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Nonlocal symmetries, Bäcklund transformation and interaction solutions for the integrable Boussinesq equation

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The nonlocal symmetry of the integrable Boussinesq equation is derived by the truncated Painlevé method. The nonlocal symmetry is localized to the Lie point symmetry by introducing auxiliary-dependent variables and the finite symmetry transformation related to the nonlocal symmetry is presented. The multiple nonlocal symmetries are obtained and localized base on the linear superposition principle, then the determinant representation of the nth Bäcklund transformation is provided. The integrable Boussinesq equation is also proved to be consistent tanh expansion (CTE) form and accurate interaction solutions among solitons and other types of nonlinear waves are given out analytically and graphically by the CTE method. The associated structure may be related to large variety of real physical phenomena.

Keywords: Integrable Boussinesq equation; nonlocal symmetry; nth Bäcklund transformation; consistent tanh expansion method; interaction solutions.

1. Introduction

In recent years, a variety of nonlinear mathematical physics models to trivial 3D or to higher derivative equations from integrable hierarchies have been paid more and more attention, which are used to describe complex physical phenomena in the real world, such as optical fiber, fluid dynamics, plasma physics and others.[1–9] Symmetry analysis is considered to be one of the most significant methods for seeking out the group invariant solutions in nonlinear mathematical physics.[10–12] The nonlocal

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symmetry whose infinitesimal generator involves integral of the dependent variable has attracted extensive attention in recent years.[13,14] Hinted at by the results of nonlocal symmetry reduction, Lou has pointed out that the residue of the Painlevé truncated expansion with the singular manifold is actually a nonlocal symmetry of the nonlinear system, and then such type of nonlocal symmetry is also known as residual symmetry.[15,16] The residual symmetry can be converted into the Lie point symmetry by defining the appropriate quantities, and it can be extended to the multiple residual symmetries which yield the nth Bäcklund transformation (BT).[17–20] Furthermore, the symmetry reduction approach with nonlocal symmetries related to Lax pair, Darboux transformations, BT has been successfully used to find new interaction solutions among different types of nonlinear excitations including the solitons, cnoidal waves, Airy waves and Bessel waves.[21–26]

On the other hand, the author established a simple effective method called consistent tanh expansion (CTE), the CTE method can be used to identify CTE solvable systems and it is a special simplified form of the consistent Riccati expansion (CRE) method in Ref. 27. Some interaction solutions between solitons and other nonlinear excitations can be found with the help of the CTE method for many integrable systems.[28–34] Recently, abundant interaction solutions among solitons and other complicated waves including periodic cnoidal waves, Boussinesq waves and Painlevé waves for many integrable systems were obtained by nonlocal symmetries reduction and the CTE method related to the Painlevé analysis.[35–37]

In this paper, we consider the following integrable Boussinesq equation:

$$u_{tt} - u_{xx} - \beta(u^2)_{xx} - \gamma u_{xxxx} + \alpha u_{xt} = 0\,, \tag{1}$$

where α, β, γ are nonzero constants. The integrable Boussinesq equation elucidates the communication (propagation) of gravity waves over the water surface, more specifically, the head-on collision of oblique wave profiles.[38] Wazwaz investigated the complete integrability of the integrable Boussinesq equation via Painlevé test, and obtained the real and complex multiple soliton solutions by mode of simplified Hirota's method, furthermore, the author employed the exponential expansion method to this equation, resulting into soliton solutions possessing rich spatial structure due to the presence of abundant arbitrary constants in Ref. 38. As far as we know, there are few studies on the integrable Boussinesq equation, and the nonlocal symmetries, BT and interaction solutions of this equation have not been studied by other scholars.

This paper is organized as follows. In Sec. 2, the nonlocal symmetry for the integrable Boussinesq equation is obtained by the truncated Painlevé expansion, and multiple nonlocal symmetries and nth BT in terms of determinant are derived. Sec. 3 provides the CTE method for the integrable Boussinesq equation and different interaction solutions among different nonlinear excitations. Conclusions and discussions are given in Sec. 4.

