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ORIGINAL PAPER

# General multi-soliton and higher-order soliton solutions for a novel nonlocal Lakshmanan–Porsezian–Daniel equation

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**Abstract** The inverse scattering transformation for a novel nonlocal Lakshmanan–Porsezian–Daniel (LPD) equation with rapidly decaying initial data is studied in the framework of Riemann–Hilbert problem. Firstly, a novel integrable nonlocal LPD equation corresponding to a $3 \times 3$ Lax pair is proposed. Secondly, the inverse scattering process with a novel left-right $3 \times 3$ matrix Riemann–Hilbert(RH) problem is constructed. The analytical properties and symmetry relations for the Jost functions and scattering data are considerably different from the local ones. Due to the special symmetry properties for the nonlocal LPD equation, the zeros of the RHP problem are purely imaginary or occur in pairs. With different types and configuration of zeros, the soliton formula is provided and the rich dynamical behaviors for the three kinds of multi-solitons for the novel nonlocal LPD equation are demonstrated. Third, by a technique of adding perturbed parameters and limiting process, the formula of higher-order solitons for the nonlocal LPD equation is exhibited. Lastly, the plots of diverse higher-order solitons and various solutions corresponding to different combinations of the following zeros: purely imaginary higher-order zeros, purely imaginary simple zeros, pairs of non-purely imaginary simple zeros and pairs of non-purely imaginary higher-order zeros are displayed.

**Keywords** Nonlocal LPD equation · Riemann–Hilbert problem · Multi-solitons · Higher-order soliton

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## 1 Introduction

The nonlinear Schrödinger (NLS) equation is a key physical model in optical and many other fields which describes the wave propagation in Kerr media. Then the higher-order integrable NLS hierarchy has been studied [1–3]. As a higher-order extension of the scalar NLS equation, the nonlinear Lakshmanan–Porsezian–Daniel (LPD) equation with the fourth nonlinear and dispersion terms was introduced [4,5] to describe the effect of discreteness of the lattice on the classical continuum limit of the isotropic Heisenberg ferromagnetic spin chain. Due to the important application in nonlinear optics, abundant optical solitons [6–11] and some new exact solutions [12] for LPD equation were explored. Besides, the rich dynamic behaviors of solitons, rogue waves and the interactional solutions for a multi-component LPD equation were studied by Darboux transformation [13–16] and inverse scattering method [17].

Then, nonlocal integrable NLS equation was introduced by Ablowitz et al. [18–20]. Due to the new spatial and temporal symmetry properties, some new physical effects and applications were found and particular attentions were given to the nonlocal integrable equations [21–39], which were mathematically distinct from the classical local ones. Some integrable nonlocal LPD equations corresponding to a $2 \times 2$ Lax pair have been proposed and studied [40–42].

Recently, a series of novel nonlocal NLS equations with clear physical motivations were proposed from the nonlocal reduction in Manakov system [43]. Different from the nonlocal NLS equation proposed in [18], the nonlocal nonlinearity-induced potential of the novel nonlocal NLS equation [43] is real and symmetric in $x$. In addition, they discovered that the structures of the Riemann–Hilbert problem (RHP) and the form of general soliton solutions were more complicated, which led to some interesting dynamical properties of soliton solutions.

Inspired by the idea of Yang [43], the following novel physically important nonlocal LPD equation is proposed:

$$\begin{aligned} \mathrm{i}q_t &= \alpha\left[q_{xx} + 2qV_1\right] + \beta\Big[q_{4x} + 4(q_{xx}V_1) \\ &\quad +2(qV_2)_x + 6q_xV_3 + 6qV_1^2\Big]. \end{aligned} \tag{1}$$

where $q$ is the potential function of $(x, t)$, $\alpha$, $\beta$ are arbitrary real numbers, the superscript $*$ represents complex conjugation. The nonlinearity-induced potentials $V_k(x, t)(k = 1, 2, 3)$ with forms

$$\begin{aligned} &V_1(x,t) = |q(x,t)|^2 + |q(x,-t)|^2, \\ &V_2(x,t) = q_x(x,-t)q^*(x,-t) + q_x^*(x,t)q(x,t), \\ &V_3(x,t) = q_x^*(x,-t)q(x,-t) + q_x(x,t)q^*(x,t). \end{aligned}$$

So $V_k$ are metrical in $t$ and $V_k(x,t) = V_k^*(x,-t)$ $(k = 1, \ldots, 3)$. The nonlocal Eq. (1) is invariant under the nonlocal reverse-time transformation $t \to -t$ and the conjugate transformation.

The nonlocal LPD Eq. (1) is deduced by implementing the following nonlocal reverse-time constraint

$$r(x,t) = q^*(x,-t), \tag{2}$$

on the following coupled LPD Eq. (3)

$$\begin{cases} \mathrm{i}q_t = \alpha\left[q_{xx} + 2q|\mathbf{q}|^2\right] + \beta[q_{4x} + 2[q(qq_x^* + rr_x^*)]_x \\ \qquad +4q_{xx}|\mathbf{q}|^2 + 6q_x(q^*q_x + r^*r_x) + 6q|\mathbf{q}|^4], \\ \mathrm{i}r_t = \alpha\left[r_{xx} + 2r|\mathbf{q}|^2\right] + \beta[r_{4x} + 2[r(qq_x^* + rr_x^*)]_x \\ \qquad +4r_{xx}|\mathbf{q}|^2 + 6r_x(q^*q_x + r^*r_x) + 6r|\mathbf{q}|^4], \end{cases} \tag{3}$$

where $q$, $r$ are complex function of $(x, t)$, $|\mathbf{q}|^2 = |q|^2 + |r|^2$.

The nonlocal LPD Eq. (1) corresponding to the following $3 \times 3$ Lax pair:

$$\Phi_x = U\Phi = (\mathrm{i}\lambda\Lambda + Q)\Phi, \tag{4a}$$

$$\Phi_t = V\Phi = \left(2\mathrm{i}\lambda^2(\alpha - 4\beta\lambda^2)\Lambda + V_0\right)\Phi. \tag{4b}$$

where $\Phi$ is a three-dimensional column vector with the elements are complex functions of $(x, t; \lambda)$ and $\lambda$ is the spectral parameter. $\Lambda = \mathbf{diag}(1, 1, -1)$ and

$$Q = \begin{bmatrix} 0 & 0 & q(x,t) \\ 0 & 0 & q^*(x,-t) \\ -q^*(x,t) & -q(-x,t) & 0 \end{bmatrix},$$

$$\begin{aligned} V_0 = \alpha\Big[&\mathrm{i}\Lambda(Q^2 + Q_x) - 2\lambda Q\Big] - \beta\Big[8Q\lambda^3 \\ &-4\mathrm{i}\left(\Lambda Q_x - Q^2\Lambda\right)\lambda^2 - 2 \\ &\times\left(Q_{xx} - 2Q^3 + [Q, Q_x]\right)\lambda \\ &+\mathrm{i}{Q_x}^2\Lambda + \mathrm{i}\Lambda Q_{3x} - \mathrm{i}[Q, Q_{xx}] \\ &-3\mathrm{i}\left(Q^2Q_x + Q_xQ^2 - Q^4\right)\Big]. \end{aligned}$$

The Lax pair is a $3 \times 3$ matrix system with two linear matrix differential equations 4a-4b. The nonlocal LPD Eq. (1) can be obtained by the compatible condition $U_t - V_x + [U, V] = 0$. This proves the integrability of the nonlocal LPD Eq. (1). When $\alpha = 1$, $\beta = 0$, Eq. (1) reduced to the nonlocal NLS equation proposed in [43]. The scattering process of the nonlocal LPD Eq. (1) will be presented with the following vanishing initial condition:

$$q(x,0) = q_0(x) \in \mathbb{S}(\mathbb{R}), \tag{5}$$

where $\mathbf{S}(\mathbb{R})$ is the Schwartz space.

The inverse scattering transformation (IST) of Manakov system with initial value condition (5) had been established in [43,44], where the RH problem corresponding to the spectral problem (4a) and the $N$-soliton formula were constructed. However, as for the time evolution part and the symmetry relations, which are quite different from the nonlocal LPD Eq. (1). The constraints of the scattering data and eigenvectors were much more complex to derive, which will lead to richer configurations of zeros and dynamical behavior of soliton solutions. Furthermore, this will lead to much richer distributions of the higher-order zeros and the structures of higher-order solitons. We will first study the formula of soliton solution under different distribution of zeros by inverse scattering process via constructing an RHP. Then we will try to get the formula of the solution corresponding to the higher-order zeros. In addition, the rich structures and characteristics of the solution under varied higher-order zeros will be considered.

The organization is as follows. Section 2 provides the inverse scattering process via an RH problem, the time evolution and the recovery of the potential function. Then the multi-soliton solution formula is obtained when the RHP is reflectless and the zeros are all simple. In Section 3, the dynamics for the general soliton solution with simple zeros are investigated in three cases: (1) $n_1 = 0, n_2 \neq 0$; (2) $n_1 \neq 0, n_2 = 0$ and (3) $n_1 \neq 0, n_2 \neq 0$. Section 4 concludes the higher-order soliton formula by a limit process and exhibits kinds of higher-order soliton solutions corresponding to different configurations of higher-order zeros and simple zeros. The last section is the conclusion.

