General solitons and higher-order solitons of the reverse-time Manakov system

Jinyan Zhu [a], Yong Chen [a,b,*]

[a] *School of Mathematical Sciences, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai, 200241, China*
[b] *College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China*

A R T I C L E   I N F O

A B S T R A C T

The main objective of this paper is to study the simple zero soliton solutions and multiple-zero soliton solutions of the reverse-time Manakov system by using Riemann–Hilbert method. It is worth noting that the symmetry of discrete scattering data for the reverse-time Manakov system is very different from the local Manakov system. In addition, in order to better show the remarkable characteristics of soliton solutions, we analyze the dynamic behavior of different solutions.

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1. Introduction

It is well known that nonlinear partial differential equations (PDEs) play an important role in nonlinear science, so it is crucial to study the properties of these PDEs [1]. Especially for a special class of integrable PDE, it has been the goal of many scholars to study the properties of their soliton solutions, breather solutions and rouge wave solutions. Among them, the most important equation was the nonlinear Schrödinger (NLS) equation, which has been widely investigated for many years because it is a universal model of many disciplines [2,3]. However, some more complex models cannot be described by NLS, so it is necessary to study higher dimensional and more complex models. As a generalization of the classical NLS equation, the Manakov system has more properties than NLS, and has clear physical significance. It represents the propagation of optical pulse in birefringent fiber and wavelength division multiplexing system [4]. Later, scholars found that the equation of PT symmetry also has good physical significance and is also worth investigating. So many nonlocal equations have been found and studied, such as nonlocal NLS, nonlocal mKdV, etc [5–7].

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* Corresponding author at: School of Mathematical Sciences, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai, 200241, China.
E-mail address: ychen@sei.ecnu.edu.cn (Y. Chen).

In this paper, we mainly consider the reverse-time Manakov system

$$iu_t(x,t) + u_{xx}(x,t) + 2[u(x,t)u(x,-t) + v(x,t)v(x,-t)]u(x,t) = 0,$$
$$iv_t(x,t) + v_{xx}(x,t) + 2[u(x,t)u(x,-t) + v(x,t)v(x,-t)]v(x,t) = 0. \tag{1}$$

The inverse scattering of the multicomponent nonlocal reverse-time NLS equations were obtained in [8], and the Darboux transformation of system (1) was also studied in [9]. Inspired by their work, we will study the simple zero N-soliton solutions and higher-order soliton solutions of the nonlocal reverse-time Manakov system (1) in more detail, and give its dynamic behavior. Here we mainly use the Riemann–Hilbert method, which is the extension of the classical inverse scattering method [10,11]. Different from the work in [12], the nonlocal equation we studied remains the original form of the Manakov system, and is still a coupled system, which will produce more abundant forms of solutions.

The order of the paper is as follows. In Section 2, the symmetric relation and asymptotic property of the scattering data matrix for the reverse-time Manakov system are analyzed, and the corresponding Riemann–Hilbert Problem (RHP) is constructed. Further, the determinant form of the N-soliton solutions of the simple zero without reflection is given, and the dynamic behavior of various solitons is analyzed in detail. In Section 3, the determinant form of multiple zeros N-soliton solutions is given by using the idea of limits, and the dynamic behavior diagram of the Higher order 2-soliton solution and Higher order 3-soliton solution are shown.

2. Inverse scattering transform for the reverse-time Manakov system

In order to maintain structural integrity, we briefly review the inverse scattering problem of the reverse-time Manakov system. The reverse-time Manakov system (1) admits the following Lax pair

$$\Phi_x = M\Phi, \quad M = M(x,t;z) := -iz\Lambda_0 + U,$$
$$\Phi_t = N\Phi, \quad N = N(x,t;z) := -3iz^2\Lambda_0 + 3zU + i\Lambda_3(U_x - U^2), \tag{2}$$

where Φ is the matrix eigenfunction, z is the spectral parameter and

$$\Lambda_0 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad U(x,t) = \begin{bmatrix} 0 & u(x,t) & v(x,t) \\ -u(x,-t) & 0 & 0 \\ -v(x,-t) & 0 & 0 \end{bmatrix}, \quad \Lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

when $x \to \pm\infty$, $u(x,0), v(x,0)$ have the following behavior $u(x,0) \to 0, v(x,0) \to 0$. Under these conditions, the spectral problems about $\Phi(x,t,z)$ satisfies $\Phi = Ye^{(-izx-3iz^2t)\Lambda_0}$, and $Y \to I$, $x \to \pm\infty$. At the same time, It is easy to find that Y satisfies the integral equation $Y_\pm = I \pm \int_{\pm\infty}^x e^{iz\Lambda_0(y-x)}UY_\pm e^{iz\Lambda_0(x-y)}\mathrm{d}y$. By analyzing the integral equation, the following properties can be obtained