2. Nonlocal Symmetry and nth BT

In this section, we give the nonlocal symmetries and corresponding finite symmetry transformation for the integrable Boussinesq equation. For the integrable Boussinesq equation (1), the truncated Painlevé expansion takes the Laurent series form

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2 \,, \tag{2}$$

where u_0, u_1, u_2 are functions of x and t to be determined, the function $\phi = \phi(x,t)$ is an arbitrary singularity manifold. Substituting the expansion (2) into Eq. (1) and vanishing all the coefficients of different powers ϕ independently. We have

$$u_0 = -\frac{6\gamma}{\beta}\phi_x^2, \quad u_1 = \frac{6\gamma}{\beta}\phi_{xx}, \quad u_2 = -\frac{4\gamma\phi_x\phi_{xxx} - \alpha\phi_x\phi_t - 3\gamma\phi_{xx}^2 + \phi_x^2 - \phi_t^2}{2\beta\phi_x^2}\,, \tag{3}$$

$$\gamma\phi_x^2\phi_{xxxx} - \alpha\phi_x^2\phi_{xt} - 4\gamma\phi_x\phi_{xx}\phi_{xxx} + \alpha\phi_x\phi_t\phi_{xx} + 3\gamma\phi_{xx}^3 - \phi_x^2\phi_{tt} + \phi_t^2\phi_{xx} = 0\,, \tag{4}$$

$$u_{2tt} - 2\beta u_{2x}^2 - 2\beta u_2 u_{2xx} - \gamma u_{2xxxx} + \alpha u_{2xt} - u_{2xx} = 0\,. \tag{5}$$

Therefore, based on the residual symmetry theorem in Ref. 15, it is clear that Eq. (5) is just Eq. (1) with the solution u_2 and the residual u_1 is the symmetry corresponding to the solution u_2. So the truncated Painlevé expansion

$$u = -\frac{6\gamma\phi_x^2}{\beta\phi^2} + \frac{6\gamma\phi_{xx}}{\beta\phi} + u_2\,, \tag{6}$$

is an auto-BT between the solutions u and u_2.

For the nonlocal symmetry $\sigma^u = \frac{6\gamma}{\beta}\phi_{xx}$, the corresponding initial value problem is

$$\frac{d\bar{u}}{d\epsilon} = \frac{6\gamma}{\beta}\bar{\phi}_{xx}, \quad \bar{u}|_{\epsilon=0} = u\,, \tag{7}$$

with ϵ being an infinitesimal parameter. However, it is very difficult to solve the Eq. (7) for the new function $\bar{u}(\epsilon)$ due to the intrusion of the function $\bar{\phi}$ and its derivatives. In order to solve this initial value problem, we need localize the nonlocal symmetry $\sigma^u = \frac{6\gamma}{\beta}\phi_{xx}$ and introduce two new-dependent auxiliary variables defined by

$$\phi_x = g, \quad g_x = h\,, \tag{8}$$

then linearized equations of the prolonged system of Eqs. (4), (5) and (8) are listed as follows:

$$\begin{aligned} &2\gamma\sigma_x^\phi\phi_x\phi_{xxxx} + \gamma\phi_x^2\sigma_{xxxx}^\phi - 2\alpha\sigma_x^\phi\phi_x\phi_{xt} - \alpha\phi_x^2\sigma_{xt}^\phi - 4\gamma\sigma_x^\phi\phi_{xx}\phi_{xxx} \\ &- 4\gamma\sigma_{xx}^\phi\phi_x\phi_{xxx} - 4\gamma\phi_x\phi_{xx}\sigma_{xxx}^\phi + \alpha\sigma_x^\phi\phi_{xx}\phi_t + \alpha\sigma_{xx}^\phi\phi_x\phi_t + \alpha\sigma_t^\phi\phi_x\phi_{xx} \\ &+ 9\gamma\sigma_{xx}^\phi\phi_{xx}^2 - 2\sigma_x^\phi\phi_x\phi_{tt} - \phi_x^2\sigma_{tt}^\phi + \sigma_{xx}^\phi\phi_t^2 + 2\sigma_t^\phi\phi_t\phi_{xx} = 0\,, \end{aligned} \tag{9}$$

$$\sigma_{tt}^u - 4\beta u_x\sigma_x^u - 2\beta\sigma^u u_{xx} - 2\beta u\sigma_{xx}^u - \gamma\sigma_{xxxx}^u + \alpha\sigma_{xt}^u - \sigma_{xx}^u = 0\,. \tag{10}$$

$$\sigma^g = \sigma_x^\phi, \quad \sigma^h = \sigma_x^g\,. \tag{11}$$

We can easily deduce that the solution of Eqs. (9)–(11) has the following form:

$$\sigma^u = \frac{6\gamma}{\beta}h, \quad \sigma^\phi = -\phi^2, \quad \sigma^g = -2\phi g, \quad \sigma^h = -2(g^2 + \phi h)\,. \tag{12}$$