## 2 Inverse scattering process and multi-soliton solutions

In this section, we will consider the scattering problems of the nonlocal Eq. (1) with initial condition (5). By the initial value condition (5), $q(x,0) \to 0, x \to \pm\infty, \quad q^*(x,0) \to 0, x \to \pm\infty$. So when $x \to \pm\infty$, the potential matrix $Q \to 0$, the solution of Eqs. (4a and 4b) is $\Phi_{bg} = \mathrm{e}^{\mathrm{i}\theta(\lambda)\Lambda}, \quad \theta(\lambda) = \lambda x + 2\lambda^2 \left(\alpha - 4\beta\lambda^2\right) t$. Define the solution of Eqs. (4a and 4b) with the form $\Phi = J\Phi_{bg}$ with $J$ is a complex function of $(x, t; \lambda)$. So $J \to \mathbb{I}, \ x \to \pm\infty$, where $\mathbb{I}$ is the third-order identity matrix and $J$ satisfies

$$J_x = \mathrm{i}\lambda\,[\Lambda, J] + QJ, \tag{6a}$$

$$J_t = 2\mathrm{i}\lambda^2(\alpha - 4\beta\lambda^2)\,[\Lambda, J] + V_0 J. \tag{6b}$$

where $[\Lambda, J] = \Lambda J - J\Lambda$ is a commutator. The adjoint Lax pair can be obtained as

$$K_x = \mathrm{i}\lambda\,[\Lambda, K] - KQ, \tag{7a}$$

$$K_t = 2\mathrm{i}\lambda^2(\alpha - 4\beta\lambda^2)\,[\Lambda, K] - KV_0. \tag{7b}$$

### 2.1 The Jost solution and the construction of the RHP

To implement the scattering process of potential function at the fixed point $t = 0$, that is, the potential function $q$ and the other related functions will be treated as functions only related to the variable $x$. Then, we will construct the corresponding scattering problem and calculate the initial scattering data at $t = 0$. To avoid confusion, it is no longer explicitly stated in this subsection. After the initial scattering data at the fixed point $t = 0$ is obtained, the scattering data at arbitrary time will be obtained through time inverse, and then, the solution $q(x, t)$ at arbitrary time will be reconstructed by using the inverse scattering process.

Recall $\Phi$ satisfies the following scattering problem (4b)

$$\Phi_x(x, 0; \lambda) = \mathrm{i}\lambda\Lambda\Phi(x, 0; \lambda) + Q(x; 0)\Phi(x, 0; \lambda), \tag{8}$$

and the corresponding adjoint scattering problem

$$K_x(x, 0; \lambda) = \mathrm{i}\lambda\Lambda K(x, 0; \lambda) - K(x, 0; \lambda)Q(x, 0). \tag{9}$$

For simplicity, $(x, 0; \lambda)$ and $(x, 0)$ will be omitted in this section.

Introduce the Jost solution $J_{1,2}$ of Eq. (6a) who satisfies the following $x$- asymptotic behavior

$$J_1 \to \mathbb{I}, \ x \to +\infty, \quad J_2 \to \mathbb{I}, \ x \to -\infty,$$

Combined with initial conditions (5), definition $\Phi_{1,2}$ and its scattering Eq. (4a). $J_{1,2}$ satisfies the following Volterra integral equation

$$\begin{aligned} J_1(x,\lambda) &= \mathbb{I} - \int_x^{+\infty} \mathrm{e}^{i\lambda\Lambda(x-y)} Q(y) \\ &\quad J_1(y,\lambda)\mathrm{e}^{-i\lambda\Lambda(x-y)}dy, \\ J_2(x,\lambda) &= \mathbb{I} + \int_{-\infty}^{x} \mathrm{e}^{i\lambda\Lambda(x-y)} Q(y) \\ &\quad J_2(y,\lambda)\mathrm{e}^{-i\lambda\Lambda(x-y)}dy. \end{aligned} \tag{10}$$

By the integral Eq. (10), to guarantee the integral of the right side coverage. Similar to the method of [44], by the structure of the potential matrix $Q$, the analytical properties of $J_k$ can be obtained.

For $J_{1,2}$ is the solution of Eq. (6a), by the definition of (4a), so there exists solution function $\Phi_{1,2}$ of (4a) that $\Phi_1 = J_1 E$, $\Phi_2 = J_2 E$, where $E = e^{\mathrm{i}\lambda x}$ is a diagram matrix. $\Phi_{1,2}$ are two different nonzero solutions of the same linear Eq. (4a), so there exists a matrix $S(\lambda) = [s_{kj}]_{3\times 3}$ who is independent of the variable $x$ and $\Phi_1 S = \Phi_2$, i.e.,

$$J_2 = J_1 e^{-\mathrm{i}\lambda x} S e^{\mathrm{i}\lambda x}. \tag{11}$$

Then by (11), the analytical properties of the elements of $S$ can be obtained. Furthermore, take $S^{-1} = (\hat{s}_{kj})_{3\times 3}$, then recall the analytical properties of matrix $J_{1,2}$, we can obtain the following properties: $s_{11}, s_{21}, s_{12}, s_{22}, \hat{s}_{33}$ are analytical when $\lambda \in \mathbb{C}_-$ and $\hat{s}_{11}, \hat{s}_{21}, \hat{s}_{12}, \hat{s}_{22}, s_{33}$ are analytical in $\lambda \in \mathbb{C}_+$. Sign $J_k = \left[J_k^{[1]}, J_k^{[3]}, J_k^{[3]}\right]$ and $J_k^{-1} = \left[\hat{J}_k^{[1]}, \hat{J}_k^{[3]}, \hat{J}_k^{[3]}\right]^T$, $k = 1, 2$, where $J_k^{[j]}$ is the $j$-th column of matrix $J_k$ and $\hat{J}_k^{[j]}$ is the $j$th row of matrix $J_k^{-1}$, , $j = 1, 2, 3$.

So the analytical properties of $S$ can be obtained. Then, we have the following lemma.

**Lemma 1** *The Jost solutions $J_k$ and the scattering matrix $S$ satisfy the following analytical properties*

- $J_2^{[1]}, J_2^{[2]}, J_1^{[3]}$ *are analytical when* $\lambda \in \mathbb{C}_+$, $J_1^{[1]}, J_1^{[2]}, J_2^{[3]}$ *are analytical when* $\lambda \in \mathbb{C}_-$.
- $\hat{J}_2^{[1]}, \hat{J}_2^{[2]}, \hat{J}_1^{[3]}$ *are analytical when* $\lambda \in \mathbb{C}_+$, $\hat{J}_1^{[1]}, \hat{J}_1^{[2]}, \hat{J}_2^{[3]}$ *are analytical when* $\lambda \in \mathbb{C}_-$.
- $s_{11}, s_{21}, s_{12}, s_{22}, \hat{s}_{33}$ *are analytical when* $\lambda \in \mathbb{C}_+$ *and* $\hat{s}_{11}, \hat{s}_{21}, \hat{s}_{12}, \hat{s}_{22}, s_{33}$ *are analytical in* $\lambda \in \mathbb{C}_-$.

Note that the trace of the matrix $U$ is i$\lambda$. It can be obtained by Abel's identity that $\det(\Phi(x;\lambda)) = e^{\mathrm{i}\lambda x}$. Then, combine the definition of $J_\pm$, and the definition of $S$, we have $\det(J_{1,2}(x,\lambda)) = \det(S(\lambda)) = 1$.

To construct a well-defined RHP, define the modified Jost solutions $P_+$ and $P_-$ with the following forms

$$P_+ = \left[J_1^{[1]}, J_1^{[2]}, J_2^{[3]}\right] = J_1 H_1 + J_2 H_2,$$

$$P_- = \left[\hat{J}_1^{[1]}, \hat{J}_1^{[2]}, \hat{J}_2^{[3]}\right] = H_1 J_1^{-1} + H_2 J_2^{-1}, \tag{12}$$

where $H_1 = \mathrm{diag}(1,1,0)$, $H_2 = \mathrm{diag}(0,0,1)$. It can be calculated that $\det(P_+) = s_{33}$, $\det(P_-) = \hat{s}_{33}$. So $P_+$ and $P_-$ are analytical in $\lambda \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_-$, respectively. By the definition of $P_\pm$ and $J_{1,2}$, it is obvious that $P_+$ is a solution of (6a), and $P_-$ is the solution of (7a). Recall the Volterra integral Eq. (10), the following $\lambda$ asymptotic property can be obtained.

$$P_\pm(x,\lambda) \to \mathbb{I}_3,\ \lambda \in \mathbb{C}_\pm \to \infty.$$

In summary, the modified matrix $P$ satisfies the following matrix RH decomposition problem.

**Riemann-Hilbert Problem 1** *The sectionally meromorphic matrix $P$ satisfies the following conditions:*

- *Analytical properties: $P_+$ is analytic in $\lambda \in \mathbb{C}_+$ and $P_-$ is analytic in $\lambda \in \mathbb{C}_-$.*
- *The canonical normalization conditions $P_\pm(x,\zeta) \to \mathbb{I}_3$, $\lambda \in \mathbb{C}_\pm \to \infty$.*
- *The jump condition*

$$\begin{aligned} &P_-(\lambda)P_+(\lambda) = G(\lambda) = E \begin{pmatrix} 1 & 0 & s_{13} \\ 0 & 1 & s_{23} \\ \hat{s}_{31} & \hat{s}_{32} & 1 \end{pmatrix} E^{-1}, \\ &E = \exp(\mathrm{i}\lambda x),\ \lambda \in \mathbb{R}. \end{aligned} \tag{13}$$

This is a matrix equation of a sectionally analytical matrix $P$ which is constructed from the initial potential at $t = 0$. So there should be some relationships between the solution of RHP and the potential function. To establish the relation between the potential function and the solution of the RHP. The relation of the potential matrix and the solution of the RHP is necessary. Thus, if the RHP 1 is solved, we can get the corresponding potential function $q$.