Proposition 1. *Dividing Y into columns as $Y = (Y^{(1)}, Y^{(2)}, Y^{(3)})$, we have*
- *The column vectors $Y_-^{(1)}$ and $Y_+^{(2)}, Y_+^{(3)}$ are analytic for $z \in \mathbb{C}_+$ and continuous for $z \in \mathbb{C}_+ \cup \mathbb{R}$;*
- *The column vectors $Y_+^{(1)}$ and $Y_-^{(2)}, Y_-^{(3)}$ are analytical for $z \in \mathbb{C}_-$ and continuous for $z \in \mathbb{C}_- \cup \mathbb{R}$;*
where $\mathbb{C}_+ = \{z \mid \arg z \in (0,\pi)\}$, $\mathbb{C}_- = \{z \mid \arg z \in (\pi, 2\pi)\}$.

Select the vector composition matrix analyzed in the upper half plane, denoted as N^+:

$$N^+ = (Y_-^{(1)}, Y_+^{(2)}, Y_+^{(3)}) = Y_- H_1 + Y_+ H_2, \tag{3}$$

where $H_1 = \mathrm{diag}\{1,0,0\}$ and $H_2 = \mathrm{diag}\{0,1,1\}$. And N^+ satisfies the asymptotic condition $N^+(x,\lambda) \to I, z \in \mathbb{C}_+ \to \infty$. In addition, the two different forms of solutions for the space part of the Lax equation must

be linearly correlated, thereby introducing the scattering matrix S through the following expression

$$Y_-(x,z,t)e^{-izx\Lambda_0} = Y_+(x,z,t)e^{-izx\Lambda_0}S(z), \quad S(z) = \begin{pmatrix} s_{11}(z) & s_{12}(z) & s_{13}(z) \\ s_{21}(z) & s_{22}(z) & s_{23}(z) \\ s_{31}(z) & s_{32}(z) & s_{33}(z) \end{pmatrix}. \tag{4}$$

Meanwhile, it is easy to verify that Y^{-1} satisfies the adjoint equation of Lax Eq. (2) and also satisfies the boundary condition $Y^{-1} \to I$ as $x \to \pm\infty$.

Proposition 2. *Taking the similar procedure as above denote matrices Y^{-1} as a collection of rows $Y^{-1} = (Y^{-1[1]}, Y^{-1[2]}, Y^{-1[3]})^T$*

- *The row vectors $Y_+^{-1[1]}$ and $Y_-^{-1[2]}, Y_-^{-1[3]}$ are analytic for $z \in \mathbb{C}_+$ and continuous for $z \in \mathbb{C}_+ \cup \mathbb{R}$.*
- *The row vectors $Y_-^{-1[1]}$ and $Y_+^{-1[2]}, Y_+^{-1[3]}$ are analytical for $z \in \mathbb{C}_-$ and continuous for $z \in \mathbb{C}_- \cup \mathbb{R}$.*

In order to construct RHP, the vector composition matrix analyzed in the lower half plane chosen as N_-^{-1}

$$N_-^{-1} = H_1 Y_-^{-1} + H_2 Y_+^{-1} \tag{5}$$

and by inverting both sides of Eq. (4), we can obtain

$$e^{izx\Lambda_0}Y_-^{-1} = S^{-1}(z)e^{izx\Lambda_0}Y_+^{-1}, \quad S^{-1}(z) = \check{S}(z) = \begin{pmatrix} \check{s}_{11}(z) & \check{s}_{12}(z) & \check{s}_{13}(z) \\ \check{s}_{21}(z) & \check{s}_{22}(z) & \check{s}_{23}(z) \\ \check{s}_{31}(z) & \check{s}_{32}(z) & \check{s}_{33}(z) \end{pmatrix}. \tag{6}$$

Through direct calculation, we can get that N_-^{-1} also satisfies the same boundary condition $N_-^{-1}(x,\lambda) \to I, z \in \mathbb{C}_- \to \infty$. In addition, Y satisfies the time equation, and the time evolution of the scattering matrix $S(z)$ and $\check{S}(z)$ can be obtained according to Eqs. (2), (4) and (6), we have

$$S_t + 3iz^2\left[\Lambda_0, S\right] = 0, \qquad \check{S}_t + 3iz^2\left[\Lambda_0, \check{S}\right] = 0.$$

Hence, the RHP of the reverse-time Manakov system is

Riemann-Hilbert Problem 1. *The matrix function $N(z;x,t)$ has the following properties:*

