT, respectively. Additionally, the small letters $u[N]$ and $v[N]$ indicate the N-step transformed solutions of Eq. (1) with the seed solutions $u[0]$ and $v[0]$ through the N-step DT separately. Hence, the $P[N]$ and $P[0]$ are the results that the elements u,v in the matrix P are replaced by $u[N]$,$v[N]$ and $u[0]$,$v[0]$, respectively.

Substituting the above concrete expressions of $|x_i>$ and $<y_i|$ into Eq. (20), it follows that

$$u[N] = u[0] + \sum_{i=1}^{N}(C_i - JC_iJ)_{21} = u[0] + 2(\chi_1, \chi_2, \ldots, \chi_N)M^{-1}\begin{pmatrix}\psi_1^*\\ \psi_2^*\\ \vdots\\ \psi_N^*\end{pmatrix} = u[0] - 2\frac{\begin{vmatrix} M & Y_1^\dagger\\ Y_2 & 0\end{vmatrix}}{|M|}, \tag{21}$$

$$v[N] = v[0] + \sum_{i=1}^{N}(C_i - JC_iJ)_{31} = v[0] + 2(\phi_1, \phi_2, \ldots, \phi_N)M^{-1}\begin{pmatrix}\psi_1^*\\ \psi_2^*\\ \vdots\\ \psi_N^*\end{pmatrix} = v[0] - 2\frac{\begin{vmatrix} M & Y_1^\dagger\\ Y_3 & 0\end{vmatrix}}{|M|}, \tag{22}$$

where

$Y_1 = (\psi_1, \psi_2, \ldots, \psi_N), \quad Y_2 = (\chi_1, \chi_2, \ldots, \chi_N),$
$Y_3 = (\phi_1, \phi_2, \ldots, \phi_N);$

besides, the subscripts "$_{21}$" and "$_{31}$" denote the second row of the first column of a matrix and the third row of the first column of a matrix, respectively.

To construct the uniform formulae (21) and (22), the following identities have been used. Suppose M is a $N \times N$ matrix and ϕ and ψ are N-tuple row vectors, then the following identities hold

$$\phi M^{-1}\psi^{\dagger} = \frac{\begin{vmatrix} M & \psi^{\dagger} \\ -\phi & 0 \end{vmatrix}}{|M|}.$$

This completes the proof. □

3 High-order semirational solutions for the coupled FL equations

In order to utilize a limiting process for constructing semirational solutions of Eq. (1), the appropriate solution of the Lax pair (2)–(3) should be derived first. The seed solution of Eq. (1) can be directly chosen as

$$u[0] = a_1 e^{i\eta}, \quad v[0] = a_2 e^{i\eta}, \tag{23}$$

where $\eta = \frac{1}{a_1^2+\sigma a_2^2}x$, a_1 and a_2 are all real constants ($a_1 \neq a_2$). Considering the above seed solution Eq. (23) and the spectrum parameter λ, a special vector solution of the Lax pair (2)–(3) can be constructed as follows

$$\Phi = \begin{pmatrix} \psi \\ \chi \\ \phi \end{pmatrix} = \begin{pmatrix} (K_1 e^{H_1} - K_2 e^{-H_1})e^{-\frac{i}{2}\eta} \\ l_1(K_2 e^{H_1} - K_1 e^{-H_1})e^{\frac{i}{2}\eta} - \gamma\sigma a_2 e^{H_2} \\ l_2(K_2 e^{H_1} - K_1 e^{-H_1})e^{\frac{i}{2}\eta} + \gamma a_1 e^{H_2} \end{pmatrix}, \tag{24}$$

where

$$H_1 = \frac{i\sqrt{\lambda^4 + 4(a_1^2+\sigma a_2^2)^2}}{4\lambda^2(a_1^2+\sigma a_2^2)}\left[2x + (a_1^2+\sigma a_2^2)\lambda^2 t + \sum_{i=1}^{N} S_i \epsilon^{2i}\right], \quad H_2 = -\frac{i(4x+\lambda^4 t)}{4\lambda^2},$$

$$l_1 = \frac{a_1}{\sqrt{a_1^2+\sigma a_2^2}}, \quad l_2 = \frac{a_2}{\sqrt{a_1^2+\sigma a_2^2}}, \quad S_i = m_i + i n_i,$$

$$K_1 = \frac{\left[\lambda^2 + 2(a_1^2+\sigma a_2^2) + \sqrt{\lambda^4 + 4(a_1^2+\sigma a_2^2)^2}\right]^{\frac{1}{2}}}{\sqrt{\lambda^4 + 4(a_1^2+\sigma a_2^2)^2}},$$

$$K_2 = \frac{\left[\lambda^2 + 2(a_1^2+\sigma a_2^2) - \sqrt{\lambda^4 + 4(a_1^2+\sigma a_2^2)^2}\right]^{\frac{1}{2}}}{\sqrt{\lambda^4 + 4(a_1^2+\sigma a_2^2)^2}}.$$

Here, m_i, n_i and γ are three arbitrary real constants. For convenience, the parameter σ in the coupled FL equations (1) is chosen as $\sigma = 1$ in the following contents.