Expanding the function $P_+$ at $\lambda \to \infty$ as $P_+(\lambda) = \mathbf{I}_3 + \lambda^{-1} P_+^{[1]}(x) + \mathscr{O}\left(\lambda^{-2}\right)$, and inserting the expansion of $P_+$ into (6a). Then comparing the same order terms of $\lambda$,

the potential matrix $Q$ can be given by

$$Q = -i\left[\Lambda, P_+^{[1]}\right] = i\left[\Lambda, P_-^{[1]}\right] \tag{14}$$

and the coefficient of $\lambda^{-1}$: $P^{[1]}_{+,x} = \mathrm{i}[\lambda, P^{[2]}_{+}] + QP^{[1]}_{+}$, $P^{[1]}_{-,x} = \mathrm{i}[\lambda, P^{[2]}_{-}] - P^{[1]}_{-}Q$. Then, together with the equations (14), we can conclude that the asymptotic expansion of $P_+$ at $\lambda = \infty$

$$P^{+}(x) = \begin{bmatrix} 1 + \frac{1}{2\mathrm{i}\lambda}\int_x^{+\infty}|q(y)|^2dy & \frac{1}{2\mathrm{i}\lambda}\int_x^{+\infty}q(y)^2dy & \frac{1}{2\mathrm{i}\lambda}q(x) \\ \frac{1}{2\mathrm{i}\lambda}\int_x^{+\infty}q^*(y)^2dy & 1 + \frac{1}{2\mathrm{i}\lambda}\int_x^{+\infty}|q(y)|^2dy & \frac{1}{2\mathrm{i}\lambda}q^*(x) \\ \frac{1}{2\mathrm{i}\lambda}q^*(x) & \frac{1}{2\mathrm{i}\lambda}q(x) & 1 - \frac{1}{\mathrm{i}\lambda}\int_{-\infty}^{x}|q(y)|^2dy \end{bmatrix} + \mathscr{O}(\lambda^{-2}).$$

Similarly, the asymptotic expansion of $P_-$ can be obtained as

$$P^{-}(x) = \begin{bmatrix} 1 + \frac{1}{2\mathrm{i}\lambda}\int_{-\infty}^{x}|q(y)|^2dy & \frac{1}{2\mathrm{i}\lambda}\int_{-\infty}^{x}q(y)^2dy & -\frac{1}{2\mathrm{i}\lambda}q(x) \\ \frac{1}{2\mathrm{i}\lambda}\int_{-\infty}^{x}q^*(y)^2dy & 1 + \frac{1}{2\mathrm{i}\lambda}\int_{-\infty}^{x}|q(y)|^2dy & -\frac{1}{2\mathrm{i}\lambda}q^*(x) \\ -\frac{1}{2\mathrm{i}\lambda}q^*(x) & -\frac{1}{2\mathrm{i}\lambda}q(x) & 1 - \frac{1}{\mathrm{i}\lambda}\int_{x}^{+\infty}|q(y)|^2dy \end{bmatrix} + \mathscr{O}(\lambda^{-2}).$$

Implementing the direct calculation on the equation above, the asymptotic expansion of the determination $\det(P_+)$, i.e., the scattering coefficient $\hat{s}_{33}$ and $s_{33}$ can be obtained

$$s_{33} = 1 + \frac{1}{\mathrm{i}\lambda}\left(\int_x^{+\infty}|q(y)|^2dy - \int_{-\infty}^{x}|q(y)|^2dy\right) + \mathscr{O}(\lambda^{-2}),$$
$$\hat{s}_{33} = 1 - \frac{1}{\mathrm{i}\lambda}\left(\int_x^{+\infty}|q(y)|^2dy - \int_{-\infty}^{x}|q(y)|^2dy\right) + \mathscr{O}(\lambda^{-2}).$$

Besides, by Eq. (14) and the form of matrix $Q$, the potential function $q$ can be recovered by the following formula

$$q = -2\mathrm{i}[P^{[1]}_{+}]_{13}. \tag{15}$$

Imposing the similar process to $P_-$, we can get $q = 2\mathrm{i}[P^{[1]}_{-}]_{13}$. If the RH problem (13) can be solved, then the potential function $q$ can be reconstructed from its solution $P_\pm$. By the definition of $Q$, it can be checked that $Q$ and the related functions satisfy the following two kinds of symmetry relations

**Proposition 1** *The modified Jost function $P_\pm$, the scattering data satisfy the symmetry properties*

– *The self-conjugate symmetry properties*

$$Q(x,t) = -Q^\dagger(x,t),\quad P^\dagger_{+}(\lambda^*) = P_-(\lambda),\quad S^\dagger(\lambda^*) = S^{-1}(\lambda),\quad \hat{s}_{33}(\lambda) = s_{33}(\lambda^*). \tag{16}$$

– *The reverse-time symmetry properties*

$$Q(x,t) = \Sigma Q^*(x,-t)\Sigma,\quad \Sigma(P_+)^*(x,-t;-\lambda^*)\Sigma = P_+(x,t;\lambda),$$

$$\Sigma S^*(-\lambda^*)\Sigma = S(\lambda),\quad s^*_{33}(-\lambda^*) = s_{33}(\lambda). \tag{17}$$

*with* $\Sigma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

*Proof* Begin from the first symmetry relation of $Q$, the anti-Hermitic symmetry (16). Take Hermitian conjugate on both sides of the scattering Eq. (6a), we can get $J^\dagger$ meet with spectral problem (9), and $J^{-1}$ also satisfies Eq. (9). So there is a constant matrix $C_0 J^\dagger(\lambda^*) = J^{-1}\lambda C_0$, make $x \to \pm\infty$, in combination with the asymptotic behavior of $J$ at infinite, then $C_0 = \mathbb{I}$, as a result, $J^\dagger(\lambda^*) = J^{-1}(\lambda)$. Consequently, the symmetry relations of $P^\pm$ and scattering matrix are $P^\dagger_+(\lambda^*) = P_-(\lambda)$, $S^\dagger(\lambda^*) = S^{-1}(\lambda)$. And the symmetry relations of the scattering data $\hat{s}_{33}(\lambda) = s_{33}(\lambda^*)$, so if $s_{33}(\lambda_k) = 0$, then we have $\hat{s}_{33}(\lambda^*_k) = 0$.

Secondly, due to the potential matrix $Q$ also satisfies the symmetry relations $Q(x,t) = -Q^\dagger(x,t)$, take conjugate of the nonlocal LPD Eq. (1). Then, multiply $\Sigma$ from both sides and replace $t$ with $-t$, then we concluded that

$$(\Sigma J^*(x,-t)\Sigma)_x = -\mathrm{i}\lambda^*[\Lambda, \Sigma J^*(x,-t)\Sigma] + Q\Sigma J^*(x,-t)\Sigma,$$

so $\Sigma J^*(x,-t;-\lambda^*)\Sigma$ satisfies the scattering Eq. (6a). Similarly, by combining the asymptotic behavior of function $J$ at infinity, $\Sigma J^*(x,-t;-\lambda^*)\Sigma = J(x,t;\lambda)$. And furthermore, by the definitions of $P^\pm$ and the matrix $S$, the following symmetric properties can be obtained

$$\Sigma(P_+)^*(x,-t;-\lambda^*)\Sigma = P_+(x,t;\lambda),\ \ \Sigma S^*(-\lambda^*)\Sigma = S(\lambda). \quad (18)$$

and $s_{33}^*(-\lambda^*) = s_{33}(\lambda)$. So if $s_{33}(\lambda_k) = 0$, then $s_{33}(-\lambda_k^*) = 0$. □

By the symmetry properties above, suppose $\det(P_+) = \hat{s}_{33}$ possess $N$ simple zeros $\lambda_k \in \mathbb{C}_+$, $k = 0,\ldots,N = n_1+2n_2$.

$$\left\{\lambda_1,\lambda_2,\ldots,\lambda_{n_1+n_2},\lambda_{n_1+n_2+1},\ldots,\lambda_{n_1+2n_2}\right\},$$

Since the scattering matrix $S$ satisfies (16), so $\hat{s}_{33}$ admits $n_1+2n_2$ simple zeros $\bar{\lambda}_k = \lambda_k^* \in \mathbb{C}_-$, $k = 0,\ldots,n_1+2n_2$.

Consequently, we have

$$\begin{aligned}
s_{33} &= \prod_{k=1}^{N_1}(\lambda-\lambda_k)\prod_{j=N_1+1}^{N_1+N_2}(\lambda-\lambda_j)(\lambda+\lambda_j^*) \\
&\quad s_{33}^{[0]}(\lambda),\ \ s_{33}^{[0]}(\lambda)\neq 0,\\
\hat{s}_{33} &= \prod_{k=1}^{N_1}(\lambda-\lambda_k^*)\prod_{j=N_1+1}^{N_1+N_2}(\lambda-\lambda_j^*)(\lambda+\lambda_j) \\
&\quad \hat{s}_{33}^{[0]}(\lambda),\ \ \hat{s}_{33}^{[0]}(\lambda)\neq 0,\\
\hat{s}_{33}(\lambda_k) &= s_{33}(\lambda_k^*) = 0,\ \ \hat{s}_{33}^{'}(\lambda_k)\neq 0, s^{'}{}_{33}(\lambda_k^*)\neq 0,
\end{aligned}$$

where

$$\lambda_k = \begin{cases}\zeta_k,\ k=1,\ldots,n_1,\\ \mu_{k-n_1},\ k=n_1+1,\ldots,n_1+n_2,\\ -\mu_{k-n_1-n_2}^*,\ k=n_1+n_2+1,\ldots,n_1+2n_2,\end{cases} \quad (19)$$

with $\zeta_k \in \mathrm{i}\mathbb{R}_+$, $k = 1,\ldots,n_1$ are purely imaginary spectral parameters and $\arg(\mu_k) \in (0,\frac{\pi}{2})$, $k = 1,\ldots,n_2$ are spectrums in the first quadrant of the complex plane. The distributions of the discrete spectrums are shown in Fig. 1.

Denote the eigenvectors for $P_+$ and $P_-$ corresponding to the eigenvalues corresponding to the eigenvalues $\lambda_k \in C_+$ and $\bar{\lambda}_k \in C_-$ are $v_k$ and $\bar{v}_k$, respectively. Since $P_+$ satisfies the scattering problem (6a) and $P_-$ satisfies the adjoint scattering problem (7a), we have $v_k = e^{\mathrm{i}\lambda_k \Lambda x}\tilde{v}_{k0}$ and $v_k^\dagger = \tilde{v}_{k0}^* e^{-\mathrm{i}\lambda_k \Lambda^* x}$.