- **Analyticity** : $N(z,x,t)$ *is analytic function in $z \in \mathbb{C}_\pm$;*
- **Jump condition***: $N_+(z,x,t) = N_-(z,x,t)V(z)$, $z \in \mathbb{R}$;*
- **Normalization** : $N(z,x,t) = I + O(z)$, *as $z \to \infty$.*

Where the jump matrix is $V = \Theta \begin{pmatrix} 1 & \check{s}_{12} & \check{s}_{13} \\ s_{21} & 1 & 0 \\ s_{31} & 0 & 1 \end{pmatrix} \Theta^{-1}$, $\Theta = e^{-izx\Lambda_0 - 3iz^2t\Lambda_0}$. If $N_\pm$ is expanded as $N_\pm = I + \frac{N_\pm^{(1)}}{z} + \frac{N_\pm^{(2)}}{z^2} + O\left(\frac{1}{z^3}\right)$, after a simple calculation, it can be obtained that

$$u(x,t) = 3i(N_+^{(1)})_{12}, \qquad v(x,t) = 3i(N_+^{(1)})_{13}.$$

Another important aspect is symmetry. As a special case of multi-component reverse-time NLS [13], the symmetry of the reverse-time Manakov system is also $Y^{-1}(x,t,z) = Y^T(x,-t,-z)$, where superscript 'T' represents the transposition of a matrix. This means that $(z,-z)$ is a pair of spectral parameters, which is different from the distribution of spectral parameters of the local Manakov system. Further,

$$N_-^{-1}(x,t,z) = N_+^T(x,-t,-z), \tag{7}$$

then we have $S^{-1}(x,t,z) = \check{S}(z) = S^T(x,-t,-z)$.

As we all know, for the regular RHP, its solution is easy to be given by the Plemelj formula as $\det(N_+) = s_{11} \neq 0$ and $\det(N_-^{-1}) = \check{s}_{11} \neq 0$. We need to pay more attention to the non-regular case. In other words, $\det(N_+) = s_{11} = 0$ and $\det(N_-^{-1}) = \check{s}_{11} = 0$ at some points. Let us first assume that these zeros are simple and there are N zeros. Therefore, each $\ker N_+(z_k)$ contains only one base column vector, which is represented by $v_k(x, t, z_k)$, and each $\ker N_-^{-1}(\check{z}_k)$ is a single base row vector, represented by $\check{v}_k(x, t, \check{z}_k)$. So we have the following expression

$$N_+(z_k)v_k(x, t, z_k) = 0, \qquad \check{v}_k(x, t, \check{z}_k)N_-^{-1}(\check{z}_k) = 0,$$

With Eq. (7), it can be deduced that $\check{v}_k(x, t, \check{z}_k) = v_k^T(x, -t, -z_k)$. With the help of Ref. [10], we know that $v_k = e^{(-iz_k x - 3iz_k^2 t)\Lambda_0} w_k$ and $\check{v}_k = \check{w}_k e^{(i\check{z}_k x + 3i\check{z}_k^2 t)\Lambda_0}$, where $w_k = (a_k, b_k, c_k)^T$ and $\check{w}_k = (\check{a}_k, \check{b}_k, \check{c}_k)$ are an arbitrary constant vector, according to the symmetry, we can deduce $\check{a}_k = a_k^T$, $\check{b}_k = b_k^T$, $\check{c}_k = c_k^T$. Based on the Yang method, we can perfectly give the solution of the reverse-time Manakov system

$$u = 3i\frac{\det F}{\det M}, \qquad v = 3i\frac{\det G}{\det M}, \qquad (8)$$

where

$$M_{jk} = \frac{a_j a_k^T e^{-2i(z_j + z_k)\left[x - 3(z_j - z_k)t\right]} + \left(b_j b_k^T + c_j c_k^T\right) e^{i(z_j + z_k)\left[x - 3(z_j - z_k)t\right]}}{z_k + z_j},$$

$$F = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} & b_1^T e^{iz_1 x - 3iz_1^2 t} \\ M_{21} & M_{22} & \cdots & M_{2n} & b_2^T e^{iz_2 x - 3iz_2^2 t} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} & b_n^T e^{iz_n x - 3iz_n^2 t} \\ a_1 e^{-2iz_1 x - 6iz_1^2 t} & a_2 e^{-2iz_2 x - 6iz_2^2 t} & \cdots & a_n e^{-2iz_n x - 6iz_n^2 t} & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} & c_1^T e^{iz_1 x - 3iz_1^2 t} \\ M_{21} & M_{22} & \cdots & M_{2n} & c_2^T e^{iz_2 x - 3iz_2^2 t} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} & c_n^T e^{iz_n x - 3iz_n^2 t} \\ a_1 e^{-2iz_1 x - 6iz_1^2 t} & a_2 e^{-2iz_2 x - 6iz_2^2 t} & \cdots & a_n e^{-2iz_n x - 6iz_n^2 t} & 0 \end{pmatrix}.$$

Next, we will analyze the specific properties of soliton solutions in detail.