For deriving the solution (24), the variable coefficient differential expressions in the Lax pair (2)–(3) with the seed solution Eq. (23) should be transformed to constant coefficient ones by the gauge transformation $\Phi = N\Psi$. The transformed Lax pair reads as

$$\Psi_x = U_0\Psi = (N^{-1}UN - N^{-1}N_x),$$
$$\Psi_t = V_0\Psi = (N^{-1}VN - N^{-1}N_t),$$

and $N = diag(e^{-\frac{2i}{3}\eta}, e^{\frac{i}{3}\eta}, e^{\frac{i}{3}\eta})$. In this paper, we consider the case that the characteristic equation of U_0 owns a double root. Under this condition, the spectrum parameter λ should be chosen as $\lambda_0 = (1+i)\sqrt{a_1^2+a_2^2}$. Substituting $\lambda = \lambda_1 = (1+i+\epsilon^2)\sqrt{a_1^2+a_2^2}$ (the parameter ϵ is a complex infinitesimal) into the vector solution (24), the Taylor expansion of $\Phi_1 = \Phi|_{\lambda=\lambda_1}$ can be expanded as

$$\Phi_1 = (\psi_1, \chi_1, \phi_1)^T = \Phi|_{\lambda=\lambda_1} = \sum_{j=0}^{N-1} \Phi_1^{[j]}\epsilon^{2j} + O(\epsilon^{2N}), \tag{25}$$

where $\Phi_1^{[j]} = (\psi_1^{[j]}, \chi_1^{[j]}, \phi_1^{[j]})^T = \frac{1}{(2j)!}\frac{\partial^j \Phi_1}{\partial \epsilon^j}|_{\epsilon=0}$.

Here, we only give the concrete expressions of the first two terms $\Phi_1^{[0]}$ and $\Phi_1^{[1]}$ in Eq. (25) as follows

$$\psi_1^{[0]} = \frac{x + ix - \rho^2 t + i\rho^2 t + \rho}{\rho\sqrt{(2+2i)\rho}} e^{-\frac{ix}{2\rho}}, \tag{26}$$

$$\chi_1^{[0]} = \frac{a_1(x + ix - \rho^2 t + i\rho^2 t - \rho)}{\rho^2\sqrt{2+2i}} e^{\frac{ix}{2\rho}} - \gamma a_2 e^{-\frac{x}{2\rho} + \frac{\rho t}{2}}, \tag{27}$$

$$\phi_1^{[0]} = \frac{a_2(x + ix - \rho^2 t + i\rho^2 t - \rho)}{\rho^2\sqrt{2(1+i)}} e^{\frac{ix}{2\rho}} + \gamma a_1 e^{-\frac{x}{2\rho} + \frac{\rho t}{2}}, \tag{28}$$

$$\begin{aligned}\psi_1^{[1]} = \frac{\sqrt{2(1+i)}}{48\rho^{\frac{7}{2}}}(&2ix^3 - 2x^3 - 6i\rho^2x^2t - 6\rho^2x^2t \\ &- 6i\rho^4xt^2 + 6\rho^4xt^2 + 2i\rho^6t^3 + 2\rho^6t^3 + 6i\rho x^2 \\ &- 12\rho^3xt - 6i\rho^5t^2 - 15\rho^2x + 21i\rho^2x + 3\rho^4t \\ &+ 9i\rho^4t - 3\rho^3 + 6i\rho^3 + 12\rho^2m_1 \\ &+ 12i\rho^2n_1)e^{-\frac{ix}{2\rho}},\end{aligned} \tag{29}$$

$$\chi_1^{[1]} = \frac{ia_1}{24\sqrt{2(1+i)}\rho^4}\Omega e^{\frac{ix}{2\rho}} + \frac{a_2\gamma(ix - x + i\rho^2 t - \rho^2 t)}{2\rho} e^{-\frac{x}{2\rho}+\frac{\rho t}{2}}, \quad (30)$$

$$\phi_1^{[1]} = \frac{ia_2}{24\sqrt{2(1+i)}\rho^4}\Omega e^{\frac{ix}{2\rho}} - \frac{a_1\gamma(ix - x + i\rho^2 t - \rho^2 t)}{2\rho} e^{-\frac{x}{2\rho}+\frac{\rho t}{2}}, \quad (31)$$

with

$$\Omega = 4ix^3 - 12\rho^2x^2t - 12i\rho^4xt^2 + 4\rho^6t^3 - 6\rho x^2 - 6i\rho x^2 + 12\rho^3xt - 12i\rho^3xt + 6\rho^5t^2 + 6i\rho^5t^2 + 6\rho^2x + 36i\rho^2x + 6i\rho^4t + 12\rho^4t - 9i\rho^3 - 12i\rho^2m_1 + 12i\rho^2n_1 + 12\rho^2m_1 + 12\rho^2n_1 - 3\rho^3, \quad \rho = \sqrt{a_1^2 + a_2^2}.$$

