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Inverse scattering transformation for generalized nonlinear Schrödinger equation☆

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ABSTRACT

Based on the robust inverse scattering method, the high-order rogue wave of generalized nonlinear Schrödinger equation with nonzero boundary is given. Using this method, we only need the elementary Darboux transformation but not with the limit progress, which is more convenient than before. By choosing different parameters c_1 and c_2 appeared in the Darboux matrix, the $2n$ and $2n-1$ order rogue waves are derived respectively. Furthermore, the general breather is also given with a different spectral parameters.

1. Introduction

In 1983, one kind of rational solution called peregrine rogue wave is given for the focusing nonlinear Schrödinger equation(NLS) [1]. Since then, the rogue wave becomes a hot topic for a long time. Meanwhile, there appear a flood to study it with various methods. The effective methods are the generalized Darboux method [2–4], KP reduction technique [5,6], Hirota bilinear method [7] and so on. Additionally, Bilmann and Miller [8] propose a robust inverse scattering transformation method, his method connect the inverse scattering and elementary Darboux method and change the jump to a circle, which is a great step for the rogue wave study. In this paper, we use the robust inverse scattering transformation method to get the

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high-order rogue waves to the following nonzero generalized NLS equation

$$\begin{aligned}&\mathrm{i}q_t+q_{xx}+2\mathrm{i}k_1q_x+2q(|q|^2-1)\\&+\gamma_1\Big(q_{xxxx}+4\mathrm{i}k_1q_{xxx}+24\mathrm{i}k_1|q|^2q_x-6k_1^2q_{xx}+12k_1^2q-12k_1^2|q|^2q-4\mathrm{i}k_1^3q_x+2q^2q_{xx}^*+4q|q_x|^2\\&+8|q|^2q_{xx}+6q^*q_x^2+6|q|^4q-6q\Big)=0\end{aligned}\tag{1}$$

This equation can be changed into the general generalized NLS equation [9]

$$\mathrm{i}u_t+u_{xx}+2|u|^2u+\gamma_1\left(u_{xxxx}+6u_x^2u^*+4u|u_x|^2+8|u|^2u_{xx}+2u^2u_{xx}^*+6|u|^4u\right)=0\tag{2}$$

with the Gauge transformation

$$q=u\mathrm{e}^{-\mathrm{i}\left(k_1x+\left(\gamma_1k_1^4-12\gamma_1k_1^2-k_1^2+6\gamma_1+2\right)t\right)}\tag{3}$$

The lax pair for (1) is

$$\mathbf{\Phi}_x=\begin{bmatrix}-\mathrm{i}\lambda-\frac{\mathrm{i}}{2}k_1 & q\\ -q^* & \mathrm{i}\lambda+\frac{\mathrm{i}}{2}k_1\end{bmatrix}\mathbf{\Phi}\quad \mathbf{\Phi}_t=\begin{bmatrix}V_{11} & V_{12}\\ V_{21} & -V_{11}\end{bmatrix}\mathbf{\Phi}\tag{4}$$

where

$$\begin{aligned}V_{11}&=8\mathrm{i}\gamma_1\lambda^4-\left(2\mathrm{i}+4\mathrm{i}|q|^2\gamma_1\right)\lambda^2+\left(4\mathrm{i}k_1\gamma_1|q|^2-2\gamma_1qq_x^*+2\gamma_1q^*q_x\right)\lambda-\mathrm{i}+\mathrm{i}|q|^2+\frac{\mathrm{i}}{2}k_1^2\\&-\gamma_1\left(\frac{\mathrm{i}}{2}k_1^4+3\mathrm{i}k_1^2|q|^2-6\mathrm{i}k_1^2+3k_1q^*q_x-3k_1qq_x^*-3\mathrm{i}|q|^4-\mathrm{i}qq_{xx}^*-\mathrm{i}q^*q_{xx}+\mathrm{i}|q_x|^2+3\mathrm{i}\right)\\V_{12}&=-8\gamma_1q\lambda^3+4\gamma_1\left(k_1q-\mathrm{i}q_x\right)\lambda^2+\left(4\mathrm{i}\gamma_1k_1q_x+4\gamma_1|q|^2q-2\gamma_1k_1^2q+2\gamma_1q_{xx}+2q\right)\lambda+6\mathrm{i}\gamma_1|q|^2q_x\\&-3\mathrm{i}\gamma_1k_1^2q_x-6\gamma_1k_1|q|^2q+\gamma_1k_1^3q+\mathrm{i}\gamma_1q_{xxx}-3\gamma_1k_1q_{xx}+\mathrm{i}q_x-k_1q\\V_{21}&=8\gamma_1q^*\lambda^3-4\gamma_1\left(\mathrm{i}q_x^*+k_1q^*\right)\lambda^2+\left(4\mathrm{i}\gamma_1k_1q_x^*-4\gamma_1|q|^2q^*+2\gamma_1k_1^2q^*-2\gamma_1q_{xx}^*-2q^*\right)\lambda\\&+6\mathrm{i}\gamma_1|q|^2q_x^*-3\mathrm{i}\gamma_1k_1^2q_x^*\\&+6\gamma_1k_1|q|^2q^*-\gamma_1k_1^3q^*+\mathrm{i}\gamma_1q_{xxx}^*+3\gamma_1k_1q_{xx}^*+\mathrm{i}q_x^*+k_1q^*\end{aligned}\tag{5}$$