$1 - $soliton   solution

The form of 1-soliton solution for the reverse-time Manakov system can be obtained as $N = 1$

$$u = -\frac{6iabze^{-9iz^2 t}}{a^2 e^{-3izx} + (b^2 + c^2)e^{3izx}}, \qquad v = -\frac{6iacze^{-9iz^2 t}}{a^2 e^{-3izx} + (b^2 + c^2)e^{3izx}}. \qquad (9)$$

When z is pure imaginary, they can be simplified as a fundamental solitons, which are stationary and bounded. When z is general, it can be seen from the expression that the amplitude of the soliton solutions are determined by the imaginary part. When $Im(z) > 0$, the amplitude of the soliton solutions increase with time. On the contrary, when $Im(z) < 0$, the amplitude of the soliton solutions decrease with time, until they disappear. In Fig. 1, we show two forms of soliton solutions. The basic 1-soliton solution parameters in Figs. 1(a) and 1(b) are $a = b = c = 1, z = i$, and the general 1-soliton solution parameters in Figs. 1(c) and 1(d) are $a = 1, b = \frac{1}{10}, c = i, z = \frac{1}{10} + \frac{3}{10}i$.

$2 - $soliton   solution

When $N = 2$ in Eq. (8), we get the two soliton solutions of the nonlocal reverse-time Manakov system. In the form of

$$u = -6i(z_1 + z_2)\frac{\mathcal{F}}{\mathcal{M}}, \qquad v = -6i(z_1 + z_2)\frac{\mathcal{G}}{\mathcal{M}}, \qquad (10)$$
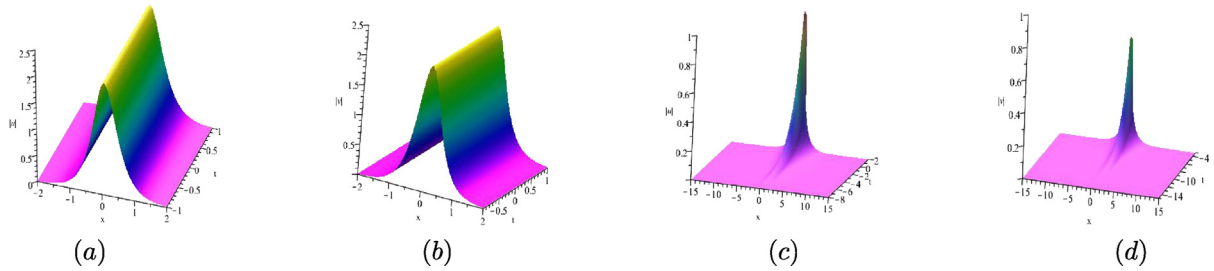
Fig. 1. 1-soliton solution for the nonlocal reverse-time Manakov system. (a) Basic 1-soliton solution of $|u|$; (b) Basic 1-soliton solution of $|v|$; (c) General 1-soliton solution of $|u|$;(d) General 1-soliton solution of $|v|$.