Then the corresponding initial value problem becomes

$$\begin{aligned} &\frac{d\bar{u}}{d\epsilon} = \frac{6\gamma}{\beta}\bar{h}, \quad \bar{u}|_{\epsilon=0} = u, \quad \frac{d\bar{\phi}}{d\epsilon} = -\bar{\phi}^2, \quad \bar{\phi}|_{\epsilon=0} = \phi\,, \\ &\frac{d\bar{g}}{d\epsilon} = -2\bar{\phi}\bar{g}, \quad \bar{g}|_{\epsilon=0} = g, \quad \frac{d\bar{h}}{d\epsilon} = -2(\bar{g}^2 + \bar{\phi}\bar{h}), \quad \bar{h}|_{\epsilon=0} = h\,, \end{aligned} \tag{13}$$

one can derive the finite transformation of the prolonged system of Eqs. (4), (5) and (8) straightforwardly, which can be stated in the following theorem.

Theorem 2.1. *If* $\{u, \phi, g, h\}$ *is a solution of the prolonged system of Eqs.* (4), (5) *and* (8), *so is* $\{\bar{u}(\epsilon), \bar{\phi}(\epsilon), \bar{g}(\epsilon), \bar{h}(\epsilon)\}$ *where*

$$\begin{aligned} &\bar{\phi} = \frac{\phi}{1+\epsilon\phi}, \quad \bar{g} = \frac{g}{(1+\epsilon\phi)^2}, \quad \bar{h} = \frac{h}{(1+\epsilon\phi)^2} - \frac{2\epsilon g^2}{(1+\epsilon\phi)^3}\,, \\ &\bar{u} = u + \frac{6\gamma\epsilon h}{\beta(1+\epsilon\phi)} - \frac{6\gamma\epsilon^2 g^2}{\beta(1+\epsilon\phi)^2}\,, \end{aligned} \tag{14}$$

where ϵ *is an arbitrary group parameter.*

Due to the linearity of the symmetry equation and there are infinitely many solutions for Eq. (4), one can derive infinitely many residual symmetries

$$\sigma_n^u = \frac{6\gamma}{\beta}\sum_{i=1}^{n} c_i\phi_{i,xx}\,, \tag{15}$$

where $\phi_i(i = 1\cdots n)$ are all different solutions of the compatibility equation (4). Here, such linear superposition of multiple nonlocal symmetries implies the follow relations

$$u = -\frac{4\gamma\phi_{i,x}\phi_{i,xxx} - \alpha\phi_{i,x}\phi_{i,t} - 3\gamma\phi_{i,xx}^2 + \phi_{i,x}^2 - \phi_{i,t}^2}{2\beta\phi_{i,x}^2}\,. \tag{16}$$

Similarly, in order to localize the nonlocal symmetries (15), some new equations need to be defined. We introduce new-dependent auxiliary variables given by

$$\phi_{i,x} = g_i, \quad g_{i,x} = h_i, \quad i = 1\cdots n\,. \tag{17}$$

Consequently, the nonlocal symmetries (15) are localized to Lie point symmetries

$$
\begin{aligned}
&\sigma_n^u = \frac{6\gamma}{\beta}\sum_{i=1}^{n} c_i h_i, \quad \sigma^{\phi_i} = -c_i\phi_i^2 - \sum_{j\neq i}^{n} c_j\phi_i\phi_j\,,\\
&\sigma^{g_i} = -2c_i\phi_i g_i - \sum_{j\neq i}^{n} c_j(g_i\phi_j + \phi_i g_j)\,,\\
&\sigma^{h_i} = -2c_i(g_i^2 + \phi_i h_i) - \sum_{j\neq i}^{n} c_j\left(\frac{g_i\phi_j h_j + 2g_i g_j^2 + \phi_i g_j h_j}{g_j}\right).
\end{aligned}
\tag{18}
$$

Considering the following initial problem:

$$
\begin{aligned}
&\frac{d\widehat{u}(\epsilon)}{d\epsilon} = \sigma_n^u\Big|_{h_i=\widehat{h}_i(\epsilon)}, \quad \frac{d\widehat{\phi}_i(\epsilon)}{d\epsilon} = \sigma^{\phi_i}\Big|_{\phi_i=\widehat{\phi}_i(\epsilon)},\\
&\frac{d\widehat{g}_i(\epsilon)}{d\epsilon} = \sigma^{g_i}\Big|_{\phi_i=\widehat{\phi}_i(\epsilon),g_i=\widehat{g}_i(\epsilon)},\\
&\frac{d\widehat{h}_i(\epsilon)}{d\epsilon} = \sigma^{h_i}\Big|_{\phi_i=\widehat{\phi}_i(\epsilon),g_i=\widehat{g}_i(\epsilon),h_i=\widehat{h}_i(\epsilon)},\\
&\widehat{u}(0) = u, \quad \widehat{\phi}_i(0) = \phi_i, \quad \widehat{g}_i(0) = g_i, \quad \widehat{h}_i(0) = h_i\,,
\end{aligned}
\tag{19}
$$

after solving out (19), one can obtain the following nth BT theorem.