Moreover, it is easy to find that m_{ij} in Theorem 1 can be expressed as

$$\begin{aligned} m_{ij}|_{(\lambda_i = \lambda_1, \lambda_j = \lambda_1)} &= \left(\frac{\Phi_i^\dagger K\Phi_j}{\lambda_i^* - \lambda_j}\frac{\Phi_i^\dagger KJ\Phi_j}{\lambda_i^* + \lambda_j}\right)|_{(\lambda_i=\lambda_1,\lambda_j=\lambda_1)} \\ &= \frac{-2[(1+i+\epsilon^2)|\psi_1|^2-(1-i+\epsilon^{*2})(|\chi_1|^2+|\phi_1|^2)]}{\sqrt{a_1^2+a_2^2}(2+\epsilon^2+\epsilon^{*2})(2i+\epsilon^2-\epsilon^{*2})} \\ &= \sum_{j,l=1}^{N} m^{[j,l]}\epsilon^{2(l-1)}\epsilon^{*2(j-1)} + O(|\epsilon|^{4N}), \end{aligned} \quad (32)$$

where

$$m^{[j,l]} = \lim_{\epsilon,\epsilon^*\to 0}\frac{1}{(2(j-1))!(2(l-1)!)}\frac{\partial^{2(j+l-2)}m_{ij}|_{(\lambda_i=\lambda_1,\lambda_j=\lambda_1)}}{\partial\epsilon^{2(l-1)}\partial\epsilon^{*2(j-1)}}.$$

In what follows, taking the limit approach of the N-step DT in Theorem 1, one can construct the Nth-order semirational solutions for the coupled FL equation (1) as

$$u[N] = u[0] - 2\frac{\begin{vmatrix} M & Y_1^\dagger \\ Y_2 & 0 \end{vmatrix}}{|M|}, \quad (33)$$

$$v[N] = v[0] - 2\frac{\begin{vmatrix} M & Y_1^\dagger \\ Y_3 & 0 \end{vmatrix}}{|M|}, \quad (34)$$

with

$$M = (m^{[j,l]})_{1\le j,l\le N},\ Y_1 = (\psi_1^{[0]}, \psi_1^{[1]}, \ldots, \psi_1^{[N-1]}),$$
$$Y_2 = (\chi_1^{[0]}, \chi_1^{[1]}, \ldots, \chi_1^{[N-1]}),$$
$$Y_3 = (\phi_1^{[0]}, \phi_1^{[1]}, \ldots, \phi_1^{[N-1]}).$$

Choosing $N = 1$ in the general expressions of N-order semirational solutions (33) and (34), we can straightway derive the explicit expressions for the first-order semirational solutions of the coupled FL equations (1) as

$$u[1] = a_1e^{\frac{ix}{\rho}} - \frac{16a_1L_1e^{\frac{ix}{\rho}} + 16a_2\gamma L_2e^{\frac{ix}{2\rho}-\frac{x}{\rho}+\frac{\rho t}{2}}}{4(1-i)(a_1+ia_2)(ia_2-a_1)\rho\gamma^2e^{\rho t-\frac{x}{\rho}} + L_3}, \quad (35)$$

$$v[1] = a_2e^{\frac{ix}{\rho}} + \frac{-16a_2L_1e^{\frac{ix}{\rho}} + 16a_1\gamma L_2e^{\frac{ix}{2\rho}-\frac{x}{\rho}+\frac{\rho t}{2}}}{4(1-i)(a_1+ia_2)(ia_2-a_1)\rho\gamma^2e^{\rho t-\frac{x}{\rho}} + L_3}, \quad (36)$$

where

$$L_1 = \frac{2ix^2 + 2i\rho^4t^2 - 2\rho x - 2\rho^3t - i\rho^2}{\sqrt{2-2i}\sqrt{2+2i}\rho^2},$$

$$L_2 = \frac{-x - ix - \rho^2t + i\rho^2t - i\rho}{\sqrt{2-2i}},$$

$$L_3 = \frac{\sqrt{2}(1+i)}{\rho^3}(2\rho x^2 + 2ia_1^2x^2 + 2ia_2^2x^2 + 2i\rho^4a_1^2t^2 + 2\rho^5t^2 + 2i\rho^4a_2^2t^2 - 2i\rho a_1^2x - 2i\rho a_2^2x + 2\rho^2x + 2i\rho^3a_2^2t + 2i\rho^3a_2^2t - 2\rho^4t + i\rho^2a_2^2 + \rho^3 + i\rho^2a_1^2).$$