where

$$
\begin{aligned}
\mathcal{F} = {}& (a_1 b_1 b_2^2 z_1^2 - a_1 b_1 b_2^2 z_1 z_2 + a_1 b_1 c_2^2 z_1^2 + a_1 b_1 c_2^2 z_1 z_2 - 2a_1 b_2 c_1 c_2 z_1 z_2) e^{-i(9tiz_1^2 + xz_1 - 2xz_2)} \\
& + \left(-a_2 b_1^2 b_2 z_1 z_2 + a_2 b_1^2 b_2 z_2^2 \right. \\
& \left. \quad -2a_2 b_1 c_1 c_2 z_1 z_2 + a_2 b_2 c_1^2 z_1 z_2 + a_2 b_2 c_1^2 z_2^2\right) e^{i(-9tz_2^2 + 2xz_1 - xz_2)} \\
& + (-a_1^2 a_2 b_2 z_1 z_2 + a_1^2 a_2 b_2 z_2^2) e^{-i(9tz_2^2 + 4xz_1 + xz_2)} \\
& + (a_2^2 b_1 z_1^2 - a_2^2 b_1 z_1 z_2) e^{-i(9tz_1^2 + xz_1 + 4xz_2)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G} = {}& (-2a_1 b_1 b_2 c_2 z_1 z_2 + a_1 b_2^2 c_1 z_1^2 + a_1 b_2^2 c_1 z_1 z_2 + a_1 c_1 c_2^2 z_1^2 - a_1 c_1 c_2^2 z_1 z_2) e^{-i(9tz_1^2 + xz_1 - 2xz_2)} \\
& + \left(a_2 b_1^2 c_2 z_1 z_2 + a_2 b_1^2 c_2 z_2^2 \right. \\
& \left. \quad -2a_2 b_1 b_2 c_1 z_1 z_2 - a_2 c_1^2 c_2 z_1 z_2 + a_2 c_1^2 c_2 z_2^2\right) e^{i(-9tz_2^2 + 2xz_1 - xz_2)} \\
& + (a_1 a_2^2 c_1 z_1^2 - a_1 a_2^2 c_1 z_1 z_2) e^{-i(9tz_1^2 + xz_1 + 4xz_2)} \\
& + (-a_1^2 a_2 c_2 z_1 z_2 + a_1^2 a_2 c_2 z_2^2) e^{-i(9tz_2^2 + 4xz_1 + xz_2)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M} = {}& (a_1^2 a_2^2 z_1^2 - 2a_1^2 a_2^2 z_1 z_2 + a_1^2 a_2^2 z_2^2) e^{-4ix(z_1+z_2)} + (a_1^2 b_2^2 z_1^2 + 2a_1^2 b_2^2 z_1 z_2 + a_1^2 b_2^2 z_2^2 + a_1^2 c_2^2 z_1^2 + 2a_1^2 c_2^2 z_1 z_2 \\
& + a_1^2 c_2^2 z_2^2) e^{-2ix(2z_1 - z_2)} \\
& + (-4a_1 a_2 b_1 b_2 z_1 z_2 - 4a_1 a_2 c_1 c_2 z_1 z_2) e^{-i(9tz_1 - 9tz_2 + x)(z_1 + z_2)} \\
& + (-4a_1 a_2 b_1 b_2 z_1 z_2 - 4a_1 a_2 c_1 c_2 z_1 z_2) e^{i(9tz_1 - 9tz_2 - x)(z_1 + z_2)} \\
& + (a_2^2 b_1^2 z_1^2 + 2a_2^2 b_1^2 z_1 z_2 + a_2^2 b_1^2 z_2^2 + a_2^2 c_1^2 z_1^2 + 2a_2^2 c_1^2 z_1 z_2 + a_2^2 c_1^2 z_2^2) e^{2ix(z_1 - 2z_2)} \\
& + \left(b_1^2 b_2^2 z_1^2 - 2b_1^2 b_2^2 z_1 z_2 + b_1^2 b_2^2 z_2^2 + b_1^2 c_2^2 z_1^2 \right. \\
& \left. \quad +2b_1^2 c_2^2 z_1 z_2 + b_1^2 c_2^2 z_2^2 - 8b_1 b_2 c_1 c_2 z_1 z_2 + b_2^2 c_1^2 z_1^2 + 2b_2^2 c_1^2 z_1 z_2 + b_2^2 c_1^2 z_2^2 + c_1^2 c_2^2 z_1^2 - 2c_1^2 c_2^2 z_1 z_2 + c_1^2 c_2^2 z_2^2\right) \\
& \times e^{2ix(z_1 + z_2)}
\end{aligned}
$$

Based on the symmetry of the above spectral parameters, we consider several relationships between the two eigenvalues:

• $Re(z_1) = Re(z_2) = 0, Im(z_1) \neq Im(z_2)$, $u(x,t)$ and $v(x,t)$ are pure imaginary two soliton solutions, and they are bound solitons. Select different values of a_k, b_k and $c_k, k = 1, 2$, and there are soliton solutions with different amplitudes. Figs. 2(a) and 2(b) show the corresponding display diagram with

$$
z_1 = 2i, a_1 = 1, b_1 = 1, c_1 = \frac{1}{10}, \qquad z_2 = i, a_2 = 1, b_2 = 1, c_2 = 1.
$$