Theorem 2.2. *If* $\{u, \phi_i, g_i, h_i, i = 1\cdots n\}$ *is a solution of the prolonged system of Eqs.* (1), (16) *and* (17), *so is* $\{\widehat{u}(\epsilon), \widehat{\phi}_i(\epsilon), \widehat{g}_i(\epsilon), \widehat{h}_i(\epsilon), i = 1\cdots n\}$ *where*

$$
\begin{aligned}
&\widehat{u}(\epsilon) = u + \frac{6\gamma}{\beta}(\ln\Delta)_{xx}, \quad \widehat{\phi}_i(\epsilon) = \frac{\Delta_i}{\Delta}\,,\\
&\widehat{g}_i(\epsilon) = \widehat{\phi}_{i,x}(\epsilon), \quad \widehat{h}_i(\epsilon) = \widehat{\phi}_{i,xx}(\epsilon)\,,
\end{aligned}
\tag{20}
$$

where Δ *and* Δ_i *being the determinants of the matrices* M *and* M_i *defined by*

$$
M = \begin{pmatrix}
c_1\epsilon\phi_1 + 1 & c_1\epsilon\omega_{12} & \cdots & c_1\epsilon\omega_{1j} & \cdots & c_1\epsilon\omega_{1n}\\
c_2\epsilon\omega_{12} & c_2\epsilon\phi_2 + 1 & \cdots & c_2\epsilon\omega_{2j} & \cdots & c_2\epsilon\omega_{2n}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
c_j\epsilon\omega_{1j} & c_j\epsilon\omega_{2j} & \cdots & c_j\epsilon\phi_j + 1 & \cdots & c_j\epsilon\omega_{jn}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
c_n\epsilon\omega_{1n} & c_n\epsilon\omega_{2n} & \cdots & c_n\epsilon\omega_{jn} & \cdots & c_n\epsilon\phi_n + 1
\end{pmatrix},
$$

$$M_i = \begin{pmatrix} c_1\epsilon\phi_1+1 & c_1\epsilon\omega_{12} & \cdots & c_1\epsilon\omega_{1,i-1} & c_1\epsilon\omega_{1i} & c_1\epsilon\omega_{1,i+1} & \cdots & c_1\epsilon\omega_{1n} \\ c_2\epsilon\omega_{12} & c_2\epsilon\phi_2+1 & \cdots & c_2\epsilon\omega_{2,i-1} & c_2\epsilon\omega_{2i} & c_2\epsilon\omega_{2,i+1} & \cdots & c_2\epsilon\omega_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i-1}\epsilon\omega_{1,i-1} & c_{i-1}\epsilon\omega_{2,i-1} & \cdots & c_{i-1}\epsilon\phi_{i-1}+1 & c_{i-1}\epsilon\omega_{i-1,i} & c_{i-1}\epsilon\omega_{i-1,i+1} & \cdots & c_{i-1}\epsilon\omega_{i-1,n} \\ \omega_{1i} & \omega_{2i} & \cdots & \omega_{i,i-1} & \phi_i & \omega_{i,i+1} & \cdots & \omega_{in} \\ c_{i+1}\epsilon\omega_{1,i+1} & c_{i+1}\epsilon\omega_{2,i+1} & \cdots & c_{i+1}\epsilon\omega_{i-1,i+1} & c_{i+1}\epsilon\omega_{i,i+1} & c_{i+1}\epsilon\phi_{i+1}+1 & \cdots & c_{i+1}\epsilon\omega_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n\epsilon\omega_{1n} & c_n\epsilon\omega_{2n} & \cdots & c_n\epsilon\omega_{i-1,n} & c_n\epsilon\omega_{in} & c_n\epsilon\omega_{i+1,n} & \cdots & c_n\epsilon\phi_n+1 \end{pmatrix},$$

and $\omega_{ij} = \sqrt{\phi_i\phi_j}$.