From the above formulae (35) and (36), we can find that these solutions are combinations of rational and exponential functions, which is called semirational solutions in many documents [11,13,19]. When the free parameter is chosen as $\gamma = 0$, the above semirational solutions are reduced to rational ones and they are all first-order RWs. When $\gamma \neq 0$, various interactional solutions can be generated in the coupled FL equations (1), which include RWs + bright solitons ('+' stands for interaction among different nonlinear waves), RWs + dark solitons and RWs + breathers.

(1) If $\gamma \neq 0$, one of the two parameters a_1 and a_2 is zero, and the semirational solutions including soliton and RW can be constructed in Eq. (1). It is shown

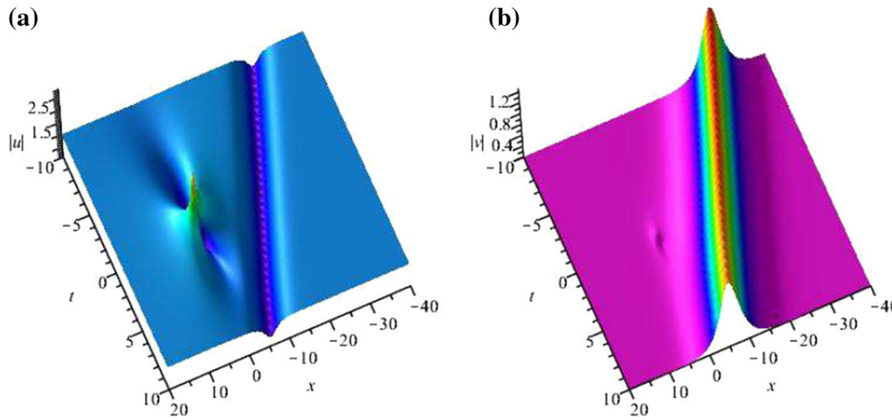

Fig. 1 **a**, **b** First-order semirational solutions of case (1) with parameters chosen by $\gamma = \frac{1}{200}, a_1 = 1, a_2 = 0$