• $Re(z_1) \neq Re(z_2), Im(z_1) = Im(z_2)$ and $Re(z_1) = -Re(z_2)$, $u(x,t)$ and $v(x,t)$ are ordinary two soliton solutions without singularity. Except that the amplitude at the intersection will collapse or become larger, the amplitude at other places will not change with time. We show the dynamic diagram in Figs. 2(c) and
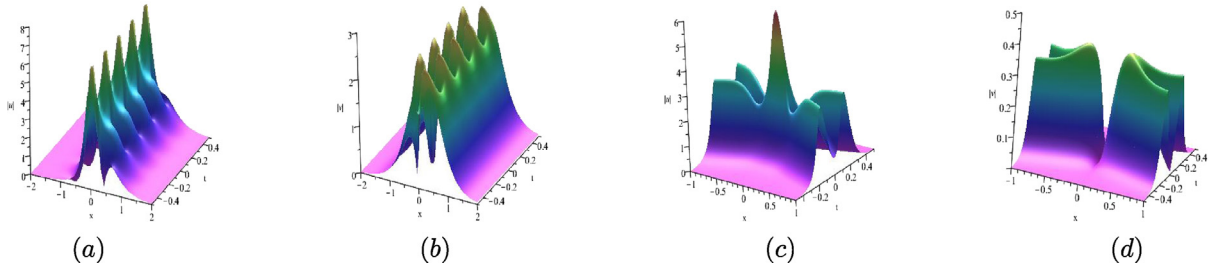
Fig. 2. Two-soliton solution for the Manakov system. (a) Pure imaginary 2 soliton solution of $|u|$; (b) Pure imaginary 2 soliton solution of $|v|$; (c) Ordinary two soliton solution of $|u|$;(d) Ordinary two soliton solution of $|v|$.
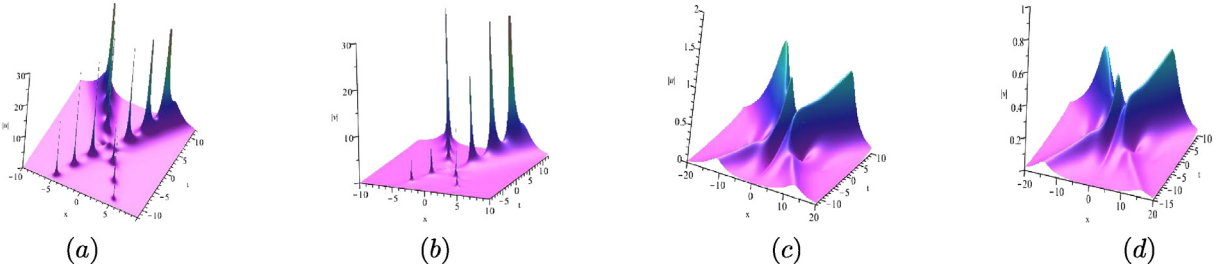

Fig. 3. Two-soliton solution for the reverse-time Manakov system. (a)Singular 2-soliton solution of $|u|$; (b) Singular 2-soliton solution of $|v|$; (c) The 2-soliton solution of the amplitude gradual transformation of $|u|$; (d) The 2-soliton solution of the amplitude gradual transformation of $|v|$.

2(d), and the corresponding parameters selection are

$$z_1 = 1 + i, a_1 = 1, b_1 = 1, c_1 = \frac{i}{10}, \qquad z_2 = -1 + i, a_2 = 1, b_2 = 1, c_2 = \frac{1}{10}.$$

• $Re(z_1) \neq Re(z_2), Im(z_1) \neq Im(z_2)$, $u(x,t)$ and $v(x,t)$ are singular two soliton solutions. We show the dynamic graph in Figs. 3(a) and 3(b), and the corresponding parameters are

$$z_1 = \frac{1}{10} + \frac{1}{4}i, a_1 = 1, b_1 = 1, c_1 = 1, \qquad z_2 = -\frac{1}{20} + \frac{1}{2}i, a_2 = 1, b_2 = 1, c_2 = \frac{1}{2}.$$

• $Re(z_1) \neq Re(z_2), Im(z_1) = Im(z_2)$ and $Re(z_1) \neq -Re(z_2)$, $u(x,t)$ and $v(x,t)$ are composed of two soliton solutions with varying amplitude. And when $|Re(z_1)| > |Re(z_2)|$, the amplitude of the soliton solution increases with time, when $|Re(z_1)| < |Re(z_2)|$, the amplitude of the soliton solution decreases with time. We display the dynamic graph in Figs. 3(c) and 3(d), and the parameters are

$$z_1 = \frac{1}{5} + \frac{1}{10}i, a_1 = 1, b_1 = i, c_1 = \frac{i}{2}, \qquad z_2 = -\frac{1}{8} + \frac{1}{10}i, a_2 = 1, b_2 = 1, c_2 = \frac{1}{3}.$$

For more simple zero point soliton solutions, after our analysis, they satisfy the nonlinear superposition. Due to the complexity of their expressions, we will not show specific forms.