3. CTE Method and Interaction Solutions

In this section, the interaction solutions of the integrable Boussinesq equation (1) are obtained by using the CTE method. The CTE method is the generalization of the traditional tanh function expansion method.[27,28] It is very effective for finding the interaction solutions between solitons and other types of nonlinear wave such as cnoidal periodic waves, resonant waves and Airy waves and so on.[33,34,39] By leading order analysis for the integrable Boussinesq equation (1), we can take following truncated tanh function expansion:

$$u = u_0 + u_1\tanh(\omega) + u_2\tanh^2(\omega)\,, \tag{21}$$

where u_0, u_1, u_2 and ω are functions of $\{x, t\}$ to be determined later. Substituting Eq. (21) into Eq. (1) and vanishing the coefficients of different powers of $\tanh(\omega)$, we have

$$\begin{gathered} u_2 = -\frac{6\gamma\omega_x^2}{\beta}, \quad u_1 = \frac{6\gamma\omega_{xx}}{\beta}\,, \\ u_0 = \frac{8\gamma\omega_x^4 + \alpha\omega_x\omega_t - 4\gamma\omega_x\omega_{xxx} + 3\gamma\omega_{xx}^2 - \omega_x^2 + \omega_t^2}{2\beta\omega_x^2}\,, \end{gathered} \tag{22}$$

$$\begin{aligned} &4\gamma\omega_x^4\omega_{xx} + \alpha\omega_x^2\omega_{xt} - \gamma\omega_x^2\omega_{xxxx} - \alpha\omega_x\omega_{xx}\omega_t \\ &\quad + 4\gamma\omega_x\omega_{xx}\omega_{xxx} - 3\gamma\omega_{xx}^3 + \omega_x^2\omega_{tt} - \omega_{xx}\omega_t^2 = 0\,. \end{aligned} \tag{23}$$

Equation (23) is the consistent condition for the integrable Boussinesq equation (1), which is also called ω-equation simplicity. Obviously, it's difficult to find the explicit solution of Eq. (23) because of the higher derivatives of the unknown function ω. We can obtain the following nonauto–BT theorem by directly calculating Eqs. (21)–(23).

Nonauto-BT

Theorem 3.1. *If* ω *is a solution of the* (23), *then*

$$u = \frac{8\gamma\omega_x^4 + \alpha\omega_x\omega_t - 4\gamma\omega_x\omega_{xxx} + 3\gamma\omega_{xx}^2 - \omega_x^2 + \omega_t^2}{2\beta\omega_x^2} + \frac{6\gamma\omega_{xx}}{\beta}\tanh(\omega) - \frac{6\gamma\omega_x^2}{\beta}\tanh^2(\omega) \tag{24}$$

is a solution of Eq. (1)*. In other words, after finding the solutions of Eq.* (23)*, the new interaction solutions u can be given from the nonauto-BT theorem directly.*

3.1. *Soliton solution*

From the consistent equation (23), a trivial straight-line solution has the form

$$\omega = k_0 x + \omega_0 t \,, \tag{25}$$

where k_0, w_0 is arbitrary constants. One can yield the soliton solution of the integrable Boussinesq equation (1) by substituting Eq. (25) into Eq. (24)

$$u = \frac{8\gamma k_0^4 + \alpha\omega_0 k_0 - k_0^2 + \omega_0^2}{2\beta k_0^2} - \frac{6\gamma k_0^2}{\beta}\tanh^2(k_0 x + \omega_0 t)\,. \tag{26}$$

Applying the identity transformation($\tanh^2\omega = 1-\mathrm{sech}^2\omega$) about the hyperbolic function to Eq. (26), one can obtain the general soliton solution form.

3.2. *Soliton-cnoidal wave solution*

According to the structure of the soliton-cnoidal wave solution, in order to possess the first type of the soliton-cnoidal wave interaction solution, we take the form

$$\omega = k_0 x + \omega_0 t + A\mathrm{arctanh}[\mathrm{sn}(k_1 x + \omega_1 t, m)]\,, \tag{27}$$

where $\mathrm{sn}(\xi, m)$ is the usual Jacobi elliptic sine function. Then substituting Eq. (27) into the consistent equation (23), there exist two kinds of different constant relations:

$$\text{(i)} \quad A = \pm\frac{1}{2}, \quad k_0 = 0, \quad \omega_1 = -\frac{1}{2}k_1\alpha, \quad \omega_0 = \frac{1}{2}k_1^2 m\sqrt{-3\gamma}\,, \tag{28}$$

where m, k_1 are arbitrary constants.

$$\text{(ii)} \quad A = \pm\frac{1}{2}, \quad k_0 = \frac{1}{2}k_1\sqrt{\frac{m^2+1}{2}}, \quad \omega_1 = -\frac{1}{2}k_1\alpha, \quad \omega_0 = \frac{1}{4}k_0\delta\,, \tag{29}$$

where m, k_1 are arbitrary constants, and the function δ satisfies the equation

$$2\delta^2 + 2\alpha\sqrt{2(m^2+1)}\delta + \alpha^2 - 6\gamma m^4 k_1^2 + 12\gamma m^2 k_1^2 + m^2\alpha^2 - 6\gamma k_1^2 = 0\,.$$

After substituting Eqs. (27)–(29) into Eq. (24), the first type of the soliton-cnoidal wave interaction solution can be obtained by choosing the appropriate constant. In Figs. 1 and 2, the corresponding images of the two situations are plotted, respectively.

