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Long-time asymptotics for the integrable nonlocal Lakshmanan–Porsezian–Daniel equation with decaying initial value data

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ABSTRACT

In this work, we study the Cauchy problem of integrable nonlocal Lakshmanan-Porsezian-Daniel equation with rapid attenuation of initial data. The basic Riemann–Hilbert problem of integrable nonlocal Lakshmanan-Porsezian-Daniel equation is constructed from Lax pair. Using Deift-Zhou nonlinear steepest descent method, the explicit long-time asymptotic formula of integrable nonlocal Lakshmanan-Porsezian-Daniel equation is derived, which is different from the local model. Besides, compared to the nonlocal nonlinear Schrödinger equation, since the increase of real stationary phase points, the long-time asymptotic formula for nonlocal Lakshmanan-Porsezian-Daniel equation becomes more complex.

1. Introduction

In 2013, a PT symmetric nonlocal integrable nonlinear Schrödinger (NLS) equation was introduced by Ablowitz and Musslimani [1]. After that, some other nonlocal integrable equations and their related properties are studied extensively [2,3]. In this work, we are committed to the long-time asymptotic behavior of the nonlocal Lakshmanan-Porsezian- Daniel (LPD) equation taking the following form [4]

$$q_t + \frac{1}{2} i q_{xx}(x,t) - i q^2(x,t) q^*(-x,t) - \delta H[q(x,t)] = 0, \ \ (x,t) \in \mathbb{R} \times (0,+\infty), \tag{1.1}$$

with

$$\begin{aligned} H[q(x,t)] = &-i q_{xxxx}(x,t) + 6 i q^*(-x,t) q_x^2(x,t) + 4 i q(x,t) q_x^*(-x,t) q_x(x,t) \\ &+ 8 i q^*(-x,t) q(x,t) q_{xx}(x,t) + 2 i q^2(x,t) q_{xx}^*(-x,t) - 6 i (q^*(-x,t))^2 q^3(x,t), \end{aligned}$$

where δ is arbitrary positive real parameter. The symbol "$*$" means the complex conjugation. The initial data is given by $q(x,0) = q_0(x)$ which belongs to the Schwartz space. The classical LPD equation was first proposed by Lakshmanan, Porsezian, and Daniel through studying the integrable properties of a classical 1-dimensional isotropic biquadratic Heisenberg spin chain in its continuum limit [5,6]. For the nonlocal LPD equation, its rational soliton solutions, periodic waves, nonsingular solution, time-periodic pure soliton solutions have been derived [4,7–9]. Recently, Rybalko and Shepelsky employed the Deift-Zhou method to analyze the long-time behavior of solutions for the Cauchy problem of nonlocal NLS equation [10]. Besides, this method was also used to discuss the long-time asymptotics for the solution of the nonlocal mKdV equation [11] and nonlocal short pulse equation [12]. In this paper,

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we focus on the long-time asymptotic behavior of the nonlocal LPD Eq. (1.1) with the initial data $q_0(x)$ rapidly decaying to 0 as $|x| \to \infty$.

Organization of the paper: In Section 2, the fundamental Riemann–Hilbert(RH) problem is constructed by the direct scattering analysis. In Section 3, through a series of deformations, we derive a model RH problem. Then, the long-time asymptotics of the solution for the nonlocal LPD equation is presented via solving the model RH problem.

2. Inverse scattering transform and the RH problem

In this section, we aim to construct the fundamental RH problem through the direct scattering analysis. The nonlocal LPD equation admits the following spectral problem

$$\Psi_x = L\Psi, \qquad L \equiv \lambda J + U, \Psi_t = M\Psi, \qquad M \equiv \lambda^2 J + \lambda U + \frac{1}{2}V + \delta V_1, \tag{2.1}$$

with

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad U = \begin{pmatrix} 0 & q(x,t) \\ q^*(-x,t) & 0 \end{pmatrix},$$
$$V = \begin{pmatrix} iq(x,t)q^*(-x,t) & -iq_x(x,t) \\ iq_x^*(-x,t) & -iq(x,t)q^*(-x,t) \end{pmatrix}, \quad V_1 = \begin{pmatrix} iA(x,t) & B(x,t) \\ -C(x,t) & -iA(x,t) \end{pmatrix}, \tag{2.2}$$

where λ is a spectral parameter. $A = -8\lambda^4 - 4q^*(-x,t)q\lambda^2 - 2i(qq_x^*(-x,t) - q^*(-x,t)q_x)\lambda - 3q^2(q^*(-x,t))^2 - q_x^*(-x,t)q_x + qq_{xx}^*(-x,t) + q^*(-x,t)q_{xx}$, $B = -8q\lambda^3 + 4iq_x\lambda^2 + 2q_{xx}\lambda - 4q^*(-x,t)q^2\lambda - iq_{xxx} + 6iqq^*(-x,t)q_x$, $C = 8q^*(-x,t)\lambda^3 + 4iq_x^*(-x,t)\lambda^2 - 2q_{xx}^*(-x,t)\lambda + 4(q^*(-x,t))^2q\lambda - iq_{xxx}^*(-x,t) + 6iq^*(-x,t)qq_x^*(-x,t)$, and $\Psi = \Psi(x,t,\lambda)$ denotes the eigenfunction. Defining the Jost solutions $\Psi_\pm = \mu_\pm e^{i[\lambda x + (\lambda^2 - 8\delta\lambda^4)t]\sigma_3}$, and we have