3. Higher-order soliton solutions of the reverse-time Manakov system

In this section, we mainly consider the case of multiple zeros, that is, $\det N_+ = \prod (z - z_j)^{r_j}$, $\det N_- = \prod (\breve{z} - \breve{z}_j)^{r_j}, r_j \geq 2$. At this time, we use the idea of limit to solve. According to the previous simple zero point, the perturbation spectrum parameters $\alpha_j = z_j + \varepsilon \in \mathbb{C}_+, \breve{\alpha}_j = \breve{z}_j + \breve{\varepsilon} \in \mathbb{C}_-, j = 1, 2..., N$ are introduced, and their corresponding kernel vectors are

$$v_k(\alpha_k) = e^{(-i\alpha_k x - 3i\alpha_k^2 t)\Lambda_0}(a_k(\varepsilon), b_k(\varepsilon), c_k(\varepsilon))^T, \qquad \breve{v}_k(\breve{\alpha}_k) = (\breve{a}_k(\breve{\varepsilon}), \breve{b}_k(\breve{\varepsilon}), \breve{c}_k(\breve{\varepsilon}))e^{(i\breve{\alpha}_k x + 3i\breve{\alpha}_k^2 t)\Lambda_0}.$$
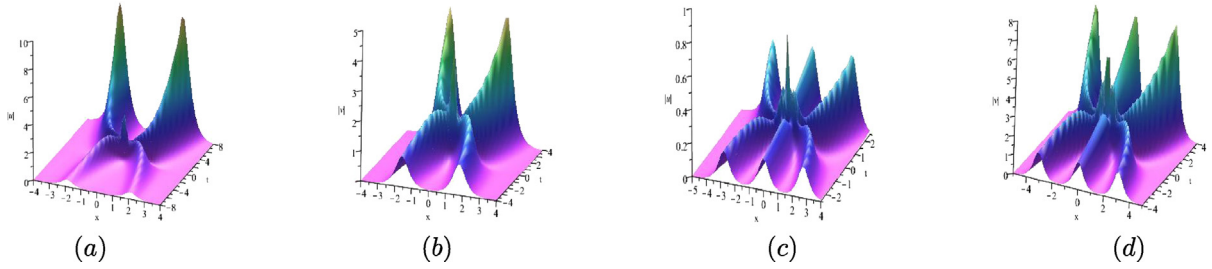
Fig. 4. Higher order soliton solutions for the reverse-time Manakov system. (a) Higher order 2-soliton solution of $|u|$; (b) Higher order 2-soliton solution of $|v|$; (c) Higher order 3-soliton solution of $|u|$; (d) Higher order 3-soliton solution of $|v|$.

Expand the above disturbance term at $\varepsilon = 0, \breve{\varepsilon} = 0$, and record them as

$$v_k(z_k, \varepsilon) = \left[\sum_{j=0}^{\infty} v_{k,1}^{[j]} \varepsilon^j, \sum_{j=0}^{\infty} v_{k,2}^{[j]} \varepsilon^j, \sum_{j=0}^{\infty} v_{k,3}^{[j]} \varepsilon^j \right]^T, \breve{v}_k\left(\breve{z}_k, \breve{\varepsilon}\right) = \left[\sum_{j=0}^{\infty} \breve{v}_{k,1}^{[j]} \breve{\varepsilon}^j, \sum_{j=0}^{\infty} \breve{v}_{k,2}^{[j]} \breve{\varepsilon}^j, \sum_{j=0}^{\infty} \breve{v}_{k,3}^{[j]} \breve{\varepsilon}^j \right].$$

In the expansion, let $\varepsilon, \breve{\varepsilon}$ tend to zero. Bring them into Eq. (8) and obtain the N-soliton solution form of multiple zeros

$$u = 3i \frac{det\tilde{F}}{det\tilde{M}}, \qquad v = 3i \frac{det\tilde{G}}{det\tilde{M}}, \tag{11}$$

where $\tilde{F}$ and $\tilde{G}$ are matrices of order $(N+1) \times (N+1)$

$$\tilde{F} = \begin{pmatrix} \tilde{M} & \mathcal{X} \\ \mathcal{Y} & 0 \end{pmatrix}, \qquad \tilde{G} = \begin{pmatrix} \tilde{M} & \mathcal{H} \\ \mathcal{Y} & 0 \end{pmatrix},$$

$$(\tilde{M})_{k,j}^{[l_1,l_2]} = \lim_{\varepsilon,\breve{\varepsilon} \to 0} \frac{1}{(l_1-1)!(l_2-1)!} \frac{\partial^{l_1+l_2-2}}{\partial \breve{\varepsilon}^{l_1-1} \partial \varepsilon^{l_2-1}} \left[\frac{\breve{v}_k\left(\breve{z}_k, \breve{\varepsilon}\right) v_j\left(z_j, \varepsilon\right)}{z_j + \varepsilon - \breve{z}_k - \breve{\varepsilon}} \right],$$