the boundary of this lax pair can be $q=1$, so the fundamental solution of this lax pair is

$$\mathbf{\Phi}_{bg}(\lambda;x)=n(\lambda)\begin{bmatrix}1 & \mathrm{i}\lambda-\mathrm{i}\rho(\lambda)+\frac{\mathrm{i}}{2}k_1\\ \mathrm{i}\lambda-\mathrm{i}\rho(\lambda)+\frac{\mathrm{i}}{2}k_1 & 1\end{bmatrix}\mathrm{e}^{-\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}=:\mathbf{E}(\lambda)\mathrm{e}^{-\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}\tag{6}$$

where

$$\begin{aligned}&\rho(\lambda)^2=\lambda^2+k_1\lambda+1+\frac{1}{4}k_1^2,\,n(\lambda)^2=\frac{2\rho(\lambda)+2\lambda+k_1}{4\rho(\lambda)},\sigma_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\\&\theta(\lambda;x,t)=\left(x+\left(-8\gamma_1\lambda^3+4\gamma_1k_1\lambda^2+\left(2-2\gamma_1k_1^2+4\gamma_1\right)\lambda+\gamma_1k_1^3-6\gamma_1k_1-k_1\right)t\right).\end{aligned}$$

The normalization factor $n(\lambda)$ ensures that $\det\left(\mathbf{\Phi}_{bg}\right)=1$. Note that the $\mathbf{E}(\lambda)$ has singularities at the branch points $\lambda=-\frac{k_1}{2}\pm\mathrm{i}$ and it is analytic for $\lambda\in\mathbb{C}\setminus\Sigma_c$, where $\Sigma_c=\left[-\frac{k_1}{2}-\mathrm{i},-\frac{k_1}{2}+\mathrm{i}\right]$. The continuous spectral Γ is the union of $\left(-\infty,-\frac{1}{2}k_1\right),\left(-\frac{1}{2}k_1,\infty\right)$ and Σ_c.

Suppose $\Delta q(x,t):=q(x,t)-1\in L^1(\mathbb{R})$, then for $\lambda\in\Gamma\setminus\left\{-\frac{1}{2}k_1-\mathrm{i},-\frac{1}{2}k_1+\mathrm{i}\right\}$, the Jost solution $\mathbf{\Phi}=\mathbf{J}^{\pm}(\lambda;x,t)$ of (4) are defined with the boundary condition

$$\mathbf{J}^{\pm}(\lambda;x,t)\mathrm{e}^{\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}=\mathbf{E}(\lambda)+o(1),\quad x\to\pm\infty\tag{7}$$

and the corresponding renormalization $\mathbf{K}^{\pm}(\lambda;x,t) := \mathbf{J}^{\pm}(\lambda;x,t)\mathrm{e}^{\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}$ can be given as the Volterra integral equations:

$$\mathbf{K}^{\pm}(\lambda;x,t) = \mathbf{E}(\lambda) + \int_{\pm\infty}^{x} \mathbf{E}(\lambda)\mathrm{e}^{-\mathrm{i}\rho(\lambda)(x-\hat{x})\sigma_3}\mathbf{E}^{-1}(\lambda)\Delta\mathbf{U}(y,t)\mathbf{K}^{\pm}(\lambda;y,t)\mathrm{e}^{\mathrm{i}\rho(\lambda)(x-\hat{x})\sigma_3}d\hat{x}, \quad \rho(\lambda)\in\mathbb{R} \tag{8}$$

where

$$\Delta\mathbf{U}(x,t) := \begin{bmatrix} 0 & \Delta q(x,t) \\ -\Delta q^*(x,t) & 0 \end{bmatrix} \tag{9}$$

It is clear the first column $\mathbf{j}^{-,1}(\lambda;x,t)$ of $\mathbf{J}^{-}(\lambda;x,t)$ and the second column $\mathbf{j}^{+,2}(\lambda;x,t)$ of $\mathbf{J}^{+}(\lambda;x,t)$ are boundary values of λ analytic in the domain $\mathrm{Im}\{\rho(\lambda)\} > 0$. and the first column $\mathbf{j}^{+,1}(\lambda;x,t)$ of $\mathbf{J}^{+}(\lambda;x,t)$ and the second column $\mathbf{j}^{-,2}(\lambda;x,t)$ of $\mathbf{J}^{-}(\lambda;x,t)$ are boundary values of λ analytic in the domain $\mathrm{Im}\{\rho(\lambda)\} < 0$. Furthermore, $\det(\mathbf{J}^{\pm}(\lambda;x,t)) = 1$ for $\lambda \in \Gamma \setminus \{-\frac{1}{2}k_1 - \mathrm{i}, -\frac{1}{2}k_1 + \mathrm{i}\}$. And they satisfy the scattering relation $\mathbf{J}^{+}(\lambda;x,t) = \mathbf{J}^{-}(\lambda;x,t)\mathbf{S}(\lambda;t)$, where $\mathbf{S}(\lambda;t)$ is the scattering matrix with the form $\begin{bmatrix} a^*(\lambda) & b^*(\lambda) \\ -b(\lambda) & a(\lambda) \end{bmatrix}$. Then the Beals–Coifman simultaneous solution of (4) is

$$\mathbf{\Phi}^{\mathrm{BC}}(\lambda;x,t) := \begin{cases} \left[a(\lambda)^{-1}\mathbf{j}^{-,1}(\lambda;x,t);\mathbf{j}^{+,2}(\lambda;x,t)\right], & \lambda\in\mathbb{C}^{+}\setminus\Sigma_c \\ \left[\mathbf{j}^{+,1}(\lambda;x,t);a^*(\lambda)^{-1}\mathbf{j}^{-,2}(\lambda;x,t)\right], & \lambda\in\mathbb{C}^{-}\setminus\Sigma_c \end{cases} \tag{10}$$