$$\mu_- = I + \int_{-\infty}^{x} e^{i\lambda(x-x')\hat{\sigma}_3}[U(x',t)\mu_1(x',t,\lambda)]\mathrm{d}x',$$
$$\mu_+ = I - \int_{x}^{+\infty} e^{i\lambda(x-x')\hat{\sigma}_3}[U(x',t)\mu_2(x',t,\lambda)]\mathrm{d}x'.$$

Suppose $q \in L^1(\mathbb{R}^\pm)$, then $\mu_\pm(x,t,\lambda)$ have the following properties: $\mu_{+1}(x,t,\lambda)$ and $\mu_{-2}(x,t,\lambda)$ are analytical and bounded in $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda > 0\}$, $\mu_{-1}(x,t,\lambda)$ and $\mu_{+2}(x,t,\lambda)$ are analytical and bounded in $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda < 0\}$. $\mu_\pm(x,t,\lambda) \to I$ as $\lambda \to \infty$. There also are $\det \mu_\pm(x,t,\lambda) = 1$ for all x, t and k. As the simultaneous solutions of spectral problem (2.1), $\Psi_\pm$ satisfy the following linear relation via defining a scattering matrix $S(\lambda), \lambda \in \mathbb{R}$, given by $\Psi_-(x,t,\lambda) = \Psi_+(x,t,\lambda)S(\lambda), \lambda \in \mathbb{R}$, where $S(\lambda) = \begin{pmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{12}^*(-\lambda^*) & s_{22}(\lambda) \end{pmatrix}, \lambda \in \mathbb{R}$, and the scattering data $s_{11}(\lambda), s_{22}(\lambda)$ meet $s_{11}(\lambda) = s_{11}^*(-\lambda^*)$, $\quad s_{22}(\lambda) = s_{22}^*(-\lambda^*)$. Similarly, according to the analyticity of $\mu_\pm$, we know that $s_{11}(\lambda)$ is analytic in the half-plane $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda < 0\}$ and continuous in $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda \leq 0\}$, $s_{22}(\lambda)$ is analytic in the half-plane $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda > 0\}$ and continuous in $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda \geq 0\}$, and $S(\lambda) \to I$ as $\lambda \to \infty$. Furthermore, $\det S(\lambda) = 1$ for $\lambda \in \mathbb{R}$.

Supposing that $s_{11}(\lambda)$ and $s_{22}(\lambda)$ have no zeros in $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda \leq 0\}$ and $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda \geq 0\}$, respectively. Then, we can construct a fundamental RH problem by defining $M_+(x,t,\lambda) = (\frac{\mu_{+1}}{s_{22}}, \mu_{-2}), M_-(x,t,\lambda) = (\mu_{-1}, \frac{\mu_{+2}}{s_{11}})$, where $\pm$ stand for analyticity in $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda > 0\}$ and $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda < 0\}$, respectively.

Riemann–Hilbert Problem *$M(x,t,\lambda)$ satisfies the following RH problem:*

$$\begin{cases} M(x,t,\lambda) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\ M_+(x,t,\lambda) = M_-(x,t,\lambda)J(x,t,\lambda), \qquad \lambda \in \mathbb{R}, \\ M(x,t,\lambda) \to I, \qquad \lambda \to \infty, \end{cases} \tag{2.3}$$

with the jump matrix $J(x,t,\lambda)$ being

$$J(x,t,\lambda) = \begin{pmatrix} 1 - r_1(\lambda)r_2(\lambda) & r_2(\lambda)e^{2i\theta(x,t,\lambda)} \\ -r_1(\lambda)e^{-2i\theta(x,t,\lambda)} & 1 \end{pmatrix}, \tag{2.4}$$

where $r_1(\lambda) = \frac{s_{12}^*(-\lambda^*)}{s_{22}(\lambda)}, r_2(\lambda) = \frac{s_{12}(\lambda)}{s_{11}(\lambda)}, \theta(x,t,\lambda) = \lambda x + (\lambda^2 - 8\delta\lambda^4)t$. The solution $q(x,t)$ of the nonlocal LPD Eq. (1.1) is expressed as $q(x,t) = -2i \lim_{\lambda\to\infty} \lambda\, [M(x,t,\lambda)]_{12}$.