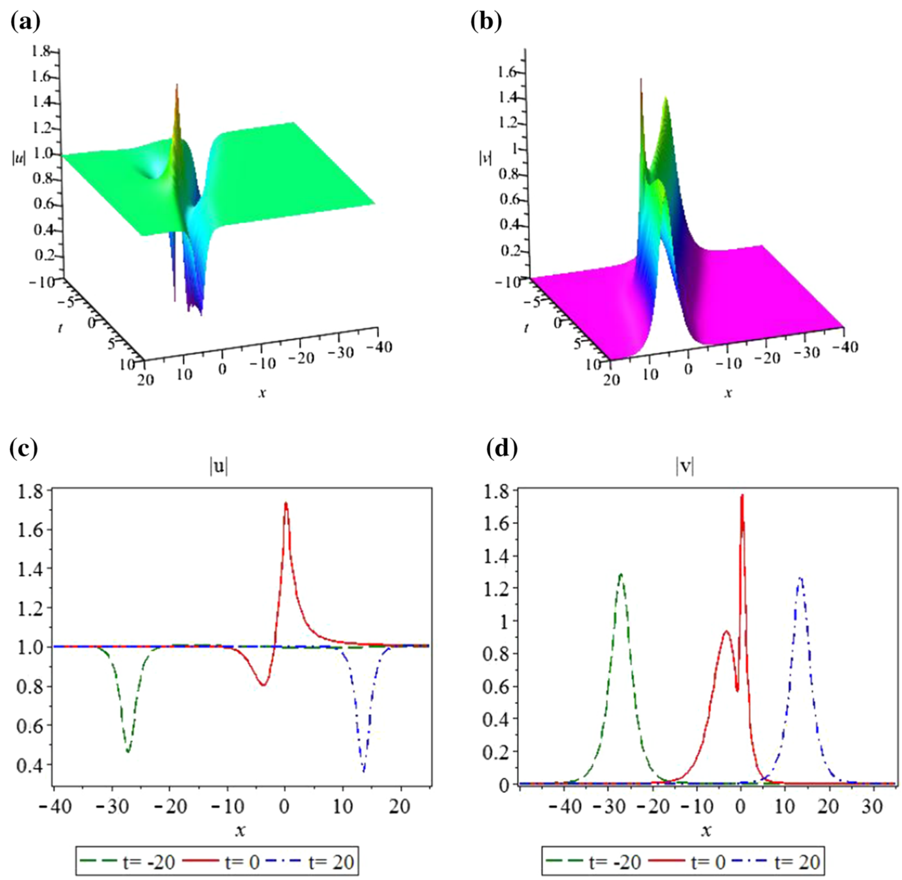

Fig. 2 One dark and bright soliton merges with the first-order RW in case (1) with parameters chosen by $\gamma = 1, a_1 = 1, a_2 = 0$: a, b the three-dimensional plots; **c**, **d** the corresponding plane evolution plots

that one component is the first-order RW + one bright soliton, and the other one is the first-order RW + one dark soliton in Figs. 1 and 2. One first-order RW exists at $t = 0$ on one dark soliton background, see Fig. 1a; one first-order RW appears on one bright soliton background, see Fig. 1b. It is shown that the first-order RW on the left of one bright soliton in Fig. 1b is only a small bump, because the RW is generated on a plane with almost zero amplitude. Besides, the first-order RW and one dark (bright) soliton separate in u and v components with a small value of $|\gamma|$, respectively.

It demonstrates that the first-order RW merges with one dark or one bright soliton by increasing the value of $|\gamma|$ in Fig. 2a, b. At this point, we can easily find the first-order RW on the top of one bright soliton in Fig. 2b, because the RW is generated on a plane with almost 1

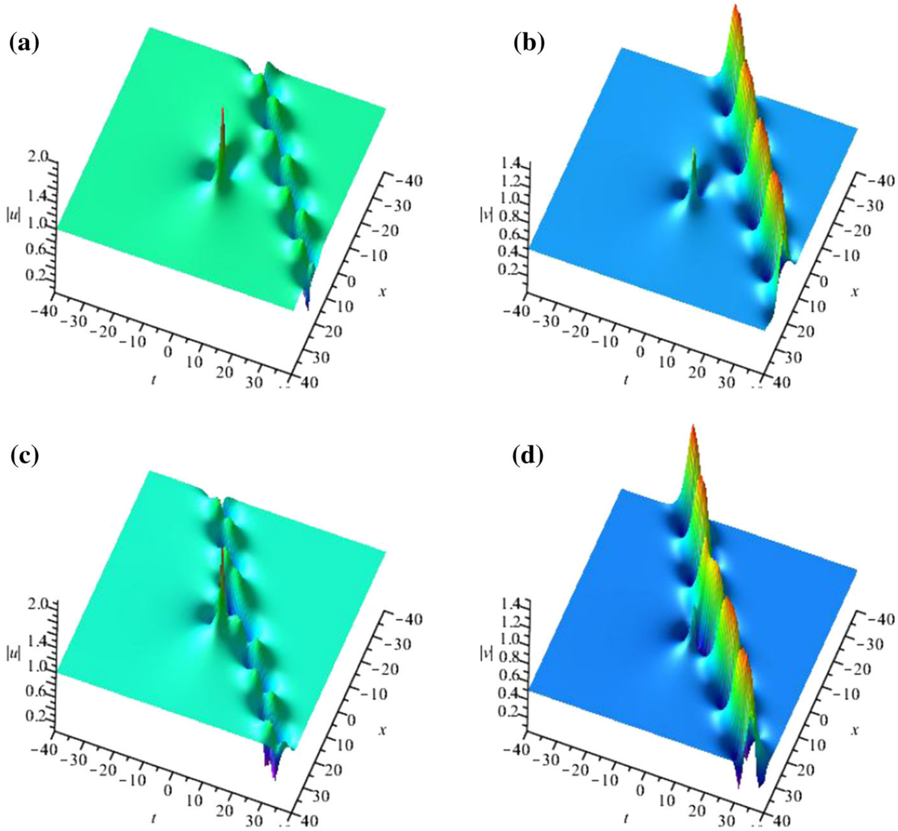

Fig. 3 First-order semirational solution of case (2) with parameters chosen by $a_1 = 1, a_2 = \frac{1}{2}$: **a**, **b** a first-order RW and a breather separate in two components with $\gamma = \frac{1}{100000}$; **c**, **d** a first-order RW merge with a breather in two components with $\gamma = \frac{1}{10}$

amplitude. The evolutionary processes of these hybrid solutions at different moments are shown in Fig. 2c, d. In Fig. 2c, only one dark soliton propagates if $t < 0$, and a first-order RW exists at $t = 0$. Then, the RW disappears and one dark soliton is still preserved if $t > 0$. Additionally, it demonstrates that the collision process in Fig. 2c is not elastic, because the amplitude of the right dark soliton is bigger than the amplitude of the left one. However, the collision between the first-order RW and one bright soliton is elastic, as shown in Fig. 2d.