Take $\mathbf{M}^{\mathrm{BC}}(\lambda;x,t) := \mathbf{\Phi}^{\mathrm{BC}}(\lambda;x,t)\mathrm{e}^{\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}$, the jump for $\mathbf{M}^{\mathrm{BC}}(\lambda;x,t)$ is the union of $\left(-\infty,-\frac{1}{2}k_1\right)$, $\left(-\frac{1}{2}k_1+\infty\right)$, $\left(-\frac{1}{2}k_1,-\frac{1}{2}k_1-\mathrm{i}\right)$ and $\left(-\frac{1}{2}k_1,-\frac{1}{2}k_1+\mathrm{i}\right)$. In [10,11], Defit and Zhou develop a method to construct another simultaneous solution of the lax pair (4) for smaller λ to make this solution no singularities. So we can use this similar method to construct a simultaneous fundamental solution matrix of the lax pair (4)

$$\mathbf{\Phi}(\lambda;x,t) = \begin{cases} \mathbf{\Phi}^{\mathrm{BC}}(\lambda;x,t), & \lambda\in D_+\cup D_- \\ \mathbf{\Phi}^{\mathrm{in}}(\lambda;x,t), & \lambda\in D_0 \end{cases} \tag{11}$$

where $\mathbf{\Phi}^{\mathrm{BC}}(\lambda;x,t)$ is the Beals–Coifman simultaneous solution and $\mathbf{\Phi}^{\mathrm{in}}(\lambda;x,t)$ is an entire function, which can be obtained as $\mathbf{\Phi}(\lambda;x,t)\mathbf{\Phi}(\lambda;L,0)^{-1}$. D_0 is an open disk with the radius r, and this choice of r should make sure the scattering data $a(\lambda)$ cannot be zero on the outside of the disk and this disk should also contain the branch cut $\left[-\frac{1}{2}k_1-\mathrm{i},-\frac{1}{2}k_1+\mathrm{i}\right]$. Set $\mathbf{M}(\lambda;x,t) = \mathbf{\Phi}(\lambda;x,t)\mathrm{e}^{\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}$, then the Riemann–Hilbert problem of the generalized NLS equation is

Riemann–Hilbert Problem 1. *The matrix function* $\mathbf{M}(\lambda;x,t)$ *has the following properties:*

- **Analyticity** : $\mathbf{M}(\lambda;x,t)$ *is analytic function in* $\lambda\in\mathbb{C}\setminus\{\Sigma\cup\Sigma_c\}$*;*
- **Jump Condition***:*

$$\mathbf{M}_+(\lambda;x,t) = \begin{cases} \mathbf{M}_-(\lambda;x,t)\mathrm{e}^{-\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}\mathbf{V}(\lambda)\mathrm{e}^{\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}, & \lambda\in\Sigma \\ \mathbf{M}_-(\lambda;x,t)\mathrm{e}^{2\mathrm{i}\rho(\lambda)_+\theta(\lambda;x,t)\sigma_3}, & \lambda\in\Sigma_c \end{cases}$$

- **Normalization** : $\lim_{\lambda\to\infty}\mathbf{M}(\lambda;x,t)\to\mathbb{I}$

where $\mathbf{V}(\lambda)$ is defined as

$$\mathbf{V}(\lambda) = \begin{cases} \left[a(\lambda)^{-1}\mathbf{j}^{-,1}(\lambda;L,0),\mathbf{j}^{+,2}(\lambda;L,0)\right], & \lambda\in\Sigma_+ \\ \left[\mathbf{j}^{+,1}(\lambda;L,0),a^*(\lambda)^{-1}\mathbf{j}^{-,2}(\lambda;L,0)\right]^{-1}, & \lambda\in\Sigma_- \\ \begin{bmatrix} 1+|r(\lambda)|^2 & r^*(\lambda) \\ r(\lambda) & 1 \end{bmatrix}, & \lambda\in\Sigma_{\mathbb{L}}\cup\Sigma_{\mathbb{R}}, r(\lambda)=b(\lambda)/a(\lambda) \end{cases} \tag{12}$$

and the contour is shown in Fig. 1. Then the function $q(x,t)$ can be obtained with the limit

$$q(x,t) = 2\mathrm{i}\lim_{\lambda\to\infty}\lambda M_{12}(\lambda;x,t) \tag{13}$$

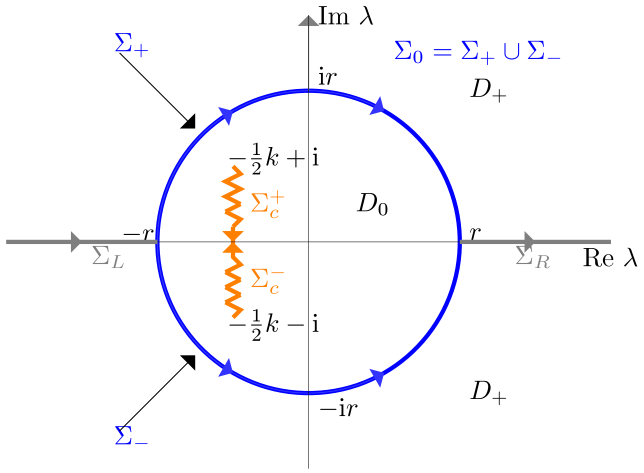


Fig. 1. Definition of the regions $D_0, D_\pm$ and the contour $\Sigma_0 = \Sigma_+ + \Sigma_-$, $\Sigma_c = \Sigma_c^+ + \Sigma_c^-$.