Then, we consider the time evolution of the inverse scattering process. To get the time dependence of the related functions, recall the time-dependent part (6b) of the Lax pair for $J$, then the time evolution of the scattering matrix can be gotten as so the corresponding Eq. (11) becomes

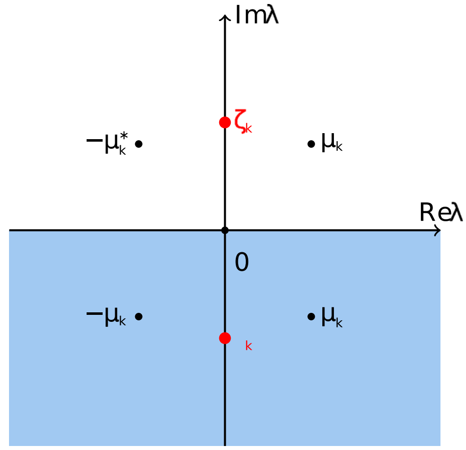



**Fig. 1** (Color online) Distribution of the discrete spectrum for the RHP on complex $\lambda$-plane, Region $D_+ = \{\lambda\in\mathbb{C}|\mathrm{Re}\lambda > 0\}$ (white region), while region $D_- = \{\lambda\in\mathbb{C}|\mathrm{Re}\lambda < 0\}$ (blue region)

$$J_2(x,t;\lambda) = J_1(x,t;\lambda)e^{-\mathrm{i}\lambda x}S(t;\lambda)e^{\mathrm{i}\lambda x}. \quad (20)$$

by Eq. (20), we can get that $J_2(x,t;\lambda)e^{-\mathrm{i}\lambda x} = J_1(x,t;\lambda)S(t;\lambda)$. Since $J_2(x,t;\lambda)$ satisfies the time part of (6b), $J_2(x,t;\lambda)E$ satisfies equation (6b) and does $J_1(x,t;\lambda)e^{-\mathrm{i}\lambda x}S(t;\lambda)$, substitute it into equation (6b), so we have

$$\begin{aligned}
&(J_1(x,t;\lambda)e^{-\mathrm{i}\lambda x}S(t;\lambda))_t = 2\mathrm{i}\lambda^2(\alpha-4\beta\lambda^2)\\
&\quad\left[\Lambda, J_1(x,t;\lambda)e^{-\mathrm{i}\lambda x}S(t;\lambda)\right]\\
&\quad+V_0(x,t;\lambda)J_1(x,t;\lambda)e^{-\mathrm{i}\lambda x}S(t;\lambda). \quad (21)
\end{aligned}$$

Let $x\to\pm\infty$ in the above equation, due to $V_0\to 0$, $J_1(x,t;\lambda)\to\mathbb{I}$, we get

$$S(t;\lambda)_t = 2\mathrm{i}\lambda^2(\alpha-4\beta\lambda^2)\left[\Lambda, S(t;\lambda)\right].$$

Similarly, the time dependence of $S^{-1}(t;\lambda)$ can be gotten

$$S^{-1}(t;\lambda)_t = 2\mathrm{i}\lambda^2(\alpha-4\beta\lambda^2)\left[\Lambda, S^{-1}(t;\lambda)\right].$$

Furthermore, the time evolutions for the scattering coefficients are $\frac{\partial s_{33}}{\partial t} = \frac{\partial \hat{s}_{33}}{\partial t} = 0$, and

$$\begin{aligned}
s_{13}(t;\lambda) &= s_{13}(0;\lambda)\exp(4\mathrm{i}\lambda^2(\alpha-4\beta\lambda^2)t),\\
s_{23}(t;\lambda) &= s_{23}(0;\lambda)\exp(4\mathrm{i}\lambda^2(\alpha-4\beta\lambda^2)t),\\
\hat{s}_{31}(t;\lambda) &= \hat{s}_{31}(0;\lambda)\exp(-4\mathrm{i}\lambda^2(\alpha-4\beta\lambda^2)t),\\
\hat{s}_{32}(t;\lambda) &= \hat{s}_{32}(0;\lambda)\exp(-4\mathrm{i}\lambda^2(\alpha-4\beta\lambda^2)t).
\end{aligned}$$

Consequently, the zeros of the RHP are dependent with $t$. Successively, by the scattering problem (6b), the time evolution of the eigenvectors $v_k$ can be taken as and the corresponding adjoint problem, respectively. Take the derivative with respect to $t$ of both sides of following equation $v_k = \mathrm{e}^{(\mathrm{i}\lambda_k x+2\mathrm{i}\lambda^2(\alpha-4\beta\lambda^2)t)\Lambda} v_{k_0}$. In addition, transform $t \to -t$ and take conjugation on the kernel equation and combine the symmetry relation (17), the corresponding eigenvector for the eigenvalue $\lambda = -\lambda_k^*$ is $\hat{v}_k = \Sigma v_k^*(x,-t;\lambda_k) = \Sigma \mathrm{e}^{(\mathrm{i}\lambda_k^* x+2\mathrm{i}\lambda^{*2}(\alpha+2\beta\lambda^*)t)\Lambda} v_{k_0}^*$. To sum up, for the eigenvalue $\lambda_k \in \mathbb{C}_+$, the corresponding eigenvectors are

$$v_k = \begin{cases} \mathrm{e}^{(\mathrm{i}\zeta_k x+2\mathrm{i}\zeta_k^2(\alpha-4\beta\zeta_k^2)t)\Lambda} v_k^{[0]}, & k=1,\ldots,n_1, \\ \mathrm{e}^{(\mathrm{i}\mu_{k-n_1} x+2\mathrm{i}\mu_{k-n_1}^2(\alpha-4\beta\mu_{k-n_1}^2)t)\Lambda} v_k^{[0]}, & k=n_1+1,\ldots,n_1+n_2, \\ \Sigma \mathrm{e}^{(-\mathrm{i}\mu_{k-n_1}^* x-2\mathrm{i}(\mu_{k-n_1}^*)^2(\alpha-4\beta(\mu_{k-n_1}^*)^2)t)\Lambda} (v_{k-n_1}^{[0]})^*, & k=n_1+n_2+1,\ldots,N, \end{cases} \tag{22}$$

and

$$\bar{v}_k = \begin{cases} (v_k^{[0]})^* \mathrm{e}^{(\mathrm{i}\zeta_k^* x+2\mathrm{i}(\zeta_k^*)^2(\alpha-4\beta(\zeta_k^*)^2)t)\Lambda}, & k=1,\ldots,n_1, \\ (v_k^{[0]})^* \mathrm{e}^{(\mathrm{i}\mu_{k-n_1}^* x+2\mathrm{i}(\mu_{k-n_1}^*)^2(\alpha-4\beta(\mu_{k-n_1}^*)^2)t)\Lambda}, & k=n_1+1,\ldots,n_1+n_2, \\ v_k^{[0]} \mathrm{e}^{(-\mathrm{i}\mu_{k-n_1} x+2\mathrm{i}\mu_{k-n_1}^2(\alpha-4\beta\mu_{k-n_1}^2)t)\Lambda} \Sigma, & k=n_1+n_2+1,\ldots\ldots,N, \end{cases} \tag{23}$$

where

$$v_k^{[0]} = \begin{cases} (a_k^{[1]}, (a_k^{[1]})^*, c_k^{[1]}), & k=1,\ldots,n_1 \\ (a_k^{[2]}, b_k^{[2]}, c_k^{[2]}), & k=n_1+1,\ldots,n_1+n_2 \end{cases}$$

with $c_k^{[1]}$ are real and $a_k^{[1]}, a_k^{[2]}, b_k^{[2]}, c_k^{[2]}$ are arbitrary complex numbers.

To solve the above problems with simple zeros, we need to transform the non-regular problem into the regular RH problem by removing simple zeros, and then, the solution of the regular RH problem can be expressed by the Sokhotski–Plemelj formula. By virtue of the above eigenvectors $v_k$ and $\bar{v}_k$, for $j = 1, \ldots, n_1$, introduce the matrix $\Theta_j$

$$\Theta_j = \mathbb{I} + \frac{\lambda_j^* - \lambda_j}{\lambda - \lambda_j^*} \frac{|v_j>< v_j^\dagger|}{< v_j^\dagger | v_j >},$$

$$\Theta_j^{-1} = \mathbb{I} - \frac{\lambda_j^* - \lambda_j}{\lambda - \lambda_j} \frac{|v_j>< v_j^\dagger|}{< v_j^\dagger | v_j >}.$$

For $k = n_1+1, \ldots, n_2$, define the matrix $\hat{\Theta}_k$

$$\hat{\Theta}_k = \left(\mathbb{I} + \frac{\lambda_k^* - \lambda_k}{\lambda - \lambda_k^*} \frac{|v_k>< v_k^\dagger|}{< v_k^\dagger | v_k >}\right) \left(\mathbb{I} + \frac{\lambda_k^* - \lambda_k}{\lambda + \lambda_k} \frac{|\hat{v}_k>< \hat{v}_k^\dagger|}{< \hat{v}_k^\dagger | \hat{v}_k >}\right),$$

$$\hat{\Theta}_k^{-1} = \left(\mathbb{I} - \frac{\lambda_k^* - \lambda_k}{\lambda - \lambda_k} \frac{|v_k>< v_k^\dagger|}{< v_k^\dagger | v_k >}\right) \left(\mathbb{I} - \frac{\lambda_k^* - \lambda_k}{\lambda + \lambda_k^*} \frac{|\hat{v}_k>< \hat{v}_k^\dagger|}{< \hat{v}_k^\dagger | \hat{v}_k >}\right).$$

By direct calculation, we obtain

$$\det(\Theta_j) = \frac{\lambda - \lambda_j}{\lambda - \lambda_j^*}, \quad \det(\hat{\Theta}_k) = \frac{(\lambda - \lambda_k)(\lambda + \lambda_k^*)}{(\lambda - \lambda_k^*)(\lambda + \lambda_k)}.$$

Then, introduce the following matrix $T$ with

$$T = \Theta_{n_1+2n_2} \ldots \Theta_{n_1+1} \Theta_{n_1} \ldots \Theta_1, \tag{24}$$

and then, we have the following lemma.