3. The long-time behavior for the nonlocal LPD equation

In this section, we aim to transform the associated original RH problem (2.3) to a solvable RH problem and then find the explicitly asymptotic formula for the nonlocal LPD Eq. (1.1). Let $\xi = \frac{x}{t}$, $f(\xi,\lambda)$ can be defined by $f(\xi,\lambda) = \lambda\xi + \lambda^2 - 8\delta\lambda^4$. Then, we take $-\sqrt{\frac{1}{27\delta}} < \xi < \sqrt{\frac{1}{27\delta}}$, it follows that there are three different real solutions for $\frac{\mathrm{d}f}{\mathrm{d}\lambda} = 0$, given by $\lambda_1 = \sqrt[3]{\frac{\xi}{64\delta} + \sqrt{(\frac{\xi}{64\delta})^2 - (\frac{1}{48\delta})^3}} + \sqrt[3]{\frac{\xi}{64\delta} - \sqrt{(\frac{\xi}{64\delta})^2 - (\frac{1}{48\delta})^3}}$, $\lambda_2 = \omega\sqrt[3]{\frac{\xi}{64\delta} + \sqrt{(\frac{\xi}{64\delta})^2 - (\frac{1}{48\delta})^3}} + \omega^2\sqrt[3]{\frac{\xi}{64\delta} - \sqrt{(\frac{\xi}{64\delta})^2 - (\frac{1}{48\delta})^3}}$, $\lambda_3 = \omega^2\sqrt[3]{\frac{\xi}{64\delta} + \sqrt{(\frac{\xi}{64\delta})^2 - (\frac{1}{48\delta})^3}} + \omega\sqrt[3]{\frac{\xi}{64\delta} - \sqrt{(\frac{\xi}{64\delta})^2 - (\frac{1}{48\delta})^3}}$, where $\omega = \frac{-1+\sqrt{3}i}{2}$. In this situation, the signature distribution for $\mathrm{Re}(if)$ is shown in Fig. 1. The following analysis of this paper restricts ξ to region $\xi \in (-\sqrt{\frac{1}{27\delta}} + \epsilon, \sqrt{\frac{1}{27\delta}} - \epsilon)$ for any positive constant ϵ.

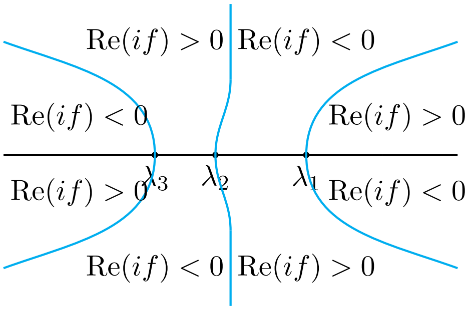

Fig. 1. The signature table for $\mathrm{Re}(if)$ in the complex λ-plane.

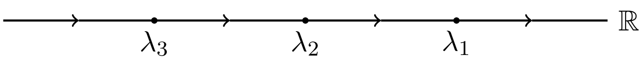

Fig. 2. The jump contour $\mathbb{R}$.

It is necessary to define the RH problem about the function $\delta(\lambda)$, given by $\delta_+(\lambda) = (1 - r_1(\lambda) r_2(\lambda))\delta_-(\lambda)$, $\lambda \in (\lambda_3, \lambda_2) \cup (\lambda_1, +\infty)$, and $\delta(\lambda) \to 1$, as $\lambda \to \infty$. Using the Plemelj formula, we get

$$\begin{aligned}\delta(\lambda) &= \exp\left\{ \left(\int_{\lambda_3}^{\lambda_2} + \int_{\lambda_1}^{\infty}\right) \frac{\ln(1 - r_1(s) r_2(s))}{2\pi i (s-\lambda)} \mathrm{d}s \right\} = (\lambda - \lambda_3)^{-i\vartheta(\lambda_3)} \left(\frac{\lambda - \lambda_1}{\lambda - \lambda_2}\right)^{-i\vartheta(\lambda_1)} e^{\chi_1(\lambda)} \\ &= (\lambda - \lambda_1)^{-i\vartheta(\lambda_1)} \left(\frac{\lambda - \lambda_3}{\lambda - \lambda_2}\right)^{-i\vartheta(\lambda_2)} e^{\chi_2(\lambda)} = (\lambda - \lambda_1)^{-i\vartheta(\lambda_1)} \left(\frac{\lambda - \lambda_3}{\lambda - \lambda_2}\right)^{-i\vartheta(\lambda_3)} e^{\chi_3(\lambda)},\end{aligned}$$