2. High-order rogue waves of the generalized NLS equation

In this section, we want to get the high-order rogue wave to the generalized NLS equation by using the Gauge transformation of the Riemann–Hilbert 1. Set the gauge transformation as

$$\widehat{\mathbf{\Phi}}(\lambda; x, t) := \begin{cases} \mathbf{G}(\lambda; x, t)\mathbf{\Phi}(\lambda; x, t), & \lambda \in D_+ \cup D_- \\ \mathbf{G}(\lambda; x, t)\mathbf{\Phi}(\lambda; x, t)\mathbf{G}(\lambda; L, 0)^{-1}, & \lambda \in D_0 \end{cases} \tag{14}$$

where $\mathbf{\Phi}(\lambda; x, t)$ is a solution of the lax pair (4) and $\mathbf{G}(\lambda; x, t) = \mathbb{I} + \frac{\mathbf{Y}(x,t)}{\lambda - \lambda_1} + \sigma_2 \frac{\mathbf{Y}^*(x,t)}{\lambda - \lambda_1^*}\sigma_2, \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$. The multiplication of $\mathbf{\Phi}(\lambda; x, t)$ on the right by $\mathbf{G}(\lambda; L, 0)^{-1}$ preserves the normalization condition of $\widehat{\mathbf{\Phi}}(\lambda; L, 0) = \mathbb{I}$ for $\lambda \in D_0$. The corresponding jump condition of $\widehat{\mathbf{M}}(\lambda; x, t) := \widehat{\mathbf{\Phi}}(\lambda; x, t)\mathrm{e}^{\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}$ only changes when $\lambda \in \Sigma_+ \cup \Sigma_-$, which change into

$$\widehat{\mathbf{V}}(\lambda; x, t) = \begin{cases} \mathbf{G}(\lambda; L, 0)\mathbf{V}(\lambda), & \lambda \in \Sigma_+ \\ \mathbf{V}(\lambda)\mathbf{G}(\lambda; L, 0)^{-1}, & \lambda \in \Sigma_- \end{cases} \tag{15}$$

and

$$\mathbf{Y}(\lambda; x, t) = \frac{-(\lambda_1 - \lambda_1^*)^2 (1 - w^*)\mathbf{s}(x,t)\mathbf{s}^{\mathrm{T}}(x,t)\sigma_2 + (\lambda_1 - \lambda_1^*) N(x,t)\sigma_2 \mathbf{s}^*(x,t)\mathbf{s}^{\mathrm{T}}(x,t)\sigma_2}{-(\lambda_1 - \lambda_1^*)^2 |1 - w(x,t)|^2 + N^2(x,t)}$$

where $\mathbf{s}(x,t) := \mathbf{\Phi}(\lambda_1; x, t)\mathbf{c}, N(x,t) := \|\mathbf{s}(x,t)\|^2 = \mathbf{s}^\dagger(x,t)\mathbf{s}(x,t), w := \mathbf{s}(x,t)\sigma_2\mathbf{\Phi}'(\lambda_1; x, t)\mathbf{c}$, and $\mathbf{c}$ is an arbitrary vector. Then the potential function $\widehat{q}(x,t)$ can be recovered from the new Riemann–Hilbert problem $\widehat{\mathbf{M}}(\lambda; x, t)$, that is $\widehat{q}(x,t) = 2\mathrm{i}\lim_{\lambda\to\infty} \lambda\widehat{\mathbf{M}}(\lambda; x, t)_{12}$.

To apply the Darboux transformation, we should know the vector $\mathbf{s}(x,t)$, we can choose $\mathbf{\Phi}_{\mathrm{bg}}^{\mathrm{in}}(\lambda; x, t) = \mathbf{\Phi}_{\mathrm{bg}}(\lambda; x, t)\mathbf{\Phi}_{\mathrm{bg}}(\lambda; 0, 0)^{-1}$ as the initial solution normalized at $(x,t) = (0,0)$, then we have

$$\mathbf{\Phi}_{\mathrm{bg}}^{\mathrm{in}}(\lambda; x, t) = \frac{\sin(\rho(\lambda)\theta(\lambda; x, t))}{\rho(\lambda)} \begin{bmatrix} -\mathrm{i}\lambda - \frac{\mathrm{i}}{2}k_1 & 1 \\ -1 & \mathrm{i}\lambda + \frac{\mathrm{i}}{2}k_1 \end{bmatrix} + \cos(\rho(\lambda)\theta(\lambda; x, t)) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{16}$$