On the other hand, we also can choose the ω function as

$$\omega = k_0 x + w_0 t + AE_\pi[\mathrm{sn}(k_1 x + w_1 t, m), n, m]\,, \tag{30}$$

where E_π is the third type of incomplete elliptic integral. Substituting (30) into the ω function (23), the consistents satisfy the relation

$$\text{(iii)} \quad A = -\frac{k_0}{k_1}, \quad n = 0, \quad w_0 = \frac{k_0 w_1}{k_1}\,, \tag{31}$$

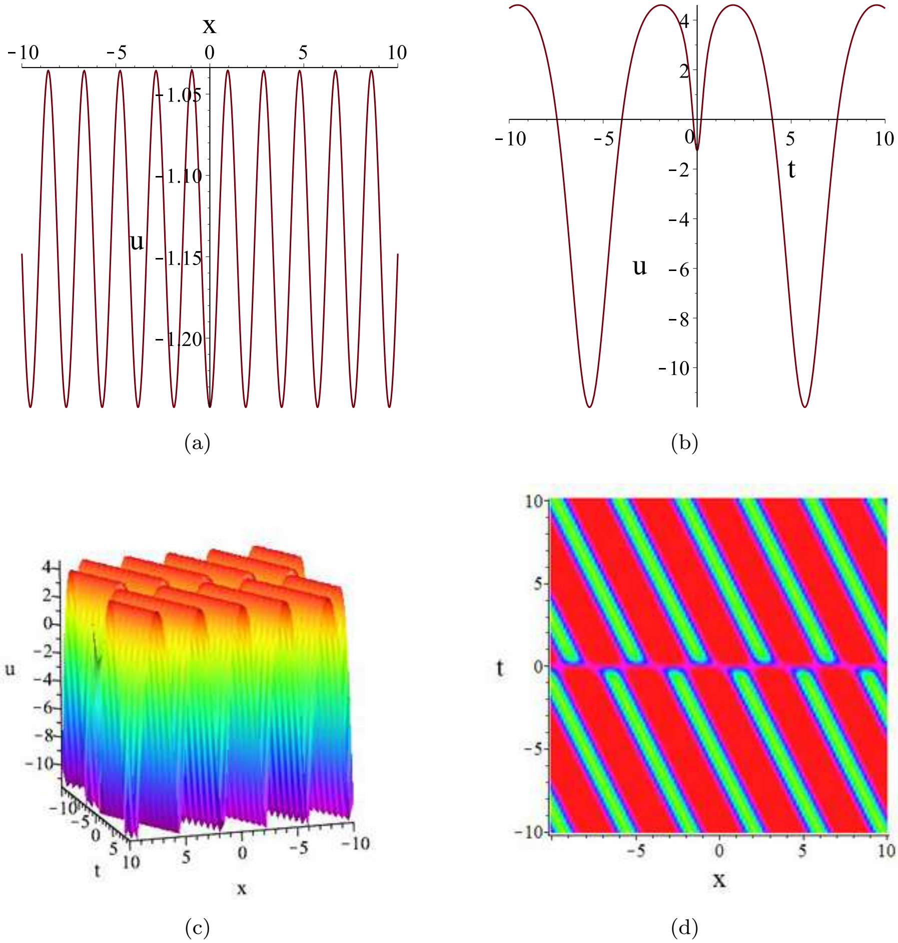

Fig. 1. First type of the soliton-cnoidal wave interaction solution for (24) with the constants chosen as $m = 1.8, k_1 = 1, \alpha = -1, \beta = 2, \gamma = -3$ in case (i). (a) The wave propagation pattern of the wave along x axis at $t = 0$; (b) the wave propagation pattern of the wave along t axis at $x = 0$; (c) the three-dimensional plots; (d) the density plots.

where m, k_0, k_1, w_1 are arbitrary constants. Substituting Eqs. (31) and (30) into Eq. (24), we can obtain the third type of the soliton-cnoidal wave interaction solution according to the proper constant selection.