(2) When $\gamma \neq 0$ and $a_1 a_2 \neq 0$, we can obtain the second kind of interactional solutions where the first-order RW interacts with a breather for the coupled FL equations (1). As shown in Figs. 3a, b, the first-order RW and one breather separate in both u and v components. Moreover, the breathers in these two figures are very different. In Fig. 3a, we can find that the amplitudes of the breather above the background plane are smaller than ones under the background plane. Nevertheless, the corresponding amplitudes of the breather are inverse in Fig. 3b. By increasing the values $|\gamma|$, the first-order RW merges with one breather obviously, see Figs. 3c, d.

Analogously, fixing $N = 2$ in the universal formulae (33) and (34), the second-order semirational solutions for the coupled FL equations (1) can be directly given as

$$u[2] = a_1 e^{\frac{ix}{a_1^2 + a_2^2}} - 2\frac{\begin{vmatrix} m^{[1,1]} & m^{[1,2]} & \psi_1^{[0]*} \\ m^{[2,1]} & m^{[2,2]} & \psi_1^{[1]*} \\ \chi_1^{[0]} & \chi_1^{[1]} & 0 \end{vmatrix}}{\begin{vmatrix} m^{[1,1]} & m^{[1,2]} \\ m^{[2,1]} & m^{[2,2]} \end{vmatrix}}, \tag{37}$$

$$v[2] = a_2 e^{\frac{ix}{a_1^2 + a_2^2}} - 2\frac{\begin{vmatrix} m^{[1,1]} & m^{[1,2]} & \psi_1^{[0]*} \\ m^{[2,1]} & m^{[2,2]} & \psi_1^{[1]*} \\ \phi_1^{[0]} & \phi_1^{[1]} & 0 \end{vmatrix}}{\begin{vmatrix} m^{[1,1]} & m^{[1,2]} \\ m^{[2,1]} & m^{[2,2]} \end{vmatrix}}. \tag{38}$$

Here, $\psi_1^{[0]*}$ and $\psi_1^{[1]*}$ stand for the complex conjugations of $\psi_1^{[0]}$ and $\psi_1^{[1]}$, respectively. The concrete expressions of $\psi_1^{[0]}, \chi_1^{[0]}, \phi_1^{[0]}$ and $\psi_1^{[1]}, \chi_1^{[1]}, \phi_1^{[1]}$ are given in Eqs. (26)–(31). Additionally, the expressions

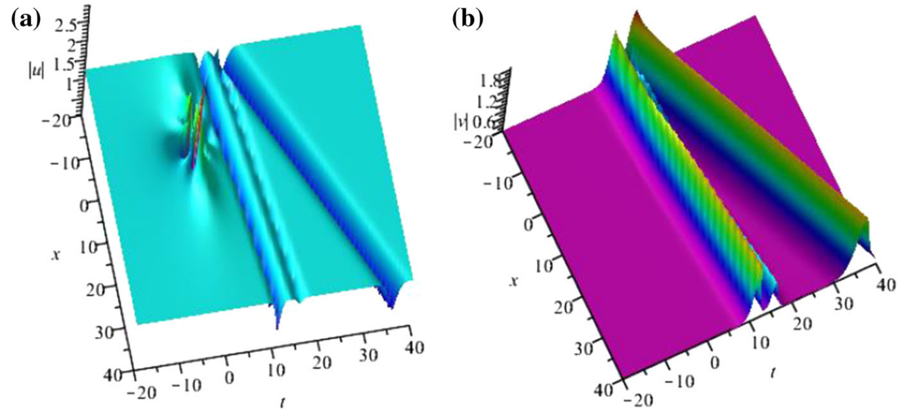

Fig. 4 **a**, **b** Second-order semirational solutions including the fundamental second-order RW interacting with two dark or bright solitons for case (1) with parameters chosen by $a_1 = \frac{5}{4}, a_2 = 0, \gamma = \frac{1}{100000}$