Then we can get the solutions of (1) from $\widehat{q}(x,t) = 2\mathrm{i}\lim_{\lambda\to\infty} \lambda\widehat{\mathbf{M}}(\lambda; x, t)_{12}$, that is

$$\widehat{q}(x,t) = 1 + \frac{(\lambda_1 - \lambda_1^*)\left(2(\lambda_1 - \lambda_1^*) s_1(x,t)^2 (1 - w^*(x,t)) - 2(\lambda_1 - \lambda_1^*) s_2^*(x,t)^2 (1 - w(x,t)) + 4\mathrm{i}s_1(x,t)s_2^*(x,t)N(x,t)^2\right)}{\left(\lambda_1 - \lambda_1^*\right)^2 |1 - w(x,t)|^2 - N(x,t)^2} \tag{17}$$

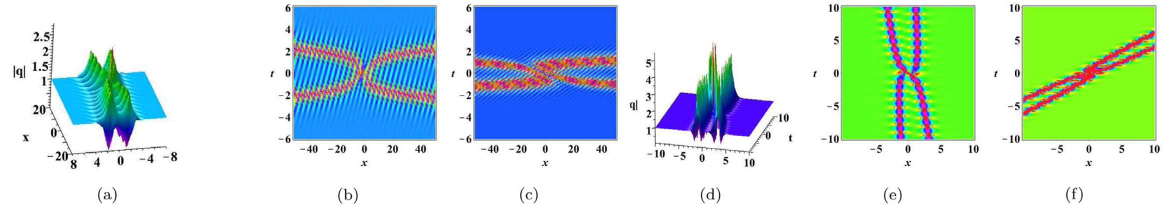


Fig. 2. (a)–(c) is under the choice of $\delta = \frac{1}{2}$, the parameter γ_1 on (a) and (b) is 0, (c) is $\gamma_1 = 1$. (d)–(f) is under the choice of $\delta = \frac{3}{2}$, in (d) and (e), $\gamma_1 = -\frac{1}{19}$, (f) is $\gamma_1 = 0$.

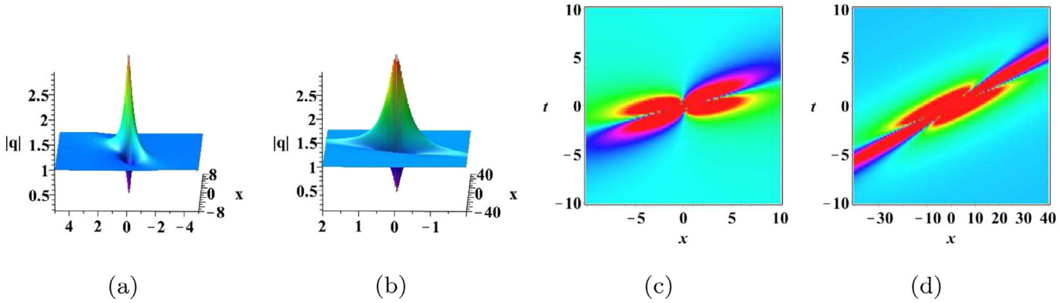


Fig. 3. The first-order rogue wave of Eq. (1). The parameter γ_1 on (a)–(b) is $\gamma_1 = 0$ and $\gamma_1 = \frac{1}{4}$. (c) and (d) is the corresponding density figures. $\gamma_1 = \frac{1}{4}$ increase the compression effects on the t direction but stretch in the x direction than $\gamma_1 = 0$.

Based on the theorem spectral analysis, when choosing different λ_1, the corresponding solution properties are very different. We list kinds of solutions under the choice of $k_1 = 1$:

Case I: When $\lambda_1 = -\frac{1}{2} + \delta\mathrm{i}(|\delta| < 1)$, it is the Akhmediev breather(AB). In this case we give two kinds of AB with different γ_1, which is shown in Fig. 2(a)–(c).

Case II: When $\lambda_1 = -\frac{1}{2} + \delta\mathrm{i}(|\delta| > 1)$, it is the Kuznetsov–Ma breather(KMB). Similarly, we also give two kinds of KMB by choosing two different parameters γ_1, which is shown in Fig. 2(d)–(f).

It is seen that, when $\delta = \frac{1}{2}$, $\gamma_1 = 1$ has more compression effects in t direction than $\gamma_1 = 0$. But it is quite different when $\delta = \frac{3}{2}$, γ_1 plays an important effect to the shape of breather which reverses the periodicity on x-direction to the t-direction.

Case III: When $\lambda_1 = -\frac{1}{2} \pm \mathrm{i}$, it is the rogue wave. Next we focus on the study of high-order rogue waves. When $\lambda_1 = -\frac{1}{2} + \mathrm{i}$, we have

$$\mathbf{s}(x,t) = \begin{bmatrix} (2\mathrm{i}t - 2t - 20\gamma_1 t + x)(c_1 + c_2) + c_1 \\ (-2\mathrm{i}t + 2t + 20\gamma_1 t - x)(c_1 + c_2) + c_2 \end{bmatrix} \tag{18}$$