3.3. *Resonant multi-soliton solution*

The resonant multi-soliton solution for the integrable Boussinesq equation (1) can be derived by choosing

$$\omega = k_0 x + \omega_0 t + c\ln\left(1 + \sum_{i=1}^{N} e^{k_i x + \omega_i t}\right), \tag{32}$$

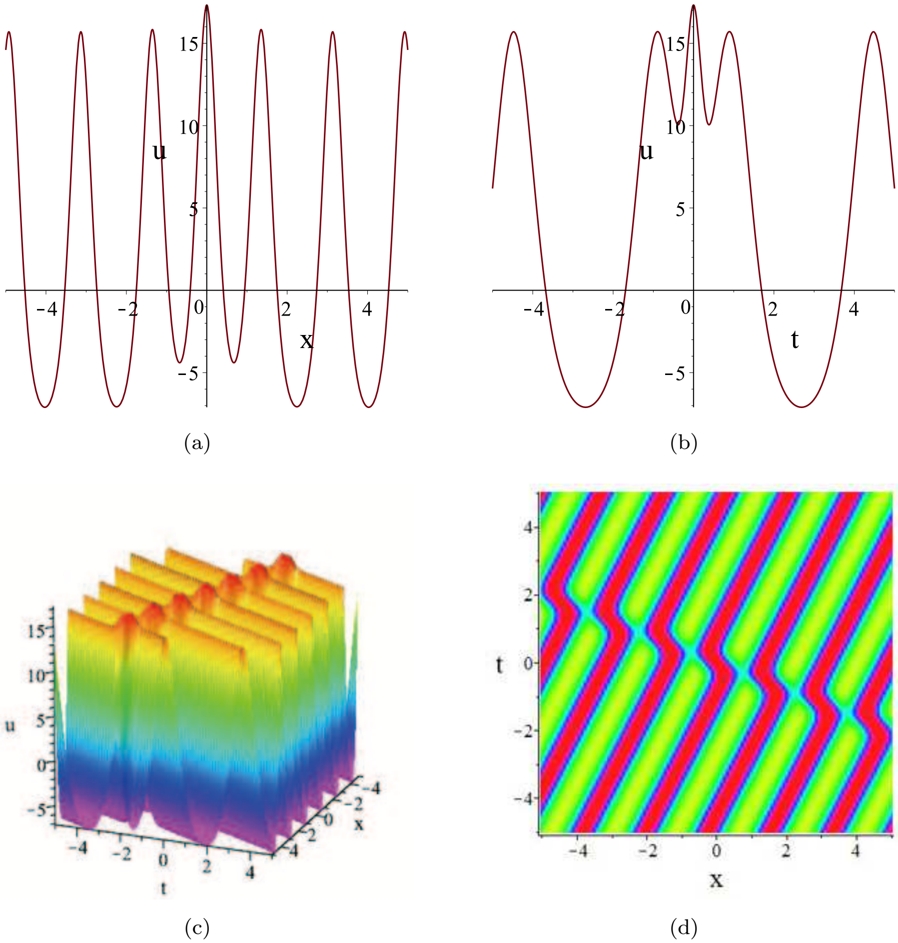

Fig. 2. First type of the soliton-cnoidal wave interaction solution for (24) with the constants chosen as $m = 1.9, k_1 = 2, \alpha = 1, \beta = 2, \gamma = 1$ in case (ii). (a) The wave propagation pattern of the wave along x axis at $t = 0$; (b) the wave propagation pattern of the wave along t axis at $x = 0$; (c) the three-dimensional plots; (d) the density plots.

After substituting Eq. (32) into Eq. (23), the constant relations are solved as

$$c = \frac{1}{2}, \quad k_i = -2k_0, \quad \omega_i = -2\omega_0\,, \tag{33}$$

where k_0, ω_0 are arbitrary constants. One can obtain the multiple resonant multi-soliton solution of the integrable Boussinesq equation (1) by substituting Eqs. (33) and (32) into Eq. (24). These parameters which are selected have rich mathematical structures in solution (33), which may be important for explaining some interesting physical properties in variety of branches. These solutions play the key role for understanding features of resonant multi-soliton waves of the investigated equations. In order to understand the structure of the resonant multi-soliton solution,