**Lemma 2** *The above product expression (24) of matrix $T$ can be written as the following expression with the sum of partial fractions*

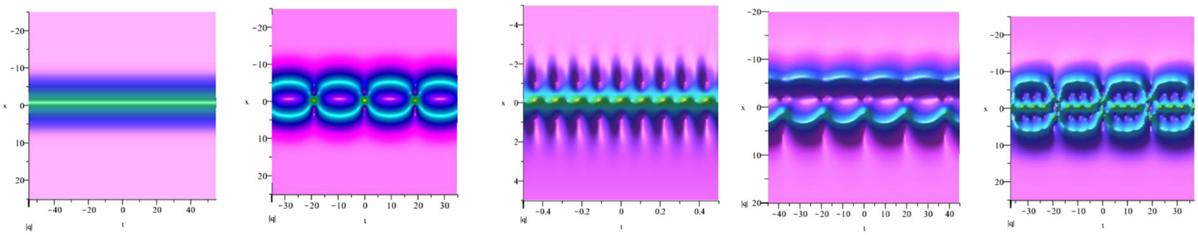
$$T(\lambda) = I + \sum_{j,k=1}^{n_1+2n_2} \frac{v_j \left(M^{-1}\right)_{jk} v_k^\dagger}{\lambda - \lambda_k^*},$$
$$T^{-1}(\lambda) = I - \sum_{j,k=1}^{n_1+2n_2} \frac{v_j \left(M^{-1}\right)_{jk} v_k^\dagger}{\lambda - \lambda_j}, \tag{25}$$

*where $M$ is a matrix of $N \times N$ with the elements*

$$M_{jk} = \frac{v_j^\dagger v_k}{\lambda_j^* - \lambda_k}, \quad 1 \le j, k \le n_1 + 2n_2,$$

*and the determination can be gotten as where*

$$\det(T) = \prod_{k=1}^{n_1} \frac{\lambda - \zeta_k}{\lambda - \zeta_k^*} \prod_{j=1}^{n_2} \frac{(\lambda - \mu_j)(\lambda + \mu_j^*)}{(\lambda - \mu_j^*)(\lambda + \mu_j)}.$$



(a) $n_1=1, n_2=0$ (b) $n_1=2, n_2=0$ (c) $n_1=2, n_2=0$ (d) $n_1=2, n_2=0$ (e) $n_1=3, n_2=0$

**Fig. 2** (Colour online) The 2d-plots of the solution Eq. (27) when $n_2=0$. The parameters are $\alpha=\beta=1$, $a_k^{[1]}=\frac{\mathrm{i}}{2}$, $k=1..4$. The spectral parameters are **a** $\eta_1=\frac{1}{4}$; **b** $\eta_1=\frac{1}{4}$, $\eta_2=\frac{1}{3}$; **c** $\eta_1=\frac{1}{4}$, $\eta_2=\frac{4}{3}$; **d** $\eta_1=\frac{1}{4}$, $\eta_2=\frac{1}{3}$; **e** same with **c** except for $\eta_3=\frac{1}{2}$

The proof of the lemma 2 can refer to the proof of theorem 2.1 in reference [44]. Here, we omit the process of details.

Furthermore, when the system is reflectless, the reflectless potential function $q$ corresponds to the soliton solution. By the solution formula to the nonregular RHP with simple zeros in [44], then the scattering coefficients of the RHP 1 are all equal to zero, i.e., $G=\mathbb{I}$, so $P_+^{-1}(\lambda)=T^{-1}(\lambda)$, $\lambda\in\mathbb{C}_+$. i.e., $P_+(\lambda)=T(\lambda)$, the sum expansion of $T(\lambda)$ have been obtained by (24), so we have

$$P_+^{[1]}=T^{[1]}(\lambda)=\sum_{j,k=1}^{n_1+2n_2} v_j\left(M^{-1}\right)_{jk} v_k^{\dagger}. \tag{26}$$

The following theorem can be obtained.

**Theorem 1** *The N soliton formula for the nonlocal LPD Eq. (1) is*

$$q(x,t)=2\mathrm{i}\frac{\begin{vmatrix} M & \bar{Y}_3^T \\ Y_1 & 0\end{vmatrix}}{|M|}, \tag{27}$$

*where $M$ is a $N\times N$ matrix with $M_{jk}=\frac{v_j^{\dagger}v_k}{\lambda_j^*-\lambda_k}$, $1\le j,k\le n_1+2n_2$, $Y_k$ and $\bar{Y}_k$ are the $k$ row of*

$$\begin{aligned} Y&=[Y_1,Y_2,Y_3]^T=\big[v_1,\dots,v_{n_1},\dots,v_{n_1+n_2},\\ &\qquad \Sigma v_{n_1+1}^*(x,-t),\dots,\Sigma v_{n_1+n_2}^*(x,-t)\big]_{3\times N},\\ \bar{Y}&=\big[\bar{Y}_1,\bar{Y}_2,\bar{Y}_3\big]^T=\big[v_1^*,\dots,v_{n_1}^*,\dots,v_{n_1+n_2}^*,\\ &\qquad \Sigma v_{n_1+1}(x,-t),\dots,\Sigma v_{n_1+n_2}(x,-t)\big]_{3\times N},\end{aligned}$$

*with $v_k$ are shown in Eq. (22).*

To illustrate the formula (27), the dynamical behaviors for the general soliton solutions will be studied in next section.

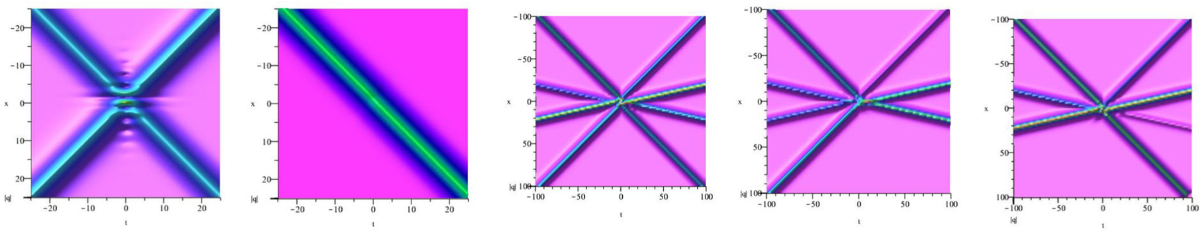
## 3 Dynamics for the general soliton solutions

For the arbitrary vectors $v_k^{[0]}$, take the parameters $c_k^{[1]}=c_k^{[2]}=1$. Then, we will study the solution with different parameter values in three cases: case 1: $n_1=0, n_2\neq 0$; case 2: $n_1\neq 0, n_2=0$ and case 3: $n_1\neq 0, n_2\neq 0$.

**Case 1** When $n_1\neq 0, n_2=0$. In this case, the solutions are standing waves breathing up and down periodically over time but does not vary with $t$. Without loss of generality, take $\zeta_k=\mathrm{i}\eta_k$ and $c_k^{[1]}=1, k=1,\dots,n_1$. We give the following examples. (1) If $n_1=1$, then $q$ have the following expression

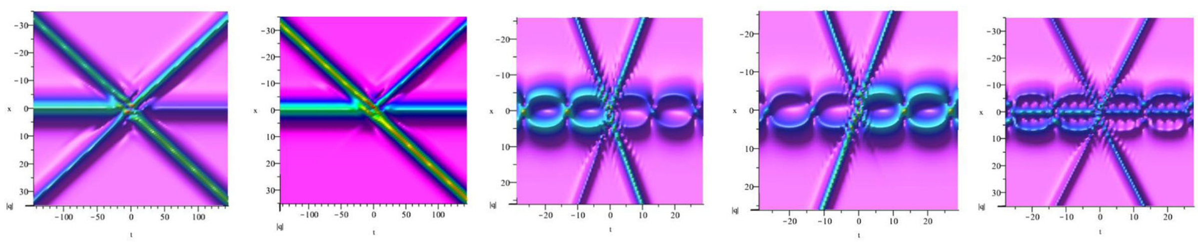
$$q=4\frac{a_1^{[1]}\eta_1 \mathrm{e}^{2\eta_1 x}}{\mathrm{e}^{4x\eta_1}+2|a_1^{[1]}|^2}\mathrm{e}^{-4\mathrm{i}\eta_1{}^2(\alpha+4\beta\eta_1{}^2)t}. \tag{28}$$

The amplitude is determined by the imaginary part of the spectrum parameter $\eta_1$ and parameter $a_1^{[1]}$. Furthermore, it can be seen that the parameters $\alpha$ and $\beta$ do not affect the amplitude of this standing wave. The plot with $\eta_1=\frac{1}{4}$, $a_1^{[1]}=\frac{\mathrm{i}}{2}$ is shown in Fig. 2a. (2) If $n_1=2$, take $\eta_1=\frac{1}{4}$, the ring breather wave shown in Fig. 2b with $\eta_2=\frac{1}{3}$ and the line breather wave in Fig. 2c with $\eta_2=\frac{4}{3}$. (3) If $n_1=3$, the ring-line breather wave shown in Fig. 2 (e) with $\eta_1=\frac{1}{4}$, $\eta_2=\frac{1}{3}$, $\eta_3=\frac{1}{2}$ is gotten. Besides, it is worth mentioned that the parameter $a_k^{[1]}$ are all taken as $\frac{\mathrm{i}}{2}$. But the shape of the waves is quite different with different $a_k^{[1]}$. For example, for $n_1=2$, when $a_1^{[1]}\neq a_2^{[1]}$, the ring breather wave will no longer be symmetric with respect to $x=0$ and



(a) $n_1=0, n_2=1$ (b) $n_1=0, n_2=1$ (c) $n_1=0, n_2=2$ (d) $n_1=0, n_2=2$ (e) $n_1=0, n_2=2$

**Fig. 3** The 2d-plots of Eq. (27) with different parameters. **a** with $\mu_1=\frac{1}{2}+\frac{\mathrm{i}}{4}$ and $a_1^{[2]}=\frac{\mathrm{i}}{2}$, $b_1^{[2]}=\mathrm{i}$; **b** with same parameters as (a) except for $b_1^{[2]}=0$; **c** with $\mu_1=\frac{1}{2}+\frac{\mathrm{i}}{4}$, $\mu_2=\frac{1}{2}+\frac{\mathrm{i}}{3}$ and $a_1^{[2]}=a_2^{[2]}=\frac{\mathrm{i}}{3}$, $b_1^{[2]}=b_2^{[2]}=1$; **d** with same parameters as (c) except for $a_1^{[2]}=0$; **e** with same parameters as (c) except for $b_1^{[2]}=0$