where

$$\begin{aligned}\chi_1(\lambda) &= \frac{1}{2\pi i}\left[\int_{\lambda_1}^{\lambda_2} \ln\left(\frac{1 - r_1(s) r_2(s)}{1 - r_1(\lambda_1) r_2(\lambda_1)}\right) \frac{\mathrm{d}s}{s - \lambda} - \int_{\lambda_3}^{\infty} \ln(\lambda - s)\, \mathrm{d}\ln(1 - r_1(s) r_2(s))\right], \\ \chi_2(\lambda) &= \frac{1}{2\pi i}\left[\int_{\lambda_3}^{\lambda_2} \ln\left(\frac{1 - r_1(s) r_2(s)}{1 - r_1(\lambda_2) r_2(\lambda_2)}\right) \frac{\mathrm{d}s}{s - \lambda} - \int_{\lambda_1}^{\infty} \ln(\lambda - s)\, \mathrm{d}\ln(1 - r_1(s) r_2(s))\right], \\ \chi_3(\lambda) &= \frac{1}{2\pi i}\left[\int_{\lambda_3}^{\lambda_2} \ln\left(\frac{1 - r_1(s) r_2(s)}{1 - r_1(\lambda_3) r_2(\lambda_3)}\right) \frac{\mathrm{d}s}{s - \lambda} - \int_{\lambda_1}^{\infty} \ln(\lambda - s)\, \mathrm{d}\ln(1 - r_1(s) r_2(s))\right], \\ \vartheta(\lambda_l) &= -\frac{1}{2\pi} \ln(1 - r_1(\lambda_l) r_2(\lambda_l)), \quad l = 1, 2, 3,\end{aligned} \tag{3.1}$$

so that $\mathrm{Im}\vartheta(\lambda_l) = -\frac{1}{2\pi}\int_{-\infty}^{\lambda_l} \mathrm{d}\arg(1 - r_1(s) r_2(s))$, $l = 1, 2, 3$. Assuming that $\int_{-\infty}^{\lambda_l} \mathrm{d}\arg(1 - r_1(s) r_2(s)) \in (-\frac{\pi}{3}, \frac{\pi}{3})$, one has $|\mathrm{Im}\vartheta(\lambda)| < \frac{1}{6}$, $\lambda \in \mathbb{R}$, which indicates that $\ln(1 - r_1(\lambda) r_2(\lambda))$ is single-valued, and the singularity of $\delta(\lambda, \xi)$ at $\lambda = \lambda_l$ is square integrable. Let $\tilde{\boldsymbol{M}}(x,t,\lambda) = \boldsymbol{M}(x,t,\lambda)\delta^{-\sigma_3}(\lambda)$, then $\tilde{\boldsymbol{M}}$ is the solution of RH problem on the jump contour $\mathbb{R}$ shown in Fig. 2, given by $\tilde{\boldsymbol{M}}_+(x,t,\lambda) = \tilde{\boldsymbol{M}}_-(x,t,\lambda)\tilde{\boldsymbol{J}}(x,t,\lambda)$, $\lambda \in \mathbb{R}$, and $\tilde{\boldsymbol{M}}(x,t,\lambda) \to \boldsymbol{I}$, as $\lambda \to \infty$, where

$$\tilde{\boldsymbol{J}}(x,t,\lambda) = \begin{cases} \begin{pmatrix} 1 & r_2(\lambda)\delta^2 e^{2ift} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -r_1(\lambda)\delta^{-2} e^{-2ift} & 1 \end{pmatrix}, & \lambda \in (\lambda_2, \lambda_1) \cup (-\infty, \lambda_3), \\ \begin{pmatrix} 1 & 0 \\ -p_1(\lambda)\delta_-^{-2} e^{-2ift} & 1 \end{pmatrix} \begin{pmatrix} 1 & p_2(\lambda)\delta_+^2 e^{2ift} \\ 0 & 1 \end{pmatrix}, & \lambda \in (\lambda_3, \lambda_2) \cup (\lambda_1, +\infty), \end{cases}$$

with $p_1(\lambda) = \frac{r_1(\lambda)}{1 - r_1(\lambda) r_2(\lambda)}$, $p_2(\lambda) = \frac{r_2(\lambda)}{1 - r_1(\lambda) r_2(\lambda)}$.

Next, we perform the first RH problem transformation by defining $\tilde{\tilde{\boldsymbol{M}}}(x,t,\lambda)$ as follows (see Fig. 3):

$$\tilde{\tilde{\boldsymbol{M}}} = \begin{cases} \tilde{\boldsymbol{M}}(\lambda), \ \lambda \in \Omega_5 \cup \Omega_6, \ \tilde{\boldsymbol{M}}(\lambda) \begin{pmatrix} 1 & 0 \\ r_1(\lambda)\delta^{-2} e^{-2ift} & 1 \end{pmatrix}, \ \lambda \in \Omega_1, \\ \tilde{\boldsymbol{M}}(\lambda) \begin{pmatrix} 1 & r_2(\lambda)\delta^2 e^{2ift} \\ 0 & 1 \end{pmatrix}, \ \lambda \in \Omega_4, \ \tilde{\boldsymbol{M}}(\lambda) \begin{pmatrix} 1 & -p_2(\lambda)\delta^2 e^{2ift} \\ 0 & 1 \end{pmatrix}, \ \lambda \in \Omega_2, \\ \tilde{\boldsymbol{M}}(\lambda) \begin{pmatrix} 1 & 0 \\ -p_1(\lambda)\delta^{-2} e^{-2ift} & 1 \end{pmatrix}, \ \lambda \in \Omega_3, \end{cases}$$

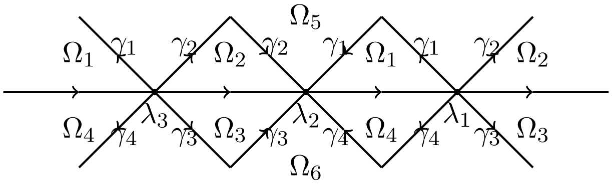

Fig. 3. The jump contour Σ and domains $\Omega_j (j = 1, \ldots, 6)$.