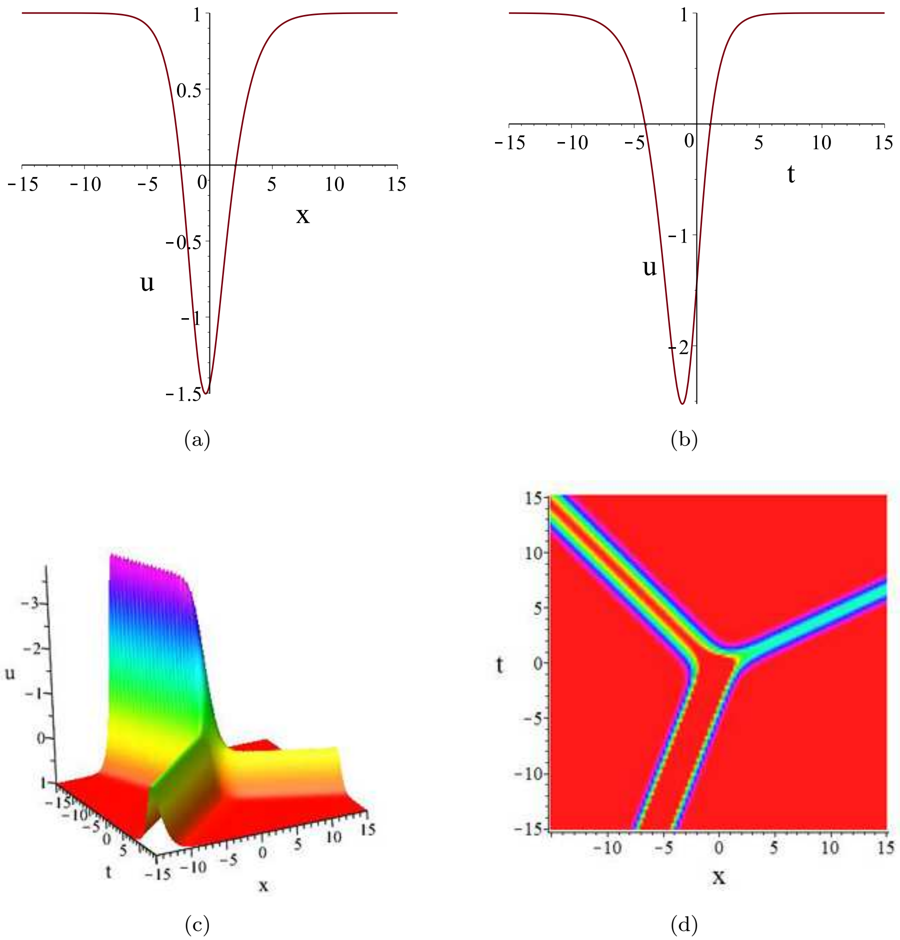

Fig. 3. A special resonant solution for (24) with the constants chosen as $c = 0.5, k_0 = -0.5, \omega_0 = -0.5, k_1 = 1.799, \omega_1 = -0.69, \alpha = 1, \beta = 1, \gamma = -1$ when $N = 2$. (a) The wave propagation pattern of the wave along x axis at $t = 0$; (b) the wave propagation pattern of the wave along t axis at $x = 0$; (c) the three-dimensional plots; (d) the density plots.

we consider the special resonant two-soliton solution when $N = 2$ in solution (33), and the special resonant two-soliton solution has been displayed in Fig. 3 by choosing parameters appropriately. The resonant two-soliton waves are used to demonstrate the wave fusion/fission, which are very useful for describing diverse types of inelastic interactions in mathematical physics. The parameter k_i, ω_i plays the key role in the inelastic mechanism. As far as we know, the generated solution (33) of the integrable Boussinesq equation and propagation of inelastic interactions have not been reported in the literature before. The novelty of constructing such resonant multi-soliton solutions can pave the way for further developments of nonlinear sciences.

4. Conclusions and Discussions

In this paper, we investigate the nonlocal symmetry and interaction solutions for the integrable Boussinesq equation. We derived the nonlocal symmetry by the truncated Painlevé expansion, and it is localized to the Lie point symmetry by introducing new proper variables. The corresponding finite symmetry transformations are presented by the Lie first principle. Then the multiple nonlocal symmetries are obtained by the linear combination of the nonlocal symmetries. Furthermore, the finite symmetry transformation which is called nth BT in terms of determinant is given. Meanwhile, the integrable Boussinesq equation is proved to be CTE solvable and abundant interaction solutions are obtained by substituting the explicit function solutions to the consistent condition according to the CTE method. The study of solitary waves and soliton-cnoidal waves are not only helpful to reveal the mechanical mechanism of diffraction, transmission and reflection of nonlinear shallow water waves in theory, but also have practical guiding significance in engineering physics. The resonant two-soliton waves are often used to demonstrate the fusion/fission of waves, and it is very practical to describe various types of inelastic interactions in mathematical physics. The more about CTE method and the BT related to the nonlocal symmetries of the integrable Boussinesq equation are worthy of further investigate. More interesting and new analytic solutions of the integrable Boussinesq equation will require further research.

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