(a) $n_1=1, n_2=1$ (b) $n_1=1, n_2=1$ (c) $n_1=2, n_2=1$ (d) $n_1=2, n_2=1$ (e) $n_1=3, n_2=1$

**Fig. 4** The 2d-plots of Eq. (27) with different parameters. **a** $\zeta_1=\frac{\mathrm{i}}{4}$, $\mu_1=\frac{1}{2}+\frac{\mathrm{i}}{3}$ and $a_1^{[1]}=b_1^{[2]}=\frac{\mathrm{i}}{2}$, $a_1^{[2]}=1+\frac{\mathrm{i}}{2}$; **b** with same eigenvalues except for $a_1^{[2]}=0$; **c** and **d** with same $\zeta_1=\frac{\mathrm{i}}{4}$, $\zeta_2=\frac{\mathrm{i}}{3}$, $\mu_1=\frac{1}{2}+\frac{\mathrm{i}}{3}$ and except for (c) $a_1^{[2]}=1+\frac{\mathrm{i}}{2}$ and **d** $a_1^{[2]}=0$; **e** with $\zeta_1=\frac{\mathrm{i}}{4}$, $\zeta_2=\frac{\mathrm{i}}{3}$, $\zeta_2=\frac{\mathrm{i}}{2}$, $\mu_1=\frac{1}{2}+\frac{\mathrm{i}}{3}$ and $a_1^{[1]}=a_1^{[2]}=a_1^{[3]}=\frac{\mathrm{i}}{2}$

when $a_1^{[1]}=\frac{\mathrm{i}}{2}, a_2^{[1]}=\frac{\mathrm{i}}{4}$, the corresponding plot in Fig. 2 (d) and other rich dynamical behaviors can be obtained with different values combination of $a_k^{[1]}$.

**Case 2** When $n_1=0, n_2\neq 0$. The solution is a $2n_2$-soliton solution which is related to a pair of non-purely imaginary eigenvalues $\{\mu_k, -\mu_k\}$, so their center positions for the $2n_2$-soliton are symmetric with respect to $t=0$. If $n_2=1$, the fundamental two-soliton solution is gotten in Fig. 3a where $\mu_1=\frac{1}{2}+\frac{\mathrm{i}}{4}$ and $a_1^{[2]}=\frac{\mathrm{i}}{2}, b_1^{[2]}=\mathrm{i}$. And with same parameters except for $b_1^{[2]}=0$, the traveling single soliton is obtained in Fig. 3 (b) which is reduced from the two-soliton solution. Similarly, an interesting phenomenon is observed when $n_2=2$. The general four-soliton can be obtained in Fig. 3c with the nonzero parameters but when the parameter $a_1^{[2]}=0$, one of the soliton disappear after the collision in Fig. 3 (d-e). The other parameters are $\mu_1=\frac{1}{2}+\frac{\mathrm{i}}{4}, \mu_2=\frac{1}{2}+\frac{\mathrm{i}}{3}$ and $a_1^{[2]}=0, a_2^{[2]}=\frac{\mathrm{i}}{3}, b_1^{[2]}=b_2^{[2]}=1$.

**Case 3** When $n_1n_2\neq 0$, the solution is a interaction wave composed by elementary $2n_2$-soliton and periodic breathing wave. But there is one exception, when $n_1=1, n_2=1$, a general two-soliton interact with a standing wave parallel to the X-axis is illustrated in Fig. 4a. Similarly, the interactive wave as shown in Fig. 4b can be obtained when $a_1^{[2]}=0$. Besides, we also give two examples of $n_1=2, n_2=1$, then the circle breather interacting with a two soliton is obtained in Fig. 4c with $a_1^{[2]}=1+\frac{\mathrm{i}}{2}$ and Fig. 4d $a_1^{[2]}=0$. Fig. 4e is the ring-line breather interacting with a two-soliton wave. In this case, both amplitudes of the breather and soliton change after collision.

## 4 Higher-order soliton formulas

If $\det(P_+)=\hat{s}_{33}(\lambda)$ possess $N=n_1+2n_2$ elementary higher-order zeros (19) (the geometric multiplication is 1). But the corresponding order of the zero $\lambda_k$ is $R[k]$ with $R=[r_1,\ldots,r_{n_1},\hat{r}_1,\ldots,\hat{r}_{n_2},\hat{r}_1,\ldots,\hat{r}_{n_2}]$

is a $N$-th vector. So the algebraic multiplications are $N_0 = \sum_{j=1}^{n_1} r_j + 2\sum_{k=1}^{n_2} \hat{r}_k$. Then, by the symmetry relation (16), $\hat{s}_{33}$ and $s_{33}$ admits the following form

$$
\begin{aligned}
\hat{s}_{33} &= \prod_{j=1}^{n_1}(\lambda-\zeta_j)^{r_j}\prod_{k=1}^{n_2}(\lambda-\mu_k)^{\hat{r}_k}(\lambda+\mu_k^*)^{\hat{r}_k}\hat{s}_{33}^{[0]}(\lambda),\\
&\quad \hat{s}_{33}^{[0]}(\lambda)\neq 0,\\
s_{33} &= \prod_{j=1}^{n_1}(\lambda-\zeta_j^*)^{r_j}\prod_{k=1}^{n_2}(\lambda-\mu_k^*)^{\hat{r}_k}(\lambda+\mu_k)^{\hat{r}_k}s_{33}^{[0]}(\lambda),\\
&\quad s_{33}^{[0]}(\lambda)\neq 0.
\end{aligned}
$$

Inspired by the idea in [45–47], for $\left\{\zeta_k, \mu_j, -\mu_j^*\right\}$, $k = 1,\ldots,n_1$, $j = 1,\ldots,n_2$, add the perturbation parameters

$$
\zeta_k \to \zeta_k+\epsilon,\quad \mu_k \to \mu_k+\epsilon,\quad -\mu_k^* \to -\mu_k^*+\hat{\epsilon}.
$$

And the corresponding eigenvectors for the perturbation eigenvalue $\zeta_j+\epsilon \in \mathrm{i}\mathbb{R}_+$ and $\zeta_j^*+\bar{\epsilon} \in \mathrm{i}\mathbb{R}_-$ are

$$
\begin{aligned}
u_j &= \left[A_1^{[j]}e^{\theta(\zeta_j)}, A_1^{[j]*}e^{\theta(\zeta_j)}, C_1^{[j]}e^{-\theta(\zeta_j)}\right]^T,\\
u_j^\dagger &= \left[A_1^{[j]*}e^{\bar{\theta}(\zeta_j)}, A_1^{[j]}e^{\bar{\theta}(\zeta_j)}, C_1^{[j]}e^{-\bar{\theta}(\zeta_j)}\right],
\end{aligned}
\tag{29}
$$

the corresponding perturbation eigenvectors for eigenvalues $[\mu_j+\epsilon, -\mu_j^*+\hat{\epsilon}] \in \mathbb{C}_+$ and $[\mu_j^*+\bar{\epsilon}, -\mu_j+\hat{\bar{\epsilon}}] \in \mathbb{C}_-$ are

$$
\begin{aligned}
w_j &= \left[A_2^{[j]}e^{\theta(\mu_j)}, B_2^{[j]}e^{\theta(\mu_j)}, C_2^{[j]}e^{-\theta(\mu_j)}\right]^T,\\
\hat{w}_j &= \left[B_2^{[j]*}e^{\hat{\theta}(\mu_j)}, A_2^{[j]*}e^{\hat{\theta}(\mu_j)}, C_2^{[j]*}e^{-\hat{\theta}(\mu_j)}\right],\\
w_j^\dagger &= \left[A_2^{[j]*}e^{\bar{\theta}(\mu_j)}, B_2^{[j]*}e^{\bar{\theta}(\mu_j)}, C_2^{[j]*}e^{-\bar{\theta}(\mu_j)}\right]^T,\\
\hat{w}_j^\dagger &= \left[B_2^{[j]}e^{\hat{\bar{\theta}}(\mu_j)}, A_2e^{\hat{\bar{\theta}}(\mu_j)}, C_2^{[j]}e^{-\hat{\bar{\theta}}(\mu_j)}\right],
\end{aligned}
\tag{30}
$$

with

$$
\begin{aligned}
\theta(\lambda_j,\epsilon) &= i\left(\lambda_j+\epsilon\right)x\\
&\quad +2i\left[\alpha\left(\lambda_j+\epsilon\right)^2-4\left(\lambda_j+\epsilon\right)^4\beta\right]t,\\
\bar{\theta}(\lambda_j,\bar{\epsilon}) &= -i\left(\bar{\lambda_j}+\bar{\epsilon}\right)x\\
&\quad -2i\left[\alpha\left(\bar{\lambda_j}+\bar{\epsilon}\right)^2-4\beta\left(\bar{\lambda_j}+\bar{\epsilon}\right)^4\right]t,\\
\hat{\theta}(\lambda_j,\bar{\epsilon}) &= -i\left(\lambda_j^*+\bar{\epsilon}\right)x\\
&\quad +2i\left[\alpha\left(\lambda_j^*+\bar{\epsilon}\right)^2-4\beta\left(\lambda_j^*+\bar{\epsilon}\right)^4\right]t,\\
\hat{\bar{\theta}}(\lambda_j,\epsilon) &= i\left(\lambda_j+\epsilon\right)x\\
&\quad -2i\left[\alpha\left(\lambda_j+\epsilon\right)^2+4\beta i\left(\lambda_j+\epsilon\right)^4\right]t.
\end{aligned}
$$