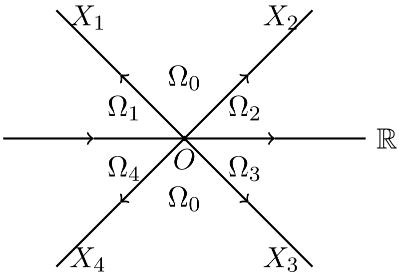

Fig. 4. The jump contour X and domains $\Omega_j (j = 0, \ldots, 4)$.

Then, the RH problem on the contour Σ is obtained $\tilde{\tilde{M}}_+(x,t,\lambda) = \tilde{\tilde{M}}_-(x,t,\lambda)\tilde{\tilde{J}}(x,t,\lambda)$, $\lambda \in \Sigma$, and $\tilde{\tilde{M}}(x,t,\lambda) \to I$, as $\lambda \to \infty$, where the jump matrix become

$$\tilde{\tilde{J}} = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_1(\lambda)\delta^{-2}e^{-2ift} & 1 \end{pmatrix}, \ \lambda \in \gamma_1, \ \begin{pmatrix} 1 & p_2(\lambda)\delta^2 e^{2ift} \\ 0 & 1 \end{pmatrix}, \ \lambda \in \gamma_2, \\ \begin{pmatrix} 1 & 0 \\ -p_1(\lambda)\delta^{-2}e^{-2ift} & 1 \end{pmatrix}, \ \lambda \in \gamma_3, \ \begin{pmatrix} 1 & -r_2(\lambda)\delta^2 e^{2ift} \\ 0 & 1 \end{pmatrix}, \ \lambda \in \gamma_4. \end{cases} \tag{3.2}$$

To separate the time t from the jump matrix, a scaling transformation is introduced by $z = T_1(\lambda) = \sqrt{4t(48\delta\lambda_1^2 - 1)}(\lambda - \lambda_1)$, $z = T_2(\lambda) = \sqrt{4t(1 - 48\delta\lambda_2^2)}(\lambda - \lambda_2)$, $z = T_3(\lambda) = \sqrt{4t(48\delta\lambda_3^2 - 1)}(\lambda - \lambda_3)$. Hence, one has $T_1(e^{itf}\delta(\lambda)) = \delta^0_{\lambda_1}\delta^1_{\lambda_1}(z)$, $T_2(e^{itf}\delta(\lambda)) = \delta^0_{\lambda_2}\delta^1_{\lambda_2}(z)$, $T_3(e^{itf}\delta(\lambda)) = \delta^0_{\lambda_3}\delta^1_{\lambda_3}(z)$. Let $t \to \infty$, we can derive the following RH problem $M^{X_l} (l = 1, 2, 3)$ in the z plane related to $X = X_1 \cup X_2 \cup X_3 \cup X_4$ (see Fig. 4), $M^{X_l}_+(\xi, z) = M^{X_l}_-(\xi, z)J^{X_l}(\xi, z)$, $z \in X$, and $M^{X_l}(\xi, z) \to I$, as $z \to \infty$, of which

$$J^{X_1} = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_1(\lambda_1)e^{\frac{iz^2}{2}}(-z)^{2i\vartheta(\lambda_1)} & 1 \end{pmatrix}, z \in X_1, \begin{pmatrix} 1 & p_2(\lambda_1)e^{-\frac{iz^2}{2}}(-z)^{-2i\vartheta(\lambda_1)} \\ 0 & 1 \end{pmatrix}, z \in X_2, \\ \begin{pmatrix} 1 & 0 \\ -p_1(\lambda_1)e^{\frac{iz^2}{2}}(-z)^{2i\vartheta(\lambda_1)} & 1 \end{pmatrix}, z \in X_3, \begin{pmatrix} 1 & -r_2(\lambda_1)e^{-\frac{iz^2}{2}}(-z)^{-2i\vartheta(\lambda_1)} \\ 0 & 1 \end{pmatrix}, z \in X_4. \end{cases}$$

$$J^{X_2} = \begin{cases} \begin{pmatrix} 1 & -p_2(\lambda_2)e^{\frac{iz^2}{2}}z^{2i\vartheta(\lambda_2)} \\ 0 & 1 \end{pmatrix}, z \in X_1, \begin{pmatrix} 1 & 0 \\ -r_1(\lambda_2)e^{-\frac{iz^2}{2}}z^{-2i\vartheta(\lambda_2)} & 1 \end{pmatrix}, z \in X_2, \\ \begin{pmatrix} 1 & r_2(\lambda_2)e^{\frac{iz^2}{2}}z^{2i\vartheta(\lambda_2)} \\ 0 & 1 \end{pmatrix}, z \in X_3, \begin{pmatrix} 1 & 0 \\ p_1(\lambda_2)e^{-\frac{iz^2}{2}}z^{-2i\vartheta(\lambda_2)} & 1 \end{pmatrix}, z \in X_4. \end{cases}$$