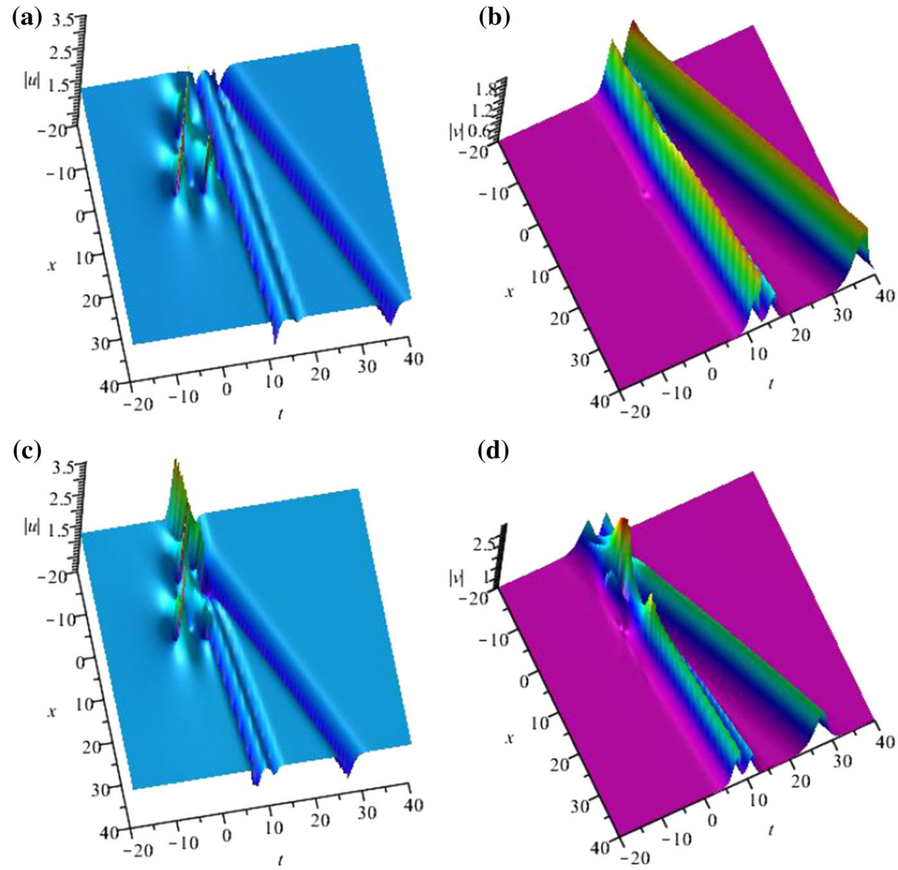

Fig. 5 Second-order semirational solutions including the second-order RW of triangular pattern interacting with two solitons for case (1) with parameters chosen by $a_1 = \frac{5}{4}, a_2 = 0, m_1 = 100, n_1 = -100$: **a**, **b** three first-order RWs and two solitons are separated with $\gamma = \frac{1}{100000}$; **c**, **d** three first-order RWs and two solitons are fused with $\gamma = \frac{1}{100}$

of $m^{[j,l]}$ $(1 \le j, l \le 2)$ are calculated out by Eq. (32). If $\gamma = 0$, the second-order semirational solutions become rational ones and they are the second-order RWs. Similar to the first-order case, the second-order semirational solutions are also classified as two types with $\gamma \neq 0$.

(1) If $\gamma \neq 0$, $a_1 \neq 0$ and $a_2 = 0$, we can get the first kind of the second-order semirational solutions consisted of the second-order RW and two dark (bright) solitons. It is shown that a second-order fundamental RW + two dark solitons exists in u component, and a second-order fundamental RW + two bright solitons appears in v component, see Fig. 4. Similar to Fig. 1b, the second-order RW is not easily seen in Fig. 4b for the reason that the amplitude of background plane on which RW appears are almost zero.