$$\mathcal{X} = \left[\breve{v}_{1,2}^{[0]}, \breve{v}_{1,2}^{[1]}, \ldots, \breve{v}_{1,2}^{[r_1-1]}, \breve{v}_{2,2}^{[0]}, \breve{v}_{2,2}^{[1]}, \ldots, \breve{v}_{2,2}^{[r_2]}, \ldots, \breve{v}_{n,2}^{[0]}, \breve{v}_{n,2}^{[1]}, \ldots, \breve{v}_{n,2}^{[r_n-1]} \right]^T,$$

$$\mathcal{Y} = \left[v_{1,1}^{[0]}, v_{1,1}^{[1]}, \ldots, v_{1,1}^{[r_1-1]}, v_{2,1}^{[0]}, v_{2,1}^{[1]}, \ldots, v_{2,1}^{[r_2]}, \ldots, v_{n,1}^{[0]}, v_{n,1}^{[1]}, \ldots, v_{n,1}^{[r_n-1]} \right],$$

$$\mathcal{H} = \left[\breve{v}_{1,3}^{[0]}, \breve{v}_{1,3}^{[1]}, \ldots, \breve{v}_{1,3}^{[r_1-1]}, \breve{v}_{2,3}^{[0]}, \breve{v}_{2,3}^{[1]}, \ldots, \breve{v}_{2,3}^{[r_2]}, \ldots, \breve{v}_{n,3}^{[0]}, \breve{v}_{n,3}^{[1]}, \ldots, \breve{v}_{n,3}^{[r_n-1]} \right]^T.$$

According to the symmetry of the above spectral parameters, $z = -z$ so $\breve{\varepsilon} = -\varepsilon$. Due to the complexity of the expression, we present the dynamic behavior diagrams of the simplest higher-order two-soliton solution and higher-order three-soliton solution of the nonlocal reverse-time Manakov system in Fig. 4 (a–d).

From the dynamic behavior diagram of higher-order soliton solution in Fig. 4, we can see that for pure imaginary higher-order two-soliton solution and higher-order three-soliton solution, the amplitude is determined by the sign of the imaginary part. At large t, it is the motion of two parallel solitons, like the mutual approximation process of two simple two-soliton solutions and three-soliton solutions, which is consistent with the limit idea in our calculation. Fig. 4 shows the most fundamental higher-order soliton solution, with common parameters of $a = b = c = 1, z = i$. In addition, $r_1 = 2, r_j = 0, j = 2..n$ in Fig. 4(a) and (b), $r_1 = 3, r_j = 0, j = 2..n$ in Fig. 4(c) and (d).

Data availability

No data was used for the research described in the article.

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References

[1] L. Debnath, Nonlinear partial differential equations for scientists and engineers, Birkhäuser, Boston, 2005.

[2] C.H. Townes R.Y. Chiao, Self-trapping of optical beams, Phys. Rev. Lett. 13 (15) (1964) 479.

[3] Y. Nakamura H. Bailung, Observation of peregrine solitons in a multicomponent plasma with negative ions, Phys. Rev. Lett. 107 (25) (2011) 255005.

[4] S.V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, Sov. Phys.-JETP 38 (2) (1974) 248–253.

[5] Z.H. Musslimani M.J. Ablowitz, Integrable nonlocal nonlinear Schrödinger equation, Phys. Rev. Lett. 110 (6) (2013) 064105.

[6] Z. Yan, Nonlocal general vector nonlinear Schrödinger equations: integrability, PT symmetribility, and solutions, Appl. Math. Lett. 62 (2016) 101–109.

[7] X. Zhou, E. Fan, Long time asymptotics for the nonlocal mKdV equation with finite density initial data, Physica D 440 (2022) 133458.

[8] W.X. Ma, Inverse scattering for nonlocal reverse-time nonlinear Schrödinger equations, Appl. Math. Lett. 102 (2020) 106161.

[9] X. Wang, C. Li, Solitons, breathers and rogue waves in the coupled nonlocal reverse-time nonlinear Schrödinger equations, J. Geom. Phys. 180 (2022) 104619.

[10] J.K. Yang, Nonlinear waves in intergrable and nonintergrable systems, SIAM soc, Ind. Appl. Math., Philadelphia (2010).

[11] J.Y. Zhu, Y. Chen, A new form of general soliton solutions and multiple zeros solutions for a higher-order Kaup–Newell equation, J. Math. Phys. 62 (12) (2021) 123501.

[12] J. Wu, Riemann–Hilbert approach and soliton classification for a nonlocal integrable nonlinear Schrödinger equation of reverse-time type, Nonlinear Dynam. 107 (1) (2022) 1127–1139.

[13] W.X. Ma, Riemann-Hilbert problems and soliton solutions of nonlocal reverse-time NLS hierarchie, Acta Math. Sci. 42 (1) (2022) 127–140.