$$J^{X_3} = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_1(\lambda_3)e^{\frac{iz^2}{2}}(-z)^{2i\vartheta(\lambda_3)} & 1 \end{pmatrix}, z \in X_1, \begin{pmatrix} 1 & p_2(\lambda_3)e^{-\frac{iz^2}{2}}(-z)^{-2i\vartheta(\lambda_3)} \\ 0 & 1 \end{pmatrix}, z \in X_2, \\ \begin{pmatrix} 1 & 0 \\ -p_1(\lambda_3)e^{\frac{iz^2}{2}}(-z)^{2i\vartheta(\lambda_3)} & 1 \end{pmatrix}, z \in X_3, \begin{pmatrix} 1 & -r_2(\lambda_3)e^{-\frac{iz^2}{2}}(-z)^{-2i\vartheta(\lambda_3)} \\ 0 & 1 \end{pmatrix}, z \in X_4. \end{cases}$$

For $l = 1, 2, 3$, defining $D_\epsilon(\lambda_l)$ as the open disk of radius ϵ centered at λ_l for a small $\epsilon > 0$ and using M^{X_l}, we introduce M^{λ_l} for $\lambda \in D_\epsilon(\lambda_l)$ $M^{\lambda_l}(x,t,\lambda) = (\delta^0_{\lambda_l})^{\sigma_3} M^{X_l}(z)(\delta^0_{\lambda_l})^{-\sigma_3}$, which is analytic function in region $\lambda \in D_\epsilon(\lambda_l) \setminus X^\epsilon_{\lambda_l}$, where $X^\epsilon_{\lambda_l} = X_{\lambda_l} \cap D_\epsilon(\lambda_l)$, $X_{\lambda_l} = X + \lambda_l$ means the cross X centered at λ_l. It is not hard to find that $M^{\lambda_l}(x,t,\lambda)$ solves following RH problem, given by $M^{\lambda_l}_+(x,t,\lambda) = M^{\lambda_l}_-(x,t,\lambda)J^{\lambda_l}$. Next, we devote to derive the explicit expression of long-time asymptotic behavior for the nonlocal LPD Eq. (1.1) on the line by introducing the approximate solution $M^{app}(x,t,\lambda)$ as follows

$$M^{app} = \begin{cases} M^{\lambda_1}, \ \lambda \in D_\epsilon(\lambda_1), \ M^{\lambda_2}, \ \lambda \in D_\epsilon(\lambda_2), \\ M^{\lambda_3}, \ \lambda \in D_\epsilon(\lambda_3), \ I, \quad \text{elsewhere.} \end{cases} \tag{3.3}$$

Let $\hat{M}(x,t,\lambda) = \tilde{\tilde{M}}(M^{app})^{-1}$, then we have $\hat{M}_+(x,t,\lambda) = \hat{M}_-(x,t,\lambda)\hat{J}(x,t,\lambda)$, $\lambda \in \hat{\Sigma}$, where the jump contour $\hat{\Sigma} = \Sigma \cup \partial D_\epsilon(\lambda_1) \cup \partial D_\epsilon(\lambda_2) \cup \partial D_\epsilon(\lambda_3)$, and the jump matrix $\hat{J}(x,t,\lambda)$ arrives at

$$\hat{J} = \begin{cases} M_-^{app}\tilde{\tilde{J}}(M_+^{app})^{-1}, & \lambda \in \hat{\Sigma} \cap (D_\epsilon(\lambda_1) \cup D_\epsilon(\lambda_2) \cup D_\epsilon(\lambda_3)), \\ (M^{app})^{-1}, & \lambda \in \partial D_\epsilon(\lambda_1) \cup \partial D_\epsilon(\lambda_2) \cup \partial D_\epsilon(\lambda_3), \\ \tilde{\tilde{J}}, & \lambda \in \hat{\Sigma} \setminus (\overline{D_\epsilon(\lambda_1)} \cup \overline{D_\epsilon(\lambda_2)} \cup \overline{D_\epsilon(\lambda_3)}). \end{cases} \tag{3.4}$$