Let $C_1^{[j]} = C_2^{[j]} = 1$ and take the following perturbation expansion $A_i^{[j]} = \sum_{k=0}^{+\infty} a_{i,j}^{[k]}\epsilon^k$, $B_i^{[j]} = \sum_{k=0}^{+\infty} b_{i,j}^{[k]}\epsilon^k$, $i = 1,2$. Expand the eigenvectors $u_k$, $u_k^\dagger$ and $w_j$, $\hat{w}_j$, $w_j^\dagger$, $\hat{w}_j^\dagger$ at $[\bar{\epsilon},\epsilon] = [0,0]$, then

$$
\begin{aligned}
u_j &= \sum_{k=0}^{r_j-1} u_j^{[k]}\epsilon^k + o(\epsilon^{r_j-1}),\\
u_j^\dagger &= \sum_{k=0}^{r_j-1} \bar{u}_j^{[k]}\bar{\epsilon}^k + o(\bar{\epsilon}^{r_j-1}),\\
w_j &= \sum_{k=0}^{\hat{r}_j-1} w_j^{[k]}\epsilon^k + o(\epsilon^{\hat{r}_j-1}),\\
w_j^\dagger &= \sum_{k=0}^{\hat{r}_j-1} \bar{w}_j^{[k]}\bar{\epsilon}^k + o(\bar{\epsilon}^{\hat{r}_j-1}),\\
\hat{w}_j &= \sum_{k=0}^{\hat{r}_j-1} \hat{w}_j^{[k]}\bar{\epsilon}^k + o(\bar{\epsilon}^{\hat{r}_j-1}),\\
\hat{w}_j^\dagger &= \sum_{k=0}^{\hat{r}_j-1} \hat{\bar{w}}_j^{[k]}\epsilon^k + o(\epsilon^{\hat{r}_j-1}).
\end{aligned}
\tag{31}
$$

To get the higher-order solution formula, add perturbation parameters $[\bar{\epsilon},\epsilon]$ into the eigenvalues and the corresponding eigenvectors of (27). At this point, the solution formula becomes an expression of the perturbation parameters. Take limit of the perturbation parameters $[\bar{\epsilon},\epsilon] \to [0,0]$, the following theorem can be obtained.

**Theorem 2** *If $\det(P_+)$ possess $n_1+2n_2$ higher-order zeros (19), then the $N_0$-th higher-order soliton formula can be gotten as*

$$
\begin{aligned}
q(x,t) &= 2\mathrm{i}\frac{\begin{vmatrix}\mathscr{F} & \bar{\chi}_3^T\\ \chi_1 & 0\end{vmatrix}}{|\mathscr{F}|},\quad r(x,t) = -q^*(x,-t)\\
&= 2\mathrm{i}\frac{\begin{vmatrix}\mathscr{F} & \bar{\chi}_3^T\\ \chi_2 & 0\end{vmatrix}}{|\mathscr{F}|}
\end{aligned}
\tag{32}
$$

*where $\chi[k]$, $\bar{\chi}[j]$ be the $k$-th column and $j$-th row of the following $\chi$ and $\bar{\chi}$, respectively.*

$$\chi = \Big[u_1^{[0]}, \ldots, u_1^{[r_1-1]}, \ldots, u_{n_1}^{[0]}, \ldots, u_{n_1}^{[r_{n_1}-1]}, w_1^{[0]}, \ldots, w_1^{[\bar r_1-1]}, \ldots, w_{n_2}^{[0]}, \ldots, w_{n_2}^{[\bar r_{n_2}-1]}, \hat w_1^{[0]}, \ldots, \hat w_1^{[\bar r_1-1]}, \ldots, \hat w_{n_2}^{[0]}, \ldots, \hat w_{n_2}^{[\bar r_{n_2}-1]}\Big]_{3\times N_0},$$

$$\bar\chi = \Big[\bar u_1^{[0]}, \ldots, \bar u_1^{[r_1-1]}, \ldots, \bar u_{n_1}^{[0]}, \ldots, \bar u_1^{[r_{n_1}-1]}, \bar w_1^{[0]}, \ldots, \bar w_1^{[\bar r_1-1]}, \ldots, \bar w_{n_2}^{[0]}, \ldots, \bar w_{n_2}^{[\bar r_{n_2}-1]}, \hat{\bar w}_1^{[0]}, \ldots, \hat{\bar w}_1^{[\bar r_1-1]}, \ldots, \hat{\bar w}_{n_2}^{[0]}, \ldots, \hat{\bar w}_{n_2}^{[\bar r_{n_2}-1]}\Big]_{N_0\times 3}$$

and $\mathscr{F} = \left[\mathscr{M}_{k,j}\right]_{N_0\times N_0}$, $k, j = 1..N_0$ is a block matrix with $\mathscr{M}_{k,j} = \left[m_{k,j}^{[l_1,l_2]}\right]_{R[k]\times R[j]}$ are $R[k]\times R[j]$ matrixes with the elements are

$$m_{k,j}^{[l_1,l_2]} = \lim_{\epsilon,\bar\epsilon\to 0} \frac{1}{(k-1)!(j-1)!} \frac{\partial^{k+j-2}}{\partial \bar\epsilon^{l_1-1}\partial\epsilon^{l_2-1}} \left[\frac{\bar\chi[k]\chi[j]}{\lambda_k^* + \bar\epsilon - \lambda_j + \epsilon}\right].$$

expressions $u_1 = u_1^{[0]} + u_1^{[1]}\epsilon + o(\epsilon)$, $u_1^\dagger = \bar u_1^{[0]} + \bar u_1^{[1]}\bar\epsilon + o(\bar\epsilon)$. $\mathscr{F} = \left[m_{1,1}^{[l_1,l_2]}\right]_{2\times 2}$ is a $2\times 2$ matrix, where the elements $m_{1,1}^{[l_1,l_2]}$ are the coefficients of the following expansion

$$\left[\frac{\bar\chi[k]\chi[j]}{\lambda_k^* + \bar\epsilon - \lambda_j + \epsilon}\right] = m_{1,1}^{[1,1]} + m_{1,1}^{[2,1]}\bar\epsilon + m_{1,1}^{[1,2]}\epsilon + m_{1,1}^{[2,2]}\bar\epsilon\epsilon + o(\bar\epsilon\epsilon).$$

By the higher-order formula (32), then the 2-order soliton solution for the nonlocal LPD Eq.is

$$q(x,t) = 2\mathrm{i}\frac{\begin{vmatrix} m_{1,1}^{[1,1]} & m_{1,1}^{[1,2]} & \bar u_1^{[0]}[3] \\ m_{1,1}^{[2,1]} & m_{1,1}^{[2,2]} & \bar u_1^{[1]}[3] \\ u_1^{[0]}[1] & u_1^{[1]}[1] & 0 \end{vmatrix}}{\begin{vmatrix} m_{1,1}^{[1,1]} & m_{1,1}^{[1,2]} \\ m_{1,1}^{[2,1]} & m_{1,1}^{[2,2]} \end{vmatrix}}.$$

Let $\zeta_1 = \mathrm{i}\xi_1$ in the equation above and simplify the equation, so we have

$$q(x,t) == \frac{\left[-8\xi_1 C_1 \mathrm{e}^{2x\xi_1+3a_{1,1}^{[0]}} - 8\xi_1\left(64i\beta t{\xi_1}^4 + 8i\alpha t{\xi_1}^2 - i\xi_1 a_{1,1}^{[1]} + 2x\xi_1 - 1\right)\mathrm{e}^{6x\xi_1+a_{1,1}^{[0]}}\right]\mathrm{e}^{-4it{\xi_1}^2(4\beta{\xi_1}^2+\alpha)}}{\left(16{a_{1,1}^{[1]}}^2{\xi_1}^2 + 4\right)\mathrm{e}^{4a_{1,1}^{[0]}} + \mathrm{e}^{8x\xi_1} + C_2\mathrm{e}^{4x\xi_1+2a_{1,1}^{[0]}}},$$

with

$$C_1 = \left(-256a_{1,1}^{[1]}{\xi_1}^5\beta + 128i\beta{\xi_1}^4 - 32a_{1,1}^{[1]}{\xi_1}^3\alpha + 16i\alpha{\xi_1}^2\right)t - 8ixa_{1,1}^{[1]}{\xi_1}^2 - 4{a_{1,1}}^2{\xi_1}^2 - 2ia_{1,1}^{[1]}\xi_1 - 4x\xi_1 - 2,$$

$$C_2 = \left(32768{\xi_1}^8\beta^2 + 8192{\xi_1}^6\alpha\beta + 512{\xi_1}^4\alpha^2\right)t^2 + 32{\xi_1}^2x^2 + 8{a_{1,1}^{[1]}}^2{\xi_1}^2 + 4.$$