Obviously, when $c_2 = -c_1$, the coefficient of the highest power for parameter x and t vanish, which is also the same with $\mathbf{\Phi}'(\lambda_1; x,t), N(x,t)$ and $w(x,t)$. So the rogue wave will take on a different behavior, which is shown in Fig. 3. Additionally, the initial solution $\mathbf{\Phi}(\lambda; x,t)$ is normalized at $(x,t) = (0,0)$, we want the peak of the rogue wave should also be placed at $(x,t) = (0,0)$. During the calculation, we find that if the peak of the rogue wave is $(x,t) = (0,0)$, c_1 should go to ∞. we can set $\mathbf{c} = \varepsilon^{-1}\mathbf{c}_\infty$, and take the limit $\varepsilon \to 0$. Then two kinds of high-order rogue waves can be given according to $c_{\infty,1} + c_{\infty,2} = 0$ or $c_{\infty,1} - c_{\infty,2} = 0$. For simplicity, set $\mathscr{G}_o(\lambda; x,t)$ denote the n-fold Gauge transformation with the data $c_{\infty,1} + c_{\infty,2} = 0$, and $\mathscr{G}_n(\lambda; x,t)$ denote $c_{\infty,1} - c_{\infty,2} = 0$. Based on the definition of (14), the n-fold $\mathbf{\Phi}^{[n]}_{\mathrm{o/n}}(\lambda; x,t)$ is obtained as

$$\mathbf{\Phi}^{[n]}_{\mathrm{o/n}}(\lambda; x,t) = \begin{cases} \mathbf{G}^{[n-1]}_{\mathrm{o/n}}(\lambda; x,t) \cdots \mathbf{G}^{[0]}_{\mathrm{o/n}}(\lambda; x,t)\mathbf{\Phi}_{\mathrm{bg}}(\lambda; x,t), & \lambda \in D_+ \cup D_- \\ \mathbf{G}^{[n-1]}_{\mathrm{o/n}}(\lambda; x,t) \cdots \mathbf{G}^{[0]}_{\mathrm{o/n}}(\lambda; x,t)\mathbf{\Phi}_{\mathrm{bg}}(\lambda; x,t)\mathbf{G}^{[0]}_{\mathrm{o/n}} & \\ \quad \times (\lambda; 0,0)^{-1} \cdots \mathbf{G}^{[n]}_{\mathrm{o/n}}(\lambda; 0,0)^{-1}, & \lambda \in D_0 \end{cases} \tag{19}$$

and $\mathbf{M}^{[n]}_{\mathrm{o/n}}(\lambda; x,t) = \mathbf{\Phi}^{[n]}_{\mathrm{o/n}}(\lambda; x,t)\mathrm{e}^{\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}$. Then the Riemann–Hilbert problem is

Riemann–Hilbert Problem 2. *The matrix function* $\mathbf{M}^{[n]}_{\mathrm{o/n}}(\lambda;x,t)$ *has the following properties:*

- **Analyticity** : $\mathbf{M}(\lambda;x,t)$ *is analytic function in* $\lambda\in\mathbb{C}\setminus\{\Sigma\cup\Sigma_c\}$*;*
- **Jump Condition** : $\mathbf{M}_+(\lambda;x,t)=\mathbf{M}_-(\lambda;x,t)\mathbf{V}^{[n]}_{\mathrm{o/n}}(\lambda;x,t)$ *on* $\lambda\in\Sigma\cup\Sigma_c$*, where*

$$\begin{aligned}
&\mathbf{V}^{[n]}_{\mathrm{o/n}}(\lambda;x,t)=\mathrm{e}^{2\mathrm{i}\rho_+(\lambda)\theta(\lambda;x,t)\sigma_3}, \quad \lambda\in\Sigma_c\\
&\mathbf{V}^{[n]}_{\mathrm{o/n}}(\lambda;x,t)=\mathrm{e}^{-\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}\mathbf{G}^{[n-1]}_{\mathrm{o/n}}(\lambda;0,0)\cdots\mathbf{G}^{[0]}_{\mathrm{o/n}}(\lambda;0,0)\mathbf{E}(\lambda)\mathrm{e}^{\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}, \quad \lambda\in\Sigma_+\\
&\mathbf{V}^{[n]}_{\mathrm{o/n}}(\lambda;x,t)=\mathrm{e}^{-\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}\mathbf{E}(\lambda)^{-1}\mathbf{G}^{[0]}_{\mathrm{o/n}}(\lambda;0,0)^{-1}\cdots\mathbf{G}^{[n-1]}_{\mathrm{o/n}}(\lambda;0,0)^{-1}\mathrm{e}^{\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}, \quad \lambda\in\Sigma_-
\end{aligned} \tag{20}$$

- **Normalization** : $\lim_{\lambda\to\infty}\mathbf{M}(\lambda;x,t)\to\mathbb{I}$

Then the rogue wave can be recovered with the usual limit

$$q_{2n-1}(x,t)=2\mathrm{i}\lim_{\lambda\to\infty}\mathbf{M}^{[n]}_{\mathrm{o},12}(\lambda;x,t),\quad q_{2n}(x,t)=2\mathrm{i}\lim_{\lambda\to\infty}\mathbf{M}^{[n]}_{\mathrm{n},12}(\lambda;x,t), \tag{21}$$