Let $\hat{\omega} = \hat{J} - I$, for $1 \le n \le \infty$, $\hat{\Sigma}_1 = \bigcup_1^4 \gamma_k \setminus (D_\epsilon(\lambda_1) \cup D_\epsilon(\lambda_2) \cup D_\epsilon(\lambda_3))$ and $\xi \in (-\sqrt{\frac{1}{27\delta}} + \epsilon, \sqrt{\frac{1}{27\delta}} - \epsilon)$, we have $\|\hat{\omega}\|_{L^n(\hat{\Sigma}_1)} \le Ce^{-ct}$, $\|\hat{\omega}^{(j)}\|_{L^n(\partial D_\epsilon(\lambda_l))} \le Ct^{-\frac{1}{2}+(-1)^{l+j}\mathrm{Im}\vartheta(\lambda_l)}$, $\|\hat{\omega}^{(j)}\|_{L^n(X_{\lambda_l}^\epsilon)} \le Ct^{-\frac{1}{2}-\frac{1}{2n}+(-1)^{l+j}\mathrm{Im}\vartheta(\lambda_l)} \ln t$.It is necessary to define the Cauchy operator $(Cf)(\lambda) = \frac{1}{2\pi i}\int_{\hat{\Sigma}} \frac{f(s)}{s-\lambda}\mathrm{d}s$, $\lambda \in \mathbb{C} \setminus \hat{\Sigma}$, and the integral operator $C_{\hat{\omega}} : L^2(\hat{\Sigma}) + L^\infty(\hat{\Sigma}) \to L^2(\hat{\Sigma})$ by $C_{\hat{\omega}} f = C_-(f\hat{\omega})$. Moreover, $\|\hat{\mu} - I\|_{L^2(\hat{\Sigma})} \le C\|\hat{\omega}\|_{L^2(\hat{\Sigma})}$, where $\hat{\mu} = I + L^2(\hat{\Sigma})$ satisfies the following integral equation $\hat{\mu} = I + C_{\hat{\omega}}\hat{\mu}$. Finally, the representation for $\hat{M}$ admits $\hat{M}(x,t,\lambda) = I + \frac{1}{2\pi i}\int_{\hat{\Sigma}} \frac{\hat{\mu}(x,t,s)\hat{\omega}(x,t,s)}{s-\lambda}\mathrm{d}s$, it follows that $\lim_{\lambda\to\infty} \lambda(\hat{M}(x,t,\lambda) - I) = -\frac{1}{2\pi i}\int_{\hat{\Sigma}} \hat{\mu}(x,t,\lambda)\hat{\omega}(x,t,\lambda)\mathrm{d}\lambda$. Finally, as $t \to \infty$, we derive the following important result

$$q(x,t) = -2i \lim_{\lambda\to\infty} \lambda[\hat{M}(x,t,\lambda) - I]_{12} = -2i \sum_{l=1}^{3} \frac{(\delta_{\lambda_l}^0)^2 [M_1^{X_l}]_{12}}{\sqrt{4t(-1)^l(1 - 48\delta\lambda_l^2)}} + R(\xi,t), \tag{3.5}$$

where

$$R(\xi,t) = \begin{cases} O(t^{-1+2\max\{|\mathrm{Im}\vartheta(\lambda_1)|,|\mathrm{Im}\vartheta(\lambda_2)|,|\mathrm{Im}\vartheta(\lambda_3)|\}}), & (-1)^l\mathrm{Im}\vartheta(\lambda_l) > 0, \\ O(t^{-1+2\max\{|\mathrm{Im}\vartheta(\lambda_1)|,|\mathrm{Im}\vartheta(\lambda_2)|\}}), & \mathrm{Im}\vartheta(\lambda_1) < 0, \mathrm{Im}\vartheta(\lambda_2) > 0, \mathrm{Im}\vartheta(\lambda_3) \ge 0, \\ O(t^{-1+2\max\{|\mathrm{Im}\vartheta(\lambda_2)|,|\mathrm{Im}\vartheta(\lambda_3)|\}}), & \mathrm{Im}\vartheta(\lambda_1) \ge 0, \mathrm{Im}\vartheta(\lambda_2) > 0, \mathrm{Im}\vartheta(\lambda_3) < 0, \\ O(t^{-1+2\max\{|\mathrm{Im}\vartheta(\lambda_1)|,|\mathrm{Im}\vartheta(\lambda_3)|\}}), & \mathrm{Im}\vartheta(\lambda_1) < 0, \mathrm{Im}\vartheta(\lambda_2) \le 0, \mathrm{Im}\vartheta(\lambda_3) < 0, \\ O(t^{-1+2|\mathrm{Im}\vartheta(\lambda_1)|}), & \mathrm{Im}\vartheta(\lambda_1) < 0, \mathrm{Im}\vartheta(\lambda_2) \le 0, \mathrm{Im}\vartheta(\lambda_3) \ge 0, \\ O(t^{-1+2|\mathrm{Im}\vartheta(\lambda_2)|}), & \mathrm{Im}\vartheta(\lambda_1) \ge 0, \mathrm{Im}\vartheta(\lambda_2) > 0, \mathrm{Im}\vartheta(\lambda_3) \ge 0, \\ O(t^{-1+2|\mathrm{Im}\vartheta(\lambda_3)|}), & \mathrm{Im}\vartheta(\lambda_1) \ge 0, \mathrm{Im}\vartheta(\lambda_2) \le 0, \mathrm{Im}\vartheta(\lambda_3) < 0, \\ O(t^{-1}\ln t), & \mathrm{Im}\vartheta(\lambda_l) = 0, (-1)^s\mathrm{Im}\vartheta(\lambda_s) \le 0, s = 1,2,3 \text{ and } s \ne l, \\ O(t^{-1}), & (-1)^l\mathrm{Im}\vartheta(\lambda_l) < 0. \end{cases} \tag{3.6}$$