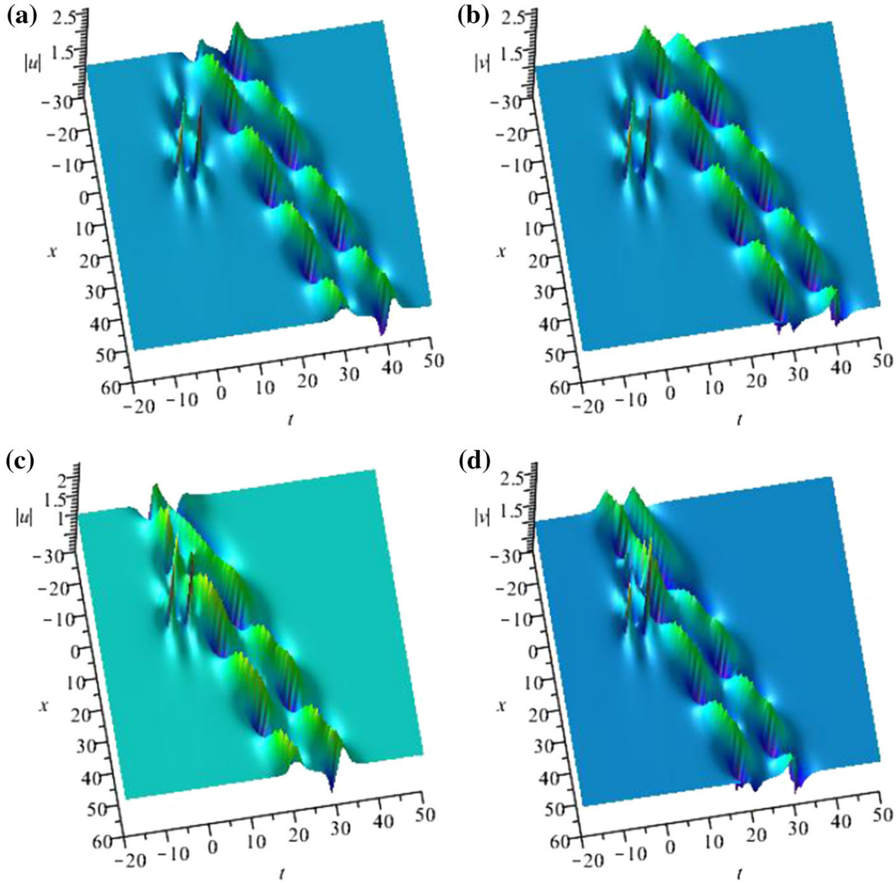

Fig. 6 Second-order semirational solution consisted of the second-order RW of triangular pattern and two breathers for case (2) with parameters chosen by $a_1 = 1, a_2 = -1$: **a**, **b** these two kinds of nonlinear waves are separated with $\gamma = \frac{1}{1000000}$; **c**, **d** they merge with each other with $\gamma = \frac{1}{100}$

The patterns of the high-order RWs in these semirational solutions are determined by the parameters S_i ($S_i = m_i + in_i$) in Eq. (24). Choosing $S_1 \neq 0$, the second-order fundamental RW in Fig. 4 is decomposed into three first-order ones in Fig. 5. Compared to Fig. 5a, b, these three first-order RWs merge with two dark or bright solitons in Fig. 5c, d through increasing the value of $|\gamma|$. These first-order RWs are easily observed in Fig. 5d, because they are generated on significant nonzero plane.

(2) If $\gamma \neq 0$ and $a_1 a_2 \neq 0$, the second kind of the second-order semirational solutions is exhibited, which includes the second-order RW and two breathers. As shown in Fig. 6a, b, a second-order RW of triangular pattern and two breathers coexist in both u and v components. Furthermore, the breathers in these two figures are same. Similar to the first-order case, it is shown that these three first-order RWs merge with two breathers through increasing the value of $|\gamma|$ in Fig. 6c, d.

From Eqs. (33)–(34), some much higher-order semirational solutions to the coupled FL equations (1) can be derived. Similar to the first- and second-order cases, these much higher-order semirational solutions can also be classified as two types with $\gamma \neq 0$: (1) One component is high-order RW + multi-dark solitons, and the other one is high-order RW + multi-bright solitons; (2) two components are all high-order RW + multi-breathers. Moreover, various patterns of high-order RWs in these semirational solutions can be generated by choosing different combinations of m_i and n_i. Here, we only discuss the first- and second-order semirational solutions in detail. We find that these semirational solutions can not be generated in single-component FL equation. It is shown that the solutions in coupled systems are more abundant and interesting than ones in uncoupled systems. Besides, we expect that these semirational solutions can be observed in the physical experiments in the future.

4 Conclusion

A special vector solution of the Lax pair (2)–(3) and the DT for the coupled FL equations (1) are constructed, respectively. Using the limiting technique, some novel interactional solutions to Eq. (1) are exhibited. During the computational processes of the vector solution (24), all solutions in the fundamental solution of the transformed U_0 are reserved and it is very important to generate various semirational solutions. Additionally, the parameter γ in Eq. (24) plays an important role in generating these semirational solutions. If $\gamma = 0$, these semirational solutions are reduced to rational ones RWs; these hybrid solutions exist only if $\gamma \neq 0$.

Moreover, these semirational solutions are mainly classified as two types: (1) One component is RWs + dark solitons, and the other one is RWs + bright solitons; (2) two components are all RWs + breathers. Here, the dynamics of the first- and second-order semirational solutions are discussed in detail. It is shown that these different nonlinear waves can merge with each other significantly by increasing the value of $|\gamma|$. Actually, wave dynamics are superposition of various kinds of nonlinear waves [11,13,32]. It is necessary to investigate the interactional solutions of nonlinear models because we believe the experimental conditions for observing hybrid solutions among first-order RW, one bright (dark) soliton and one breather in [11] in the future could depend on high-order semirational solutions of coupled FL equations.

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Compliance with ethical standards

Conflict of interests The authors declare that they have no conflict of interests regarding the publication of this paper.

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