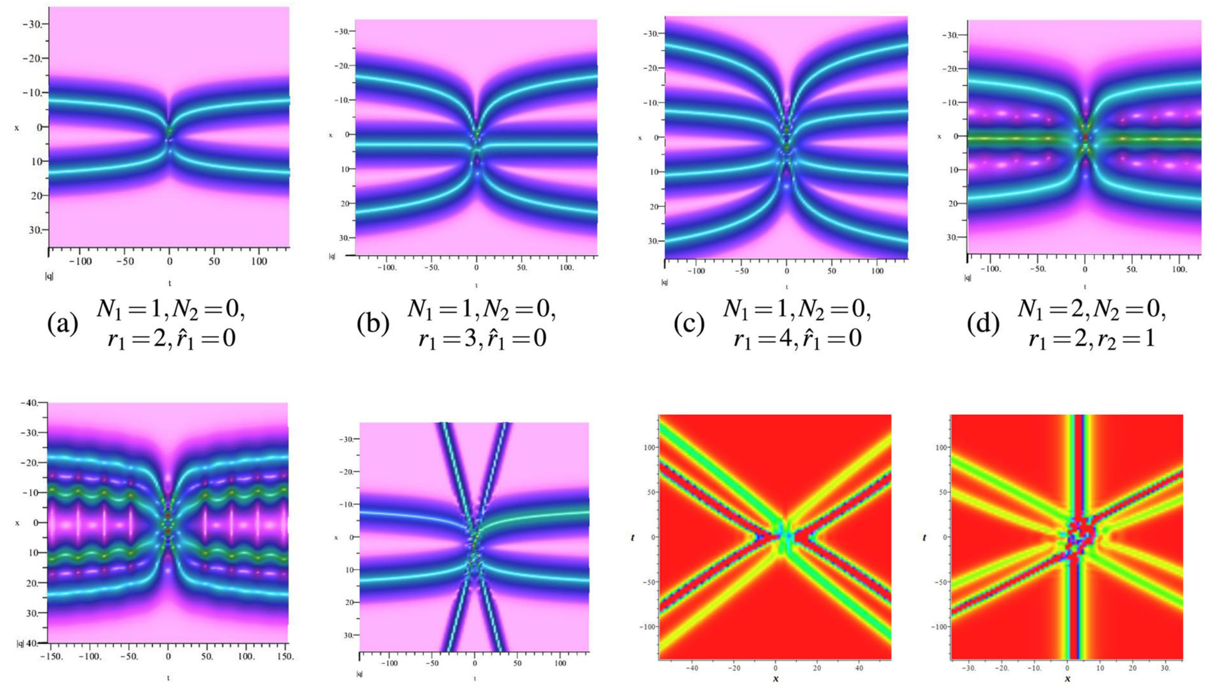
Take $\zeta_1 = \frac{1}{4}$, $a_{1,1}^{[0]} = a_{1,1}^{[1]} = 0$ and the plot of 2-order soliton for the nonlocal LPD Eq. (1) is shown in Fig. 5 (a). It can be seen that the branches of higher order solitons propagate forward in almost parallel directions with time, and then return to the original direction after the collision near the zero point and continue to propagate forward. Besides, the 3-order and 4-order soliton corresponding to a purely imaginary higher-order zero for the nonlocal LPD Eq. (1) are shown in Fig. 5b and c, respectively.

The proof of the theorem can be referred to the details in [45,47], where the nonlocal NLS equation was corresponded to $2\times 2$ linear equation. The process for this nonlocal LPD Eq. (1) is similar except for the symmetry properties. Thus, the detailed process of the higher-order soliton formula for such nonlocal LPD Eq. (1) is omitted here.

*Remark* The biggest difficulty and difference here is that the symmetry properties for the perturbed parameters. Due to the special nonlocal properties for such nonlocal LPD Eq. (1) corresponding to a $3\times 3$ Lax pair, the symmetry properties for the perturbed parameters are quite different and difficult to conclude.

To illustrate the higher-order soliton formula (32), we give some examples as follows.

When $N_1 = 1$, $N_2 = 0$, the RHP (1) only have a purely imaginary higher-order zero $\zeta_1$. In this case, the solution is a higher-order soliton. If $r_1 = 2$, at this time, implement the Taylor expansions of the following



(a) $N_1=1, N_2=0, r_1=2, \hat{r}_1=0$ (b) $N_1=1, N_2=0, r_1=3, \hat{r}_1=0$ (c) $N_1=1, N_2=0, r_1=4, \hat{r}_1=0$ (d) $N_1=2, N_2=0, r_1=2, r_2=1$

(e) $N_1=2, N_2=0, r_1=2, r_2=2$ (f) $N_1=1, N_2=1, r_1=2, \hat{r}_1=1$ (g) $N_1=0, N_2=1, r_1=0, \hat{r}_1=2$ (h) $N_1=1, N_2=1, r_1=1, \hat{r}_1=2$

**Fig. 5** The 2d-plots of the higher-order soliton formula (32) for nonlocal LPD Eq. (27) with different parameters. The parameters of the waves are: $\alpha = \beta = 1$ and the eigenvalues are **a**, **b** and **c**:$\zeta_1 = \frac{i}{4}$, **d** and **e**:$\zeta_1 = \frac{i}{4}, \zeta_1 = \frac{i}{6}$, **f**:$\mu_1 = \frac{1}{3} + \frac{i}{5}$ and **g**, **h**:$\zeta_1 = \frac{i}{4},\ \mu_1 = \frac{1}{3} + \frac{i}{5}$. Other parameters are all 0

In addition, in Fig. 5d, we also give the mixed higher-order soliton solution of a two-order soliton and a standing wave, which is correspond to a purely imaginary higher-order zero and a purely imaginary simple zero. And Fig. 5e is a higher-order soliton solution which is correspond to two purely imaginary 2-order zeros. Fig. 5c and e shows two different kinds of 4-order soliton corresponded to different spectral configurations. Besides, Fig. 5f and g shows the other two kinds of 4-order solitons with different higher-order zeros, respectively. The wave shown in Fig. 5f is correspond to a two second-order purely imaginary zeros and a pair of non-purely imaginary simple zeros. And the solution of Fig. 5g is correspond to a pair of second-order non-purely imaginary zeros. Finally, Fig. 5e gives a standing wave interact with a fourth-order soliton shown in Fig. 5d, which is correspond to a purely imaginary simple zero and a pair of non-purely imaginary second-order zeros .

## 5 Conclusion

In summary, we proposed a novel nonlocal LPD Eq. (1) corresponding to a $3 \times 3$ Lax pair (4a and 4b), which is reduced from a two-component coupled LPD Eq. (3) by a reverse-time reduction. Then, the inverse scattering transformation for initial value problem of the nonlocal LPD Eq. (1) was studied with the help of an RHP. The RHP was established after constructing the Jost function with specific analytic properties and asymptotic properties. The symmetry relations were then provided for the Jost function and the scattering data.

The zeros of the RHP were purely imaginary or appear in pairs as $(\mu_k, -\mu_k^*)$. Then, the corresponding reflectionless potential formulas for the RHP were given as the following three different types of simple zeros: Case 1: all the zero are purely imaginary; Case 2: all the zero are non-purely imaginary or in pairs; Case 3: $n_1$ zeros are purely imaginary and $n_2$ pairs of non-purely imaginary zeros .

At the same time, the corresponding plots and dynamical behaviors for the above solutions were given. For case 1, when $n_1 = 1, n_2 = 0$, the solution is a standing wave which did not travel with time. When $n_1 \leq 2, n_2 = 0$, the solutions were breathing with time and kinds of rich structures were given with different parameters. For case 2, when $n_1 = 0, n_2 \neq 0$, the $2n_2$ solitons occurring symmetrically were obtained with suitable parameters. Besides, the $2n_2 - 1$ solitons and special soliton structures shown in Fig. 3c and e can be obtained in this case. For Case 3, when $n_1 n_2 \neq 0$, the interaction solutions of a two-soliton solutions and a standing wave were obtained. It is worth mentioning that the amplitude of these two waves changed markedly after the collision. The plots of the symmetric two-solitons interacting with a standing wave and breathers are shown in Fig. 4a, c and e. Besides, with special parameters, the special interacting waves as shown in Fig. 4b and d were obtained.

Ultimately, the higher-order soliton formula was obtained by adding perturbations and taking limit of the perturbations. Then by some matrix transformations, the higher-order formula in determinant form (5) was given. It is worth noting that this formula can be reduced to the solution formula (27) corresponding to simple zeros when the order of all higher-order zeros is one. In addition, through this formula (5), we can also get some solutions when the high-order zeros and the simple zeros exist simultaneously. The corresponding solutions corresponding to different combinations of the following zeros: purely imaginary higher-order zeros, purely imaginary simple zeros, pairs of non-purely imaginary simple zeros and pairs of non-purely imaginary higher-order zeros are shown in Fig. 5, respectively.

Furthermore, the inverse scattering transformation and kinds of soliton solutions for the following nonlocal integrable LPD equation will be considered in the future.

(1) The nonlocal reverse-space LPD equation

$$
\begin{aligned}
&\mathrm{i}q_t(x,t)\\
&= \mathrm{i}\beta\left[q_{xxx}(x,t) + 3q_x(x,t)\left(|q(x,t)|^2\right.\right.\\
&\quad \left.+|q(-x,t)|^2\right) + 3q(x,t)\left(q_x(-x,t)q^*\right.\\
&\quad \left.\left.\times(-x,t) + q_x(x,t)q^*(x,t)\right)\right] + \alpha\left[q_{xx}(x,t)\right.\\
&\quad \left.+2q\left(|q(x,t)|^2 + |q(-x,t)|^2\right)\right].
\end{aligned}
\tag{33}
$$

(2) The nonlocal reverse-time LPD equation

$$
\begin{aligned}
&\mathrm{i}q_t(x,t)\\
&= \mathrm{i}\beta\left[q_{xxx}(x,t) + 3q_x(x,t)\left(|q(x,t)|^2 + |q(x,-t)|^2\right)\right.\\
&\quad \left.+3q(x,t)\left(q_x^*(x,-t)q(x,-t) + q_x(x,t)q^*(x,t)\right)\right]\\
&\quad + \alpha\left[q_{xx}(x,t) + 2q(x,t)\left(|q(x,t)|^2 + |q(x,-t)|^2\right)\right].
\end{aligned}
\tag{34}
$$

(3) The nonlocal LPD equation of reverse-spacetime type

$$
\begin{aligned}
&\mathrm{i}q_t(x,t)\\
&= \mathrm{i}\beta\left[q_{xxx}(x,t) + 3q_x(x,t)\left(|q(x,t)|^2 + |q(-x,-t)|^2\right)\right.\\
&\quad + 3q(x,t)\left(q_x^*(-x,-t)q(-x,-t)\right.\\
&\quad \left.\left.+q_x(x,t)q^*(x,t)\right)\right]\\
&\quad + \alpha\left[q_{xx}(x,t) + 2q(x,t)\left(|q(x,t)|^2 + |q(-x,-t)|^2\right)\right].
\end{aligned}
\tag{35}
$$

The nonlocal equations above were obtained from the coupled LPD Eq. (3) by the following nonlocal reductions $r(x,t) = q(-x,t)$, $r(x,t) = q^*(x,-t)$, and $r(x,t) = q^*(-x,-t)$.

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**Data Availability** The datasets generated and analyzed during the current study are available from the corresponding author on reasonable request.

## Declarations

**Conflicts of interest** The authors have no relevant financial or nonfinancial interests to disclose. The authors declare that there is no conflict of interests regarding the publication of this paper.

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