and $[M_1^{X_l}]_{12}$ can be explicitly solved by using the Liouville's theorem and parabolic cylinder functions, given by $[M_1^{X_1}]_{12} = \frac{\sqrt{2\pi}e^{-\frac{3\pi i}{4}+\frac{\pi\vartheta(\lambda_1)}{2}}}{ir_1(\lambda_1)\Gamma(i\vartheta(\lambda_1))}$, $[M_1^{X_2}]_{12} = \frac{\sqrt{2\pi}ie^{-\frac{\pi i}{4}+\frac{\pi\vartheta(\lambda_2)}{2}}}{r_1(\lambda_2)\Gamma(-i\vartheta(\lambda_2))}$, $[M_1^{X_3}]_{12} = \frac{\sqrt{2\pi}e^{-\frac{3\pi i}{4}+\frac{\pi\vartheta(\lambda_3)}{2}}}{ir_1(\lambda_3)\Gamma(i\vartheta(\lambda_3))}$. Substituting them into (3.5), we finally achieve the main result of Theorem 3.1.

Theorem 3.1. *Suppose that $q(x,t)$ be the solution of the Cauchy problem of the nonlocal LPD Eq. (1.1) with $q_0(x)$ lying in the Schwartz space. Assume that the scattering data associated with $q_0(x)$ satisfy: (i)$s_{11}(\lambda)$ and $s_{22}(\lambda)$ have no zeros in $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda \le 0\}$ and $\{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda \ge 0\}$, respectively. (ii)$\int_{-\infty}^{\zeta} \mathrm{d}\arg(1 - r_1(s)r_2(s)) \in (-\frac{\pi}{3}, \frac{\pi}{3})$ for all $\zeta \in \mathbb{R}$(i.e. $|\mathrm{Im}\vartheta(\lambda)| < \frac{1}{6}$), where $r_1(\lambda) = \frac{s_{12}^*(-\lambda^*)}{s_{22}(\lambda)}, r_2(\lambda) = \frac{s_{12}(\lambda)}{s_{11}(\lambda)}$. Then, for any positive constants $\delta > 0, \epsilon > 0$, when $t \to \infty, \xi = \frac{x}{t} \in (-\sqrt{\frac{1}{27\delta}} + \epsilon, \sqrt{\frac{1}{27\delta}} - \epsilon)$, the long-time asymptotics of the solution $q(x,t)$ is*

$$q(x,t) = \sum_{l=1}^{3} t^{-\frac{1}{2}+(-1)^l\mathrm{Im}\vartheta(\lambda_l)} P_l e^{48\delta i\lambda_l^4 t - 2i\lambda_l^2 t - (-1)^l i\mathrm{Re}\vartheta(\lambda_l)\ln t} + R(\xi,t),$$

where

$$P_1 = -\frac{2\sqrt{2\pi}e^{(i\vartheta(\lambda_1)-\frac{1}{2})\ln(192\delta\lambda_1^2-4)-2i\vartheta(\lambda_3)\ln(\lambda_1-\lambda_3)+2i\vartheta(\lambda_1)\ln(\lambda_2-\lambda_1)+2\chi_1(\lambda_1)+\frac{\pi}{2}\vartheta(\lambda_1)-\frac{3}{4}\pi i}}{r_1(\lambda_1)\Gamma(i\vartheta(\lambda_1))},$$

$$P_2 = \frac{2\sqrt{2\pi}e^{-(i\vartheta(\lambda_2)+\frac{1}{2})\ln(4-192\delta\lambda_2^2)-2i\vartheta(\lambda_1)\ln(\lambda_2-\lambda_1)-2i\vartheta(\lambda_2)\ln(\lambda_2-\lambda_3)+2\chi_2(\lambda_2)+\frac{\pi}{2}\vartheta(\lambda_2)-\frac{1}{4}\pi i}}{r_1(\lambda_2)\Gamma(-i\vartheta(\lambda_2))},$$

$$P_3 = -\frac{2\sqrt{2\pi}e^{(i\vartheta(\lambda_3)-\frac{1}{2})\ln(192\delta\lambda_3^2-4)-2i\vartheta(\lambda_1)\ln(\lambda_3-\lambda_1)+2i\vartheta(\lambda_3)\ln(\lambda_2-\lambda_3)+2\chi_3(\lambda_3)+\frac{\pi}{2}\vartheta(\lambda_3)-\frac{3}{4}\pi i}}{r_1(\lambda_3)\Gamma(i\vartheta(\lambda_3))},$$

and Γ is Euler's Gamma function, $\chi_l(\lambda_l), \vartheta(\lambda_l)$ are given in (3.1), and the error estimation $R(\xi,t)$ is given in (3.6).

Data availability

Data will be made available on request.

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