Next, we want to rewrite the high-order rogue wave to an algebraic representation. Set $\mathbf{D}_{\mathrm{o/n}}(\lambda;x,t)$ to the ordered product of the gauge transformation matrices

$$\mathbf{D}_{\mathrm{o/n}}(\lambda;x,t):=\mathbf{G}^{[n-1]}_{\mathrm{o/n}}(\lambda;x,t)\cdots\mathbf{G}^{[1]}_{\mathrm{o/n}}(\lambda;x,t)\mathbf{G}^{[0]}_{\mathrm{o/n}}(\lambda;x,t). \tag{22}$$

then this matrix can be factorized into the following form

$$\mathbf{D}_{\mathrm{o/n}}(\lambda;x,t)=\mathbb{I}+\sum_{m=1}^{n}\left(\frac{\mathbf{D}^+_{\mathrm{o/n,k}}(x,t)}{\left(\lambda-(-\frac{1}{2}+\mathrm{i})\right)^m}+\frac{\mathbf{D}^-_{\mathrm{o/n,k}}(x,t)}{\left(\lambda-(-\frac{1}{2}-\mathrm{i})\right)^m}\right) \tag{23}$$

With the Riemann–Hilbert Problem 2, we have

$$\mathbf{D}_{\mathrm{o/n}}(\lambda;x,t)\mathbf{E}(\lambda)\mathrm{e}^{-\mathrm{i}\rho(\lambda)\theta(\lambda;x,t)\sigma_3}\mathbf{E}(\lambda)^{-1}\mathbf{G}^{[0]}_{\mathrm{o/n}}(\lambda;0,0)^{-1}\cdots\mathbf{G}^{[n-1]}_{\mathrm{o/n}}(\lambda;0,0)^{-1}=\mathbf{\Phi}^{[n],\mathrm{in}}_{\mathrm{o/n}}(\lambda;x,t) \tag{24}$$

Obviously, $\mathbf{\Phi}^{\mathrm{in}}_{\mathrm{bg}}(\lambda;x,t)$ have a Taylor expansion at $\lambda=-\frac{1}{2}+\mathrm{i}$ and $\lambda=-\frac{1}{2}-\mathrm{i}$,

$$\mathbf{\Phi}^{\mathrm{in}}_{\mathrm{bg}}(\lambda;x,t)=\sum_{j=0}^{\infty}\mathbf{T}^{\pm}_j(x,t)\left(\lambda-\left(-\frac{1}{2}\pm\mathrm{i}\right)\right)^j \tag{25}$$

For simplicity, defining some vectors as

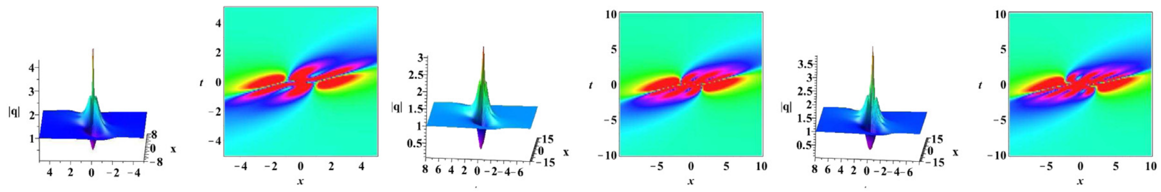
$$\mathbf{w}^+_{\mathrm{o/n},k}(x,t)=\mathbf{T}^+_k(x,t)\mathbf{c}_{\infty,\mathrm{o/n}},\quad \mathbf{w}^-_{\mathrm{o/n},k}(x,t)=\mathbf{T}^+_k(x,t)\mathbf{c}_{\infty,\mathrm{n/e}}. \tag{26}$$

According to the analyticity of Eq. (23) at $\lambda=-\frac{1}{2}\pm\mathrm{i}$ of the left-hand side and the right-hand side, we can get the unknown coefficients $\mathbf{D}^+_{\mathrm{o/n,k}},\mathbf{D}^-_{\mathrm{o/n,k}}$, then the high-order rogue waves can be obtained as

$$\begin{aligned}
q_{2n-1}(x,t)&=1+2\mathrm{i}\frac{\det(\mathscr{D}_{\mathrm{o},4n-1}(x,t))+\det(\mathscr{D}_{\mathrm{o},4n}(x,t))}{\det(\mathscr{D}_{\mathrm{o}}(x,t))},\\
q_{2n}(x,t)&=1+2\mathrm{i}\frac{\det(\mathscr{D}_{\mathrm{n},4n-1}(x,t))+\det(\mathscr{D}_{\mathrm{n},4n}(x,t))}{\det(\mathscr{D}_{\mathrm{n}}(x,t))}
\end{aligned} \tag{27}$$

where $\mathscr{D}_{\mathrm{o/n}}(x,t)$ is a coefficient matrix that consists of $2n, n\times n$ blocks:

$$\mathscr{D}_{\mathrm{o/n}}:=
\begin{bmatrix}
\mathbf{P}^{[1]}_0 & \mathbf{P}^{[2]}_0 & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0}\\
\mathbf{P}^{[1]}_1 & \mathbf{P}^{[2]}_1 & \mathbf{P}^{[1]}_0 & \mathbf{P}^{[2]}_0 & \mathbf{0} & \cdots & \cdots & \mathbf{0}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\mathbf{P}^{[1]}_{n-1} & \mathbf{P}^{[2]}_{n-1} & \mathbf{P}^{[1]}_{n-2} & \mathbf{P}^{[2]}_{n-2} & \cdots & \cdots & \mathbf{P}^{[1]}_0 & \mathbf{P}^{[2]}_0\\
\mathbf{P}^{[1]}_n+\mathbf{Q}^{[1]}_{0,n} & \mathbf{P}^{[2]}_n+\mathbf{Q}^{[2]}_{0,n} & \mathbf{P}^{[1]}_{n-1}+\mathbf{Q}^{[1]}_{0,n-1} & \mathbf{P}^{[2]}_{n-1}+\mathbf{Q}^{[2]}_{0,n-1} & \cdots & \cdots & \mathbf{P}^{[1]}_1+\mathbf{Q}^{[1]}_{0,1} & \mathbf{P}^{[2]}_1+\mathbf{Q}^{[2]}_{0,1}\\
\mathbf{P}^{[1]}_{n+1}+\mathbf{Q}^{[1]}_{1,n} & \mathbf{P}^{[2]}_{n+1}+\mathbf{Q}^{[2]}_{1,n} & \mathbf{P}^{[1]}_{n}+\mathbf{Q}^{[1]}_{1,n-1} & \mathbf{P}^{[2]}_{n}+\mathbf{Q}^{[2]}_{1,n-1} & \cdots & \cdots & \mathbf{P}^{[1]}_2+\mathbf{Q}^{[1]}_{1,1} & \mathbf{P}^{[2]}_2+\mathbf{Q}^{[2]}_{1,1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\mathbf{P}^{[1]}_{2n-1}+\mathbf{Q}^{[1]}_{n-1,n} & \mathbf{P}^{[2]}_{2n-1}+\mathbf{Q}^{[2]}_{n-1,n} & \mathbf{P}^{[1]}_{2n-2}+\mathbf{Q}^{[1]}_{n-1,n-1} & \mathbf{P}^{[2]}_{2n-2}+\mathbf{Q}^{[2]}_{n-1,n-1} & \cdots & \cdots & \mathbf{P}^{[1]}_n+\mathbf{Q}^{[1]}_{n-1,1} & \mathbf{P}^{[2]}_n+\mathbf{Q}^{[2]}_{n-1,1}
\end{bmatrix} \tag{28}$$



(a) (b) (c) (d) (e) (f)

Fig. 4. (a)–(b) is the second order rogue wave, (c)–(d) is the three-order and (e)–(f) is the four-order.

where

$$\mathbf{P}_k^{[j]} = \begin{bmatrix} \left(\mathbf{w}^{+}_{\mathrm{n/o},k}(x,t)\right)_j & 0 \\ 0 & \left(\mathbf{w}^{-}_{\mathrm{n/o},k}(x,t)\right)_j \end{bmatrix},$$

$$\mathbf{Q}_{m,k}^{[j]} = \begin{bmatrix} 0 & \sum_{l=0}^{m} \gamma_{lk} \left(\mathbf{w}^{+}_{\mathrm{n/o},m-l}(x,t)\right)_j \\ \sum_{l=0}^{m} (-1)^{l+m} \gamma_{lk} \left(\mathbf{w}^{-}_{\mathrm{n/o},m-l}(x,t)\right)_j & 0 \end{bmatrix}, \; j = 1,2$$

with $\gamma_{lm} = \frac{(-1)^k}{(2\mathrm{i})^{m+k}} \begin{pmatrix} m+k-1 \\ k \end{pmatrix}$, $\mathscr{D}_{\mathrm{o/n},k}(x,t)$ stands for the matrix $\mathscr{D}_{\mathrm{o/n}}$ with the kth column changed into a vector of $\left(0,\ldots,0,-\left(\mathbf{w}^{+}_{\mathrm{n/o},0}(x,t)\right)_1, -\left(\mathbf{w}^{-}_{\mathrm{n/o},0}(x,t)\right)_1,\ldots,-\left(\mathbf{w}^{+}_{\mathrm{n/o},n-1}(x,t)\right)_1, -\left(\mathbf{w}^{-}_{\mathrm{n/o},n-1}(x,t)\right)_1\right)$.

When $n = 2$, $\gamma_1 = 0$. Eq. (27) is

$$q = 1 + \frac{A}{64x^6 - 768tx^5 + \left(4608t^2+48\right)x^4 - \left(16384t^3+384t\right)x^3 + \left(36864t^4+108\right)x^2 - \left(49152t^5-3072t^3+432t\right)x + 32768t^6+3072t^4+2016t^2+9} \tag{29}$$

where

$$\begin{aligned} A = &-\left(768\mathrm{i}t - 192\right)x^4 + \left(6144\mathrm{i}t^2 + 1536t\right)x^3 - \left(24576\mathrm{i}t^3 + 1152\mathrm{i}t - 9216t^2 - 288\right)x^2 \\ &+ \left(49152\mathrm{i}t^4 - 4608\mathrm{i}t^2 + 24576t^3 + 1152t\right)x \\ &- 49152\mathrm{i}t^5 + 3072\mathrm{i}t^3 - 36864t^4 + 720\mathrm{i}t - 4608t^2 + 36 \end{aligned}$$

Then we give the second-order, third-order and fourth-order rogue wave in Fig. 4 by choosing $\gamma_1 = 0$.

3. Conclusion

In conclusion, we get the high-order rogue waves to the generalized Schrödinger equation with the robust inverse scattering transformation, and analyze effects of the parameter γ_1 appeared in Eq. (1). It can change the rogue wave compression form and affect the shape of the breather. Furthermore, we will discuss the behavior of infinity-order rogue waves to this equation with the similar method for the Schrödinger equation